On the inhomogeneous Aizenman-Bak model

A nonlocal repulsion-aggregation model: Steady States, Stability, Bifurcations

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[Aizenman, Bak]'79 inhomogeneous

Continuous coagulation/fragmentation of polymers

$$\partial_t f - a(y)\triangle_x f = Q_{coag}(f,f) + Q_{frag}(f)$$

polymer density: $f(t,x,y) \geq 0$

time $t \geq 0$, space $x \in \Omega$ with $|\Omega| = 1$, size/length $y \in [0,\infty)$

$$\begin{aligned} Q_{coal}(f,f) &= \int_0^y f(y-y')\,f(y')dy' - 2f(y)\int_0^\infty f(y')\,dy' \\ Q_{frag}(f) &= 2\int_y^\infty f(y')\,dy' - y\,f(y) \end{aligned}$$

homogeneous Neumann $\nabla_x f(t,x,y)\cdot\nu(x) = 0$ on $\partial\Omega$

[Aizenman, Bak]'79 inhomogeneous

Macroscopic densities

amount of monomers N, number density M

$$N = \int_0^\infty y' f(y')\, dy' \,, \quad M = \int_0^\infty f(y')\, dy'$$

conservation of the total mass

$$\begin{aligned} \partial_t N - \triangle_x \left(\int_0^\infty y' a(y') f(y')\, dy' \right) &= 0 \\ \partial_t M - \triangle_x \left(\int_0^\infty a(y') f(y')\, dy' \right) &= N - M^2 \end{aligned}$$

[Aizenman, Bak]'79 inhomogeneous

Entropy (free energy functional)

entropy

$$H(f)(t,x) = \int_0^\infty (f \ln f - f)\; dy\,,$$

entropy dissipation

$$\begin{aligned}
\frac{d}{dt}\int_\Omega H(f)\,dx &= -D_H(f) \\
D_H(f) &= \int_\Omega \int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\
&\quad + \int_\Omega \int_0^\infty \int_0^\infty (f'' - f f')\ln\left(\frac{f''}{f f'}\right) dy\,dy' dx
\end{aligned}$$

Inequality by [Aizenman, Bak]'79

$$\int_0^\infty\int_0^\infty (f(y+y') - f(y)f(y'))\ln\left(\frac{f(y+y')}{f(y)f(y')}\right)dy\,dy' \geq$$

$$M\,H(f|f_{\sqrt{N},N}) + 2(M-\sqrt{N})^2$$

entropy dissipation

$$\begin{aligned} D_H(f) \quad &\geq \quad \int_\Omega\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\ &\quad +M\,H(f|f_{\sqrt{N},N}) + 2(M-\sqrt{N})^2 \end{aligned}$$

[Aizenman, Bak]'79 inhomogeneous

local and global equilibria

intermediate equilibria with the very moments N and $M = \sqrt{N}$

$$f_{\sqrt{N},N} = e^{-\frac{1}{\sqrt{N}}y}$$

global equilibrium

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}}$$

constant in x satisfying $M_\infty^2 = N_\infty$

conservation of mass $N_\infty = \int_0^\infty N(x)\,dx$

[Aizenman, Bak]'79 inhomogeneous

relative entropy, additivity

relative entropy

$$H(f|g) = H(f) - H(g)$$

additivity

$$H(f|f_\infty) = H(f|f_{\sqrt{N},N}) + H(f_{\sqrt{N},N}|f_\infty)$$

$f_{\sqrt{N},N}$ and f_∞ do not need to have the same L^1_y-norm, but nevertheless

$$\int_\Omega H(f_{\sqrt{N},N}|f_\infty)\,dx = 2\left(\sqrt{\int_\Omega N\,dx} - \int_\Omega \sqrt{N}\,dx\right) \geq 0$$

[Aizenman, Bak]'79 inhomogeneous

Existence results

[Am, AW] global existence and uniqueness of classical solutions (1D, not [Aizenman, Bak])

