ON A CONSTRAINED NAVIER STOKES FLOW

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1 Introduction

This talk concerns some ideas concerning contrained gradient flows in the Wasserstein metric developed by myself and Wilfrid Gango, but applied to some problems concerning the two dimensional Navier–Stokes equation that have recently been introduced investigated by Cagliotti, Pulvirenti and Rousset.

1.1 About 2-d Navier–Stokes and Euler

The 2-d Navier–Stokes equation in the vorticity formulation is

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) \omega = \nu \Delta \omega \ ,$$

where $\nu > 0$ and

$$u(x,t) = \nabla^{\perp}\psi(x,t) \qquad \text{and} \qquad \psi(x,t) = \int_{\mathbb{R}^2} G(x-y)\omega(y)\mathrm{d}y$$

with

$$G(x) = -\frac{1}{2\pi}\ln(|x|) \ ,$$

the Green's function for the plane. (So that $\psi = (-\Delta)^{-1}\omega$.)

The Euler equation is the case $\nu = 0$:

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) \omega = 0 \ .$$

Setting $\nu = 0$ changes the character of the equation quite substantially: It becomes a Hamiltonian flow with many conserved quantities.

1.2 The Hamiltonian structure of the Euler flow

The *energy* $E(\omega)$ defined by

$$E(\omega) = \frac{1}{2}\int_{\mathbb{R}^2} \omega(x)G(x-y)\omega(d)\mathrm{d}x\mathrm{d}y = \frac{1}{2}\langle \omega, (-\Delta)^{-1}\omega\rangle_{L^2}$$

is conserved for the Euler flow, and in fact is a Hamiltonian for it:

The Euler equation can be written as

$$\frac{\partial}{\partial t}\omega = -\mathrm{div}\left(\omega\nabla^{\perp}\frac{\delta E}{\delta\omega}\right) := J_W(\nabla_W E(\omega)) \ .$$

If we suppose ω to be a probability measure, this is a Hamiltonian flow in Wasserstein space, as discussed by Ambrosio and Gangbo.

1.3 Conserved quantities for the Euler flow

In addition to the energy,

$$M(\omega) = \int_{\mathbb{R}^2} x\omega(x)\mathrm{d}x \qquad \text{and} \qquad I(\omega) = \frac{1}{2}\int_{\mathbb{R}^2} |x - M(\omega)|^2\omega(x)\mathrm{d}x$$

are conserved, as are the functionals

$$F_\phi(\omega) = \int_{\mathbb{R}^2} \phi(\omega(x)(\mathrm{d}x \ .$$

For the Navier Stokes flow, from among these, only $M(\omega)$ is conserved. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}I(\omega) = 2\nu \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}E(\omega) = -\nu\int_{\mathbb{R}^2} \omega^2\mathrm{d}x \ ,$$

while

$$\frac{\mathrm{d}}{\mathrm{d}t}F_\phi(\omega) = -\nu\int_{\mathbb{R}^2} \phi''(\omega)|\nabla\omega|^2\mathrm{d}x \ .$$

Notice that if ϕ is convex, F_ϕ is dissipated at a rate involving the derivatives of ω, in contrast to the dissipation of E and $-I$.

1.4 The intermediate asymptotics proposal of CPR

Cagliotti, Pulvirenti and Rousset have made an interesting intermediate asymptotics proposal based on the observation that it is possible for

$$\int_{\mathbb{R}^2} \phi''(\omega)|\nabla\omega|^2 \mathrm{d}x$$

to be quite large compared with the rates of change of $I(\omega)$ and $E(\omega)$, and that for such initial data there might be an interesting intermediate asymptotic regime that one can make emerge as a final asymptotic regime by constraining the Navier Stokes flow to conserve E and I.

There are many ways one might do this. Their proposal is to use Otto's differential geometric framework for gradient flows in the Wasserstein metric. Specifically, introducung the entropy

$$S(\omega) = \int_{\mathbb{R}^2} \omega \ln \omega \mathrm{d}x \ ,$$

they observe that the Navier-Stokes flow can be written in the form

$$\frac{\partial}{\partial t}\omega = -\mathrm{div}\left(\omega\nabla^{\perp}\frac{\delta E}{\delta\omega}\right) + \nu\mathrm{div}\left(\omega\nabla\frac{\delta S}{\delta\omega}\right) \ .$$

Let $\mathcal{M}$ denote the set of probability densities on $\mathbb{R}^2$ for which E, I, and S are finite, and $M(\omega) = 0$. For a smooth path $t \to \rho(\cdot, t)$ in $\mathcal{M}$ through ρ_0 at $t = 0$ with

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\nabla\phi\rho) \ ,$$

Otto's metric is

$$\left\langle \frac{\partial}{\partial t}\rho, \frac{\partial}{\partial t}\rho \right\rangle_{W_2} = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\phi(x)|^2 \rho_0(x)\mathrm{d}x \ .$$

For positive numbers E and I, let $\mathcal{M}_{E,I}$ be define by

$$\mathcal{M}_{E,I} = \{\ \omega \in \mathcal{M}\ :\ E(\omega) = E\ , I(\omega) = I\ \}\ .$$

This is the "constraint manifold" to which we wish to restrict the Navier–Stokes flow.

1.5 The constraint prescription for the flow

The prescription for doing this in the work of Caglioti, Pulvirenti and Rousset is to use the orthogonal projection P in the tangent space to ω onto the orthogonal complement of the span of the gradients of $I(\omega)$ and $E(\omega)$. These gradients are

$$\nabla_W I(\omega) = -\mathrm{div}\,(\omega x) \qquad \text{and} \qquad \nabla_E I(\omega) = -\mathrm{div}\left(\omega \nabla \frac{\delta E}{\delta \omega}\right) = -\mathrm{div}\,(\omega \nabla \psi)\ .$$

Since

$$-\nabla_W S(\omega) = \mathrm{div}(\omega \nabla \ln \omega) = \Delta \omega\ ,$$

$$P[-\nabla_W S(\omega)] = \Delta \omega - a\mathrm{div}\,(\omega x) - b\mathrm{div}\,(\omega \nabla \psi)$$

where a and b are chosen to make the right hand side orthogonal to $\mathrm{div}\,(\omega x)$ and $\mathrm{div}\,(\omega \nabla \psi)$ in Otto's inner product.

Since

$$\int_{\mathbb{R}^2} \omega x \cdot \nabla \psi = -\frac{1}{4\pi} \ ,$$

one easily finds

$$a = -4\pi \frac{8\pi \int \omega |\nabla \psi|^2 - \int \omega^2}{1 - 32\pi^2 I(\omega) \int \omega |\nabla \psi|^2}$$

and

$$b = 4\pi \frac{8\pi I(\omega) \int \omega^2 - 2}{1 - 32\pi^2 I(\omega) \int \omega |\nabla \psi|^2} \ .$$

This leads to the evolution equation

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla \right) \omega = \nu \mathrm{div} \left(\nabla \omega - b \omega \nabla \psi - a \omega x \right) \ .$$

1.6 Entropy dissipation

The entropy is still dissipated for the constrained flow, only at a lower rate due to the projection:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(\omega) &= -\langle \nabla_W S, \nu P \nabla_W S + J_W \nabla_W E\rangle_W \\
&= -\langle P\nabla_W S, \nu P \nabla_W S\rangle_W \\
&= -\int_{\mathbb{R}^2} \omega |\nabla \omega - b\omega \nabla \psi - a\omega x|^2 \mathrm{d}x \ .
\end{aligned} \tag{1}$$

This suggests that the eventual steady state of the constrained evolution will be the vorticity density $\omega_{E,I}$ given by

$$\omega_{E,I} = \mathrm{argmin}\left\{ S(\omega) \ : \ \omega \in \mathcal{M}_{E,I} \ \right\} \ .$$

This variational problem had been studied in previous work of Caglioti, Lions, Marchioro and Pulvirienti, who proved the existence of a unique minimizer.

In their recent work, CPR raise the question as to whether solutions of

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) \omega = \nu \mathrm{div}\left(\nabla \omega - b\omega \nabla \psi - a\omega x\right)$$

tend towards the corresponding optimizers $\omega_{E,I}$, and if so, at what rate.

This question is largely open, especially concerning the rate, though they have proved a number of very interesting results on global solutions for data already close to equilibrium.

The first problem that one encounters in studying this equation is that the equations defining a and b can degenerate: The denominator in a and b will vanish whenever $\mathrm{div}\,(\omega x)$ and $\mathrm{div}\,(\omega \nabla \psi)$ are proportional. This happens for a vorticity patch in which ω is a multiple of the characteristic function of a centered disc.