[LM] global existence of weak solutions satisfying the entropy dissipation inequality

$$\int_\Omega H(f(t))\,dx + \int_0^t D_H(f(s))\,ds \le \int_\Omega H(f_0)\,dx$$

Diffusivity $a(y) \in L^\infty([1/R, R])$ for all $R > 0$

Without rate: Equilibrium states attract all global weak solutions

[Aizenman, Bak]'79 inhomogeneous

Theorems

Nonnegative initial data $(1 + y + \ln f_0)f_0 \in L^1((0,1) \times (0,\infty))$ with positive initial mass $\int_0^1 N_0(x)\,dx = N_\infty > 0$

For $\Omega = (0,1)$ and $0 < a_* \le a(y) \le a^*$ or $\Omega \in \mathbb{R}^d$ and $a = const$

$$\|f(t,\cdot,\cdot) - f_\infty\|^2_{L^1_{x,y}} \quad \le \quad C_2 \int_0^1 H(f_0|f_\infty)\,dx\; e^{-\alpha\, t}$$

$$\int_0^\infty (1+y)^q\, \|f(t,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\, dy \le C_3\, e^{-\alpha\, t}\,, \quad q \ge 1$$

For $\Omega = (0,1)$ and $a(y) \in L^\infty([1/R, R])$ for all $R > 0$ with $a(y) = O(y^{-\gamma})$ for $\gamma < 1$: algebraic decay

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$**,** $\mathcal{M}_*$**,** a_*

Step 1) Additivity

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f_{\sqrt{N},N}|f_\infty)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$**,** $\mathcal{M}_*$**,** a_*

Step 2) "Reacting" Moments N and M

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f_{\sqrt{N},N}|f_\infty)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \leq \frac{2}{\sqrt{N_\infty}}\left[\|M - \sqrt{N}\|^2_{L^2_x} + \|M - \overline{M}\|^2_{L^2_x}\right].$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$**,** $\mathcal{M}_*$**,** a_*

Step 2) "Reacting" Moments N **and** $M > \mathcal{M}_* > 0$

$$
\begin{aligned}
\int_0^1 H(f|f_\infty)\,dx &\le C\left[\int_0^1 M\,H(f|f_{\sqrt{N},N})\,dx + 2\|M-\sqrt{N}\|^2_{L^2_x}\right] \\
&\quad + \frac{4}{\sqrt{N_\infty}}\|M-\overline{M}\|^2_{L^2_x} \\
&\le C\int_0^1\int_0^\infty\int_0^\infty (f''-f\,f')\ln\left(\frac{f''}{f\,f'}\right)dy\,dy'\,dx \\
&\quad + \frac{4}{\sqrt{N_\infty}}\|M-\overline{M}\|^2_{L^2_x}\,,
\end{aligned}
$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\mathcal{M}_*$, a_*

Step 3) Diffusion

$$
\begin{aligned}
\int_0^1\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dydx &\geq \frac{a_*}{\|M\|_{L^\infty_x}}\int_0^1\left[\int_0^\infty \frac{|\nabla_x f|^2}{f}\,dy\right]\int_0^\infty f\,dydx \\
&\geq \frac{a_*}{\|M\|_{L^\infty_x}}\int_0^1\left|\int_0^\infty \nabla_x f\,dy\right|^2 dx \\
&= \frac{a_*}{\|M\|_{L^\infty_x}}\int_0^1 |\nabla_x M|^2\,dx \\
&\geq \frac{a_*}{P(\Omega)\|M(t,\cdot)\|_{L^\infty_x}}\|M-\overline{M}\|^2_{L^2_x}\,.
\end{aligned}
$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\mathcal{M}_*$, a_*