To avoid this problem, CPR work with a modified problem in which only a linear combination of E and I is constrained, or with the original problem in a small neighborhood of $\omega_{E,I}$ where the degeneracy does not occur.

1.7 Continuity of the constraints

Of the two constraints,

$$E(\omega) = E \qquad \text{and} \qquad I(\omega) = I \ ,$$

the second one poses greater challenges. Indeed, while $\omega \mapsto E(\omega)$ is not weakly continuous on $L^1_+(\mathbb{R}^2)$, it is on subsets with bounded entropy. This is an easy consequence of the inequality

$$st \leq e^{s-1} + t \ln t \ .$$

However, the abscence of control on higher moments means that if $\omega_n \to \omega$ weakly in $\mathcal{M}_{E,I}$ one in general has

$$I(\omega) < \liminf_{n\to\infty} I(\omega_n) \ .$$

This is one of the main obstacles in the proof by Caglioti, Lions, Marchioro and Pulvirienti of the existence and uniqueness of $\omega_{E,I}$. It was also one of the main difficulties in problem on constrained gradient flow considered by myself and Gangbo.

2 Gradient flow for the entropy with the first and second moments constrained

In this section we recall results by myself and Gangbo on a closely related constrained gradient flow.

Let $\mathcal{P}_{M,I}$ denote the set of probabilty densities ρ on $\mathbb{R}^d$ with $I(\rho) = I$ and $M(\rho) = M$. Given a number $h > 0$, and a density $\rho_0 \in \mathcal{P}_{M,I}$, consider the functional

$$\rho \mapsto \left[W_2^2(\rho, \rho_0) + hS(\rho)\right]$$

on $\mathcal{P}_{M,I}$.

If one ignores the constraint, the prescription

$$\rho_n = \text{argmin}\left\{W_2^2(\rho, \rho_{n-1}) + hS(\rho)\right\}$$

is the well known Jordan-Kinderlehrer-Otto discrete scheme for solving the heat equation; is is a discrete version of

$$\frac{\partial}{\partial t}u = -\nabla_W S(\rho) = \text{div}(\rho\nabla \ln \rho) = \Delta\rho\ .$$

2.1 Formal incorporation of the constraint

Formally, we can incorporate the constraint by projecting out the components of $\nabla_W S(\rho)$ along $\nabla_W I(\rho)$ and $\nabla_W M(\rho)$. Since

$$\nabla_W M_i(\rho) = -\mathrm{div}(\rho \hat{e}_i) \ ,$$

each component of $\nabla_W M$ is orthogonal to $\nabla_W S(\rho)$, so we can ignore this in the projection. Then since

$$\nabla_W I(\rho) = -\mathrm{div}(\rho(x - M(\rho)) \ ,$$

we deduce that

$$a = -\frac{1}{dI(\rho)} \ .$$

Since $I(\rho) = I$ and $M(\rho) = M$, the equation os the Ornstien—Uhlenbeck equation

$$\frac{\partial}{\partial t} u = \mathrm{div}\left(\nabla \ln \rho + \frac{1}{dI}(x - M) \right) \ .$$

In this case, there is no problem with determining the asymptotic behavior: The solution tends exponentially fast towards

$$\rho_{M,I}(x) = \text{argmin}\,\{S(\rho) \ : \ \rho \in \mathcal{P}_{M,I}\} = (2\pi I)^{-d/2} e^{-|x-M|^2/2I} \ .$$

The exponential convergence comes from the fact that the equation

$$\frac{\partial}{\partial t} u = \text{div}\left(\nabla \ln \rho + \frac{1}{dI}(x - M)\right)$$

is gradient flow *on the whole, unconstrained manifold* for the free energy functional

$$F(\rho) = S(\rho) + \frac{1}{2dI} I(\rho) \ .$$

The second term on the right is strictly displacement convex, and this leads to the exponential decay by Otto's method. Notice that the constraint effectively renders the entropy strictly convex.

2.2 The kinetic Fokker–Planck equation

The motivation for considering the Fokker–Planck equation in terms of constrained flow was to study the *kinetic Fokker–Planck equation*

$$\frac{\partial}{\partial t} f(x,v,t) + \nabla_x \cdot (v f(x,v,t)) = \mathcal{L}_f f(x,v,t)$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L}_f \phi = \theta \nabla_v \cdot \left(M_f \nabla_v \left(\frac{\phi}{M_f} \right) \right) = \theta \nabla_v \cdot \left(\phi \nabla_v \ln \left(\frac{\phi}{M_f} \right) \right)$$

and

$$M_{u,\theta}(v) = (2\pi\theta)^{-d/2} \, e^{-|v-u|^2/2\theta}$$

is the *Maxwellian* denisity having the same mean u and variance θ as $f(x,v)/\rho(x)$.