Entropy Entropy Dissipation Estimate

Let $f := f(x,y) \geq 0$ be a measurable function with moments satisfying $0 < \mathcal{M}_* \leq M(x) \leq \|M\|_{L^\infty_x}$ and $0 < N_\infty = \overline{N}$. Assume $0 < a_* \leq a(y) \leq a^*$. Then,

$$\int_0^1 H(f|f_\infty)\,dx \leq C(\mathcal{M}_*, N_\infty, a_*, P(\Omega))\|M\|_{L^\infty_x} D(f)\,.$$

Assume $a(y) = O(y^{-\gamma})$, $\gamma < 1$. Then, for all $A > 0$

$$\int_0^1 H(f|f_\infty)\,dx \leq C(\mathcal{M}_*, \gamma, A)\| \int_0^\infty a(y) f(y)\,dy\|_{L^\infty_x} D + \frac{C}{A^2}\|N\|_{L^2_x}$$

A-priori Estimates

$L^1_t + L^\infty_t$ bounds

Lemma

$$\sup_{0<x<1} \int_0^\infty a(y)\, f(t,x,y)\, dy \le m_\infty + m_1(t)$$

1D: $\sqrt{f}(t,x,y) - \sqrt{f}(t,\tilde{x},y) = \int_{\tilde{x}}^x \partial_x \sqrt{f}(t,\xi,y)\, d\xi$

$$\begin{aligned} \int_0^\infty \sup_{0<x<1} a(y)\, f(t,x,y)\, dy \quad &\le \quad 2 \int_0^\infty \int_0^1 a(y) \left(\partial_x \sqrt{f}(t,\xi,y)\right)^2 d\xi\, dy \\ &\quad +2 \int_0^\infty \int_0^1 a(y)\, f(t,\tilde{x},y)\, d\tilde{x}\, dy \end{aligned}$$

if $0 < a_* \le a(y)$: $\|M\|_{L^\infty_x}$ bound in $L^1_t + L^\infty_t$

A-priori Estimates

$0 < a_* \le a(y)$: **Moments** $M_p(f)(t) := \int_0^1 \int_0^\infty y^p\, f\, dy\, dx$

Lemma: For $p > 1$ and for a.a. $t > t_* > 0$

$$M_p(f)(t) \le \mathcal{M}_p^*(M_p(f)(t_*), m_\infty, m_1, p)$$

Idea: fragmentation produces moments

$$\begin{aligned} \int_0^\infty y^p Q(f,f)\, dy \quad &\le \quad 2(C_p - 1) \int_0^\infty y^p f(y)\, dy\, (m_\infty + m_1(t)) \\ &\quad -\frac{p-1}{p+1} \int_0^\infty y^{p+1}\, f(y)\, dy \end{aligned}$$

A-priori Estimates

$0 < a_* \leq a(y)$: **Moments** $M_p(f)(t) := \int_0^1 \int_0^\infty y^p \, f \, dy \, dx$

Interpolation

$$M_p(f)(t) \leq \frac{1}{\epsilon^{p-1}} \int_\Omega N_0(t,x)\, dx + \epsilon \, M_{p+1}(f)(t)$$

Thus

$$\frac{d}{dt} M_p(f)(t) \leq -\frac{1}{2\epsilon} M_p(f)(t) + 2(C_p - 1)\, m_1(t)\, M_p(f)(t) + C_\epsilon$$

Vallée-Poussin for $(1+y) f_0 \in L^1$

A-priori Estimates

$a(y) = O(y^\gamma)$, $\gamma < 1$: Moments up to $y^2/a(y)$

$$\int_0^\infty Q\, y^2\, dy = \int_0^\infty\!\!\int_0^\infty 2yy' f(y) f(y')\, dy dy' - \frac{1}{3}\int_0^\infty y^3\, f(y)\, dy$$

and $2yy' \leq \frac{y^2}{a(y)} a(y') + \frac{y'^2}{a(y')} a(y)$

$$\|M\|^2_{L^2_x} \leq \int_\Omega \int_0^\infty \frac{f(y)}{a(y)}\, dy \int_0^\infty a(y) f(y)\, dy\, dx \quad \text{in } L^1_t + L^\infty_t$$

$$\|N\|^2_{L^2_x} \leq \int_\Omega \int_0^\infty \frac{y^2 f(y)}{a(y)}\, dy \int_0^\infty a(y) f(y)\, dy\, dx \quad \text{in } L^1_t + L^\infty_t$$

Moreover $\int_0^\infty Q(1+y)^2\, dy \leq N - M^2 + 2N^2$ is controlled

A-priori Estimates

$\|M\|_{L^\infty_x}$ via bootstrap

heat equation: $\partial_t f - a(y)\partial_{xx} f = g$

for all $q \in [1,3)$: $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$$\|f\|_{L^r([0,T]\times\Omega)} \leq C_T a(y)^{\frac{1-q}{2q}} \|f_{in}\|_{L^p_x} + C_T a(y)^{\frac{1-q}{2q}} \|g\|_{L^p_{t,x}}$$

and while $a(y)^{\frac{1-q}{2q}} \leq (1+y)^{1/3}$ for y large

$$\|M\|_{L^\infty_x} \leq C_T$$

Inhomogeneous Aizenman-Bak

Fast reaction limit

$$\partial_t f^\varepsilon - a(y)\triangle_x f^\varepsilon = \frac{1}{\varepsilon}\left(Q_{coag}(f^\varepsilon, f^\varepsilon) + Q_{frag}(f^\varepsilon)\right)$$

formal limit: $f^\varepsilon \to e^{-\frac{y}{\sqrt{N^0}}}$ satisfying

$$\partial_t N^0 - \triangle_x n(N^0) = 0$$

where $n(N) := \int_0^\infty a(y) y e^{-\frac{y}{\sqrt{N}}}\, dy$ with $0 < a_* N \leq n(N) \leq a^* N$

Theorems:

- convergence without rate using compactness
- assuming lower bound: convergence with rate in ε

A nonlocal repulsion-aggregation model

Nonlocal Fokker-Planck type evolution equation

$\int_{\mathbb{R}} \rho = 1$ conserved, measure valued solutions

$$\partial_t \rho = \partial_x(\rho \partial_x [a(\rho) - G * \rho + V])$$

Consider only a repulsion-aggregation potential

1D: $u(z)$, $z \in [0,1]$ pseudo-inverse of distribution function

$$\partial_t u(z) = \int_0^1 G'(u(z) - u(\zeta))\, d\zeta\,, \qquad z \in [0,1]\,,$$

smooth $G(x) = G(-x)$ even

local minimum $x = 0$, local maximum $x = 2x_0$

$$G'(0) = 0\,,\ G''(0) = \beta > 0\,, \quad G'(2x_0) = 0\,,\ G''(2x_0) = -\alpha < 0\,.$$

A nonlocal repulsion-aggregation model

conservation law, steady state

Conservation of (centre of) mass $\int_0^1 u_{in}(z)\,dz$

$$\int_0^1 u(z,t)\,dz = \int_0^1 u_{in}(z)\,dz = 0 \qquad t \geq 0\,,$$

One-parameter family of monotone increasing two-valued steady states

$$u_\infty(z,z_0) = \begin{cases} -2(1-z_0)x_0 & z < z_0\,, \\ 2z_0x_0 & z > z_0\,. \end{cases}$$

Parameter $z_0 \in (0,1)$

A nonlocal repulsion-aggregation model

Linear stability

Linearised nonlocal operator

$$F'(u_\infty(z_0))(v(z)) = \begin{cases} \lambda_1\, v(z) - (\alpha+\beta)\int_0^{z_0} v(z)\,dz & z < z_0\,, \\ \lambda_2\, v(z) + (\alpha+\beta)\int_0^{z_0} v(z)\,dz & z > z_0\,. \end{cases}$$

where λ_1 and λ_2 denote

$$\lambda_1 := z_0\beta - (1-z_0)\alpha\,, \qquad \lambda_2 := (1-z_0)\beta - z_0\alpha\,.$$

λ_1 and λ_2 are convex combinations of β and $-\alpha$.
Consider mass preserving perturbations $v(z)$: $\int_0^1 v(z)\,dz = 0$.