2.3 Rigorous results on the constrained variational problem

Let us return to the consideration of determining

$$\inf \left\{ W_2^2(\rho, \rho_0) + hS(\rho) \ : \ \rho \in \mathcal{P}_{M,I} \right\} \ ,$$

and the description of the minimizers, if any.

We fix ρ_0 and h and define the functional

$$K(\rho) = W_2^2(\rho, \rho_0) + hS(\rho) \ ,$$

and will solve the problem of minimizing $K(\rho)$ over $\mathcal{P}_{M,I}$ by introducing a dual problem. First, we determine the Euler-Lagrange equation.

2.4 The Euler–Lagrange equation

Theorem (C., Cangbo) *Suppose that ρ_1 is a minimizer of the functional $K(\rho)$ subject to the constraint that $\rho_1 \in \mathcal{P}_{M,I}$. Let φ be the convex function on $\mathbb{R}^d$ such that*

$$\nabla\varphi \# \rho_1 = \rho_0 \ .$$

Then

$$\int_{\mathbb{R}^d} |\nabla \ln \rho_1|^2 \rho_1(x) \mathrm{d}x < \infty$$

and

$$\nabla\varphi(x) = x + h\nabla\left(\ln \frac{\rho_1}{\rho_{M,I}}\right) + (M - x)\left[W_2^2(\rho_1, \rho_0)\right] \ .$$

The last term is formal $\mathcal{O}(h^2)$. Otherwise, this is what we deduced by formal projection.

2.5 Existence of the minimizer

Our goal is to introduce a dual variational problem of computing

$$\sup\{J(a,b,\phi,\varphi) \ : \ (a,b,\phi,\varphi) \in \mathcal{U}\}$$

for some J and $\mathcal{U}$ such that the sup is a max, and such that

$$\inf\{K(\rho) \ : \ \rho \in \mathcal{P}_{M,I}\} = \max\{J(a,b,\phi,\varphi) \ : \ (a,b,\phi,\varphi) \in \mathcal{U}\} \ ,$$

and such that for the maximizing (a,b,ϕ,φ),

$$\rho_1 := \nabla\varphi \# \rho_0$$

is the optimizer for the original problem.

In what follows, we suppose for simplicity that $I = 1$ and $M = 0$. The starting point is the fact that

$$1 - W_2^2(\rho_0, \rho)$$

is given by

$$\inf \left\{ \int_{\mathbb{R}^d} \phi(v)\rho_0(v)\mathrm{d}v + \int_{\mathbb{R}^d} \varphi(w)\rho(w)\mathrm{d}w \;\middle|\; \phi(v) + \varphi(w) \geq v \cdot w \text{ a.e.} \right\} .$$

Furthermore, the minimizing pair, which exists, consists of a dual pair of convex functions. That is, we may assume that ϕ and φ are Legendre transforms of one another. The gradients of the minimizing pair provide the optimal transport plans; i.e., $\nabla\phi\#\rho_0 = \rho$ and $\nabla\varphi\#\rho = \rho_0$.

Then for any dual convex pair of functions ϕ and φ,

$$K(\rho) \geq hS(\rho) + 1 - \left(\int_{\mathbb{R}^d} \phi(x)\rho_0(x)\mathrm{d}x + \int_{\mathbb{R}^d} \varphi(y)\rho(y)\mathrm{d}y \right) .$$

Temporary strong assumptions on ρ_0: We suppose that ρ_0 is supported in B_R, the centered ball of radius R, and that on B_R it is bounded below by some strictly positive number α. Then for any other density ρ in $\mathcal{P}$, these hypotheses impose some regularity on the optimal map $\nabla\varphi\#\rho = \rho_0$. In particular,

$$|\nabla\varphi(x)| \le R$$

for all x.