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

1) $\lambda_1 \neq \lambda \neq \lambda_2$: Then φ is piecewise constant and

$$\lambda = -\alpha < 0\,, \qquad v(z) = \begin{cases} -\frac{1-z_0}{z_0} v_r & z < z_0\,, \\ v_r & z > z_0\,. \end{cases}$$

for all constants $v_r \neq 0$.

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

2) $\lambda_1 = \lambda = \lambda_2$, $z_0 = \frac{1}{2}$: Then,

$$\lambda = \frac{\beta-\alpha}{2}\,, \qquad v(z) = \begin{cases} v_l(z) & z < \frac{1}{2}\,, \\ v_r(z) & z > \frac{1}{2}\,. \end{cases}$$

$v_l(z)$ and $v_r(z)$ such that $\int_0^{1/2} v_l\,dz = 0 = \int_{1/2}^1 v_r\,dz$

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

3) $\lambda_1 = \lambda \neq \lambda_2$, $z_0 \neq \frac{1}{2}$: Then $\int_0^{z_0} v\,dz = 0 = \int_{z_0}^1 v\,dz$

$$\lambda = \lambda_1 = z_0\beta - (1-z_0)\alpha\,, \qquad v(z) = \begin{cases} v_l(z) & z < z_0\,, \\ 0 & z > z_0\,. \end{cases}$$

$v_l(z)$ such that $\int_0^{z_0} v_l\,dz = 0$.

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,\varphi(z) = +(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z < z_0\,, \\ (\lambda_2 - \lambda)\,\varphi(z) = -(\alpha+\beta)\int_0^{z_0}\varphi(z)\,dz & z > z_0\,. \end{cases}$$

4) $\lambda_2 = \lambda \neq \lambda_1$, $z_0 \neq \frac{1}{2}$: Like 3) after mirroring $z_0 \to (1 - z_0)$.

Summary: Given $\beta - \alpha < 0$ there exists an open interval of parameters z_0 with linearly stable steady states $u_\infty(z_0)$:

$$\max\{\lambda(z_0)\} < 0 \qquad \forall z_0 \in (1 - z_0^*, z_0^*)\,, \quad z_0^* := \frac{\alpha}{\alpha+\beta} > \frac{1}{2}\,.$$

A nonlocal repulsion-aggregation model

local asymptotic stability for $\beta < \alpha$

Assume for a constant A and a δ small enough.

$$|G''(x) - \beta| < A\delta \quad \text{for} \quad |x| < 2\delta\,,$$

$$|G''(x) + \alpha| < A\delta \quad \text{for} \quad |x - 2x_0| < 2\delta\,,$$

$$w(t,z) = e^{\mu t}(u((t,z) - u_\infty(z))\,, \qquad \int_0^1 w(z)\,dz = 0\,,$$

rate $\mu > 0$, Denote $\Delta u_\infty := u_\infty(z) - u_\infty(\zeta)$, $\Delta w := w(z) - w(\zeta)$

$$\partial_t w = \mu w + e^{\mu t} \int_0^1 G'(\Delta u_\infty + e^{-\mu t}\Delta w)\,d\zeta\,.$$

local asymptotic stability for $\beta < \alpha$

For a B small enough

$$S_\delta = \left\{ w(z) : ||w(z)||_{L^\infty([0,1])} \leq \delta \,,\ \left| \int_0^{z_0} w(z)\, dz \right| \leq B\delta \right\} \,,$$

Denote $S := \mathrm{sign}(\int_0^{z_0} w(z)\, dz)$

$$\frac{d}{dt} \left| \int_0^{z_0} w\, dz \right| = \mu \left| \int_0^{z_0} w\, dz \right| + e^{\mu t} \int_0^{z_0}\!\! \int_0^{z_0} S\, G'(e^{-\mu t} \Delta w)\, dz d\zeta$$
$$+ e^{\mu t} \int_0^{z_0}\!\! \int_{z_0}^{1} S\, G'(-2x_0 + e^{-\mu t} \Delta w)\, dz d\zeta \,.$$

Taylor expansion $G'(e^{-\mu t} \Delta w) = G''(\xi_1(z,\zeta))\, e^{-\mu t} \Delta w$

A nonlocal repulsion-aggregation model

local asymptotic stability for $\beta < \alpha$

For a B small enough

$$S_\delta = \left\{ w(z) : ||w(z)||_{L^\infty([0,1])} \leq \delta \, , \, \left| \int_0^{z_0} w(z) \, dz \right| \leq B\delta \right\} \, ,$$

Denote $S := \mathrm{sign}(\int_0^{z_0} w(z) \, dz)$

$$\begin{aligned} \frac{d}{dt} \left| \int_0^{z_0} w \, dz \right| &\leq (\mu - \alpha) \left| \int_0^{z_0} w \, dz \right| + A 2 z_0 \delta^2 \\ &\leq B\delta \left(\mu - \alpha + \frac{2 z_0 A}{B} \delta \right) . \end{aligned}$$

A nonlocal repulsion-aggregation model

local asymptotic stability for $\beta < \alpha$

For a B small enough

$$S_\delta = \left\{ w(z) : ||w(z)||_{L^\infty([0,1])} \le \delta \, , \; \left| \int_0^{z_0} w(z)\, dz \right| \le B\delta \right\} \, ,$$

Denote $s := \operatorname{sign}(w(z_0))$

$$\begin{aligned} \partial_t |w|(t_0, z_0) =& \mu|w| + e^{\mu t} \int_0^{z_0} s\, G'(2x_0 + e^{-\mu t} \Delta w(z_0, \zeta))\, d\zeta \\ & + e^{\mu t} \int_{z_0}^{1} s\, G'(e^{-\mu t} \Delta w(z_0, \zeta))\, d\zeta \\ \le& \delta \left(\mu + \lambda_1 + A\delta + (\alpha + \beta) B \right) \, , \end{aligned}$$

Choose $\mu < \frac{\alpha - \beta}{2} < \alpha$, B, δ small

A nonlocal repulsion-aggregation model

Bifurcation for $z_0 \neq \frac{1}{2}$

Formal expansion if $\varepsilon := \lambda_1 \ll 1$

$$u(z) = u_\infty(z, z_0 + \varepsilon a) + \varepsilon v(z)\,, \qquad \int_0^1 v(z)\, dz = 0\,.$$

Denote $G'''(0) = 0$, $G'''(2x_0) = \gamma$, $z_0^\varepsilon := z_0 + \varepsilon a$

$$\begin{cases} \varepsilon \left[-(\alpha + \beta) \int_0^{z_0} v\, dz \right] + O(\varepsilon^2) & z < z_0^\varepsilon \\ \varepsilon \left[(\beta - \alpha) v + (\alpha + \beta) \int_0^{z_0} v\, dz \right] + O(\varepsilon^2) & z > z_0^\varepsilon \end{cases}$$

$$O(\varepsilon): \quad \int_0^{z_0} v\, dz = \varepsilon V \qquad \text{and} \qquad v(z) = \varepsilon \tilde{v}(z)\,,\; z > z_0^\varepsilon\,,$$

A nonlocal repulsion-aggregation model

Bifurcation for $z_0 \neq \frac{1}{2}$

Integrating over $(0, z_0^\varepsilon)$ and $(z_0^\varepsilon, 1)$

$$V = -\frac{\gamma}{2}\frac{1-z_0}{2}\int_0^{z_0} v^2\,dz + O(\varepsilon)\,.$$

Reinsert

$$\begin{cases} \varepsilon^2\left[(1+\frac{a\alpha}{z_0})v - \frac{\gamma}{2}(1-z_0)v^2 + \frac{\gamma}{2}\frac{1-z_0}{z_0}\int_0^{z_0} v^2\,dz\right] + O(\varepsilon^3) & z < z_0^\varepsilon \\ \varepsilon^2\left[\frac{1-2z_0}{z_0}\alpha\tilde{v} - \frac{\gamma}{2}\frac{1-2z_0}{z_0}\int_0^{z_0} v^2\,dz\right] + O(\varepsilon^3) & z > z_0^\varepsilon \end{cases}$$

A nonlocal repulsion-aggregation model

Bifurcation for $z_0 \neq \frac{1}{2}$

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v can assume at most two different values for $z < z_0$.

$$v(z) = \begin{cases} v_1 := -\frac{z_0 - z_1}{z_1} v_2 & 0 < z < z_1 \\ v_2 & z_1 < z < z_0 \end{cases} .$$

for a constant $v_2 \neq 0$ and $\int_0^{z_0} v \, dz = 0$.

$$v_1 = -\frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_0 - z_1}{2z_1 - z_0} , \qquad v_2 = \frac{2}{\gamma} \frac{z_0 + a\alpha}{(1 - z_0) z_0} \frac{z_1}{2z_1 - z_0} .$$

$$\tilde{v} = \frac{\gamma}{2\alpha} \frac{z_0 (z_0 - z_1)}{z_1} v_2^2 , \qquad V = \frac{\gamma}{2\alpha} \frac{(1 - z_0) z_0 (z_0 - z_1)}{z_1} v_2^2 .$$

Three-valued steady states

Take $0 < u_1, u_2 < 2x_0$ and $u_1 + u_2 > 2x_0$

Denote $G'(u_1) = g_3$, $G'(u_1 + u_2) = -g_2$, and $G'(u_2) = g_1$

Then, there are values $0 < z_1, z_2 < 1$ and $z_1 + z_2 < 1$ and

$$u_\infty(z, u_1, u_2) = \begin{cases} u_l & 0 < z < z_1\,, \\ u_l + u_1 & z_1 < z < z_1 + z_2\,, \\ u_l + u_1 + u_2 & z_1 + z_2 < z < 1\,, \end{cases}$$

are steady states with zero mass.

$u_1 = 2x_0$ or $u_2 = 2x_0$ yields the two-valued steady states

A nonlocal repulsion-aggregation model

Linear stability

Denote $G''(0) = h_1, G''(u_1) = h_2, G''(u_1 + u_2) = h_3, G''(u_2) = h_4$

Linearised operator $F'(u_\infty)(v)$

$$\begin{cases} \lambda_1\, v(z) - h_1 \int_0^{z_1} v - h_2 \int_{z_1}^{z_1+z_2} v - h_3 \int_{z_1+z_2}^{1} v & z \in (0, z_1) \\ \lambda_2\, v(z) - h_2 \int_0^{z_1} v - h_1 \int_{z_1}^{z_1+z_2} v - h_4 \int_{z_1+z_2}^{1} v & z \in (z_1, z_1 + z_2) \\ \lambda_3\, v(z) - h_3 \int_0^{z_1} v - h_4 \int_{z_1}^{z_1+z_2} v - h_1 \int_{z_1+z_2}^{1} v & z \in (z_1 + z_2, 1) \end{cases}$$

where

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_2 & h_1 & h_4 \\ h_3 & h_4 & h_1 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ 1 - z_1 - z_2 \end{pmatrix}.$$

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

$$\begin{cases} (\lambda_1 - \lambda)\,v(z) - (h_1 - h_3)\int_0^{z_1} v\,dz - (h_2 - h_3)\int_{z_1}^{z_1+z_2} v\,dz = 0 \\ (\lambda_2 - \lambda)\,v(z) - (h_2 - h_4)\int_0^{z_1} v\,dz - (h_1 - h_4)\int_{z_1}^{z_1+z_2} v\,dz = 0 \\ (\lambda_3 - \lambda)\,v(z) - (h_3 - h_1)\int_0^{z_1} v\,dz - (h_4 - h_1)\int_{z_1}^{z_1+z_2} v\,dz = 0 \end{cases}$$

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

1) $\lambda \neq \lambda_i$, $i = 1, 2, 3$: v is piecewise constant. Eigenvalues

$$\mu_{1/2} = \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - \Delta}\,,$$

$$\begin{cases} \tau = (h_2 + h_3)z_1 + (h_2 + h_4)z_2 + (h_3 + h_4)z_3\,, \\ \Delta = h_2 h_3 z_1 + h_2 h_4 z_2 + h_3 h_4 z_3 \end{cases}$$

$$\mu_3 = h_3 z_1 + h_4 z_2 + h_1 z_3\,,$$

(in)stability of $\mu_{1/2}$ is not obvious.

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

1) $\lambda \neq \lambda_i$, $i = 1, 2, 3$:

Relating back to two-valued steady states

$u_2 = 2x_0$ such that $g_1 = 0 = z_1$

$$\mu_1 = -\alpha\,, \qquad \mu_2 = \beta(1 - z_2) - \alpha z_2\,,$$

with only μ_1 satisfying $v_2 z_2 + v_3(1 - z_2) = 0$

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

2) $\lambda = \lambda_1$ and $\lambda \neq \lambda_j$, $j = 2, 3$

$$(h_1 - h_3)\int_0^{z_1} v\,dz = (h_3 - h_2)\int_{z_1}^{z_1+z_2} v\,dz$$

while v is piecewise constant on $(z_1, z_1 + z_2)$ and $(z_1 + z_2, 1)$.

$$v = \begin{cases} v_1(z) & z \in (0, z_1) \\ v_2 & z \in (z_1, z_1 + z_2) \\ v_3 & z \in (z_1 + z_2, 1) \end{cases}$$

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\,dz = 0$

Signs of the eigenvalues λ_i not obvious.

Relating back: $u_2 = 2x_0$ with $z_1 = 0$ and $h_1 = \beta$ and $h_4 = -\alpha$

$$\lambda_1 = h_2 z_2 + h_3(1 - z_2)\,,\ \lambda_2 = \beta z_2 - \alpha(1 - z_2)\,,\ \lambda_3 = \alpha z_2 - \beta(1 - z_2)\,,$$

λ_1 is spurious since $z_1 = 0$.

A nonlocal repulsion-aggregation model

Linear stability

Ansatz $v(z) = e^{\lambda t}\varphi(z)$

eigenproblem $\lambda\varphi = F'(u_\infty(z_0))(\varphi)$ with $\int_0^1 \varphi(z)\, dz = 0$

For small perturbations $u_2 - 2x_0 = -\delta < 0$

$$\lambda_2 = \beta z_2 - \alpha(1 - z_2) + (h_2 \frac{\alpha}{g} - G'''(2x_0)(1 - z_2) + \frac{\alpha^2}{g})\delta + O(\delta^2)\,.$$

At bifurcation $z_2 = \frac{\alpha}{\alpha+\beta}$

$$(h_2 \frac{\alpha}{g} - G'''(2x_0)\frac{\beta}{\alpha + \beta} + \frac{\alpha^2}{g})$$

has sign depending on G.

A nonlocal repulsion-aggregation model

To Dos and open problems

- rigorous bifurcation analysis
- G with cascades of pitchfork bifurcations?
- continuous steady states?
- more well, other bifurcation? Hopf?

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