Now define $\eta(t)$ by

$$\eta(t) = t\ln t \qquad \text{for} \qquad t > 0 \ , +\infty \quad \text{otherwise} \ ,$$

so that

$$S(\rho) = \int_{\mathbb{R}^d} \eta(\rho)\mathrm{d}x.$$

The Legendre transform $\eta^*(s)$ of $\eta(t)$ is $\eta^*(s) = e^{s-1}$. By Young's inequality, $\eta(t) + \eta^*(s) \geq st$, and so for any $a \in \mathbb{R}^d$ and any $b \in \mathbb{R}$,

$$\eta(\rho) + \eta^* \left(\frac{a \cdot w + b|w|^2/2 + \varphi(w)}{h} \right) \geq \frac{a \cdot w + b|w|^2/2 + \varphi(w)}{h} \rho \ .$$

Integrating yields

$$hS(\rho) - \int_{\mathbb{R}^d} \varphi(y)\rho(y)\mathrm{d}y \geq b - h \int_{\mathbb{R}^d} \eta^* \left(\frac{a \cdot y + b|y|^2/2 + \varphi(y)}{h} \right) \mathrm{d}y \ .$$

Therefore, introduce the functional

$$J(a,b,\phi,\varphi) = 1 - \int_{\mathbb{R}^d} \phi(v)\rho_0(x)\mathrm{d}x + b - h\int_{\mathbb{R}^d} \eta^*\left(\frac{a\cdot y + b|y|^2/2 + \varphi(y)}{h}\right)\mathrm{d}y\ .$$

Note that ϕ is bounded below and η^* is positive, and hence $J(a,b,\phi,\varphi)$ is well–defined. It then follows that or any dual convex pair of functions ϕ and φ, $a \in \mathbb{R}^d$ and any $b \in \mathbb{R}$,

$$K(\rho) \geq J(a,b,\phi,\varphi)\ .$$

We let $\mathcal{U}$ denote the set of all quadruplets (a,b,ϕ,φ) where $a \in \mathbb{R}^d$, $b \in \mathbb{R}$, and ϕ and φ are a pair of dual convex functions with

$$\phi(x) = \infty \qquad \text{for} \quad |x| > R\ .$$

Theorem (C., Cangbo) *There exists* $(a_0, b_0, \phi_0, \varphi_0) \in \mathcal{U}$ *such that*

$$J(a_0, b_0, \phi_0, \varphi_0) \geq J(a, b, \phi, \varphi)$$

for all $(a, b, \phi, \varphi) \in \mathcal{U}$ *. Furthermore, if we define*

$$\rho_1(y) = (\eta^*)' \left(\frac{a_0 \cdot y + b_0 |y|^2/2 + \varphi_0(y)}{h} \right)$$

then $\rho_1 \in \mathcal{P}_{0,1}$*,*

$$\nabla \varphi_0 \# \rho_1 = \rho_0$$

and

$$\nabla \varphi_0(y) = y + h \nabla \ln(\rho_1) + hy - W_2^2(\rho_0, \rho_1) \ .$$

Note that this gives us a solution of the Euler–Lagrange equation for the minimum of $K(\rho)$ that we derived in the last section. And indeed, since $\eta(t) + \eta^*(s) = st$ with

$$t = \rho_1 \quad \text{and} \quad s = \frac{a_0 \cdot y + b_0|y|^2/2 + \varphi_0(y)}{h}$$

with $\rho = \rho_1$, $\varphi = \varphi_0$, there is equality in the integrated form of Young's inequality. From this one can conclude that $K(\rho_1) = J(a_0, b_0, \phi_0, \varphi_0)$, and this proves that ρ_1 minimizes K on $\mathcal{P}_{0,1}$.

The advantage of the J functional lies in the compactness properties of the dual convex pairs.

2.6 Why the constraints are satisified

First, suppose that the maximizer $(a_0, b_0, \phi_0, \varphi_0)$ does exist. Observe that for any real number λ, $(a_0, b_0, \phi_0 + \lambda, \varphi_0 - \lambda) \in \mathcal{U}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(a_0, b_0, \phi_0 + \lambda, \varphi_0 - \lambda)\bigg|_{\lambda=0} = 0$$

and this clearly leads to

$$1 = \int_{\mathbb{R}^d} (\eta^*)' \left(\frac{a_0 \cdot y + b_0 |y|^2/2 + \varphi_0(y)}{h} \right) \mathrm{d}y \ .$$

Hence we see that

$$\rho_1(y) = (\eta^*)' \left(\frac{a_0 \cdot y + b_0 |y|^2/2 + \varphi_0(y)}{h} \right)$$

does define a probability density.

Next, for some $\epsilon > 0$,

$$\int_{\mathbb{R}^d} e^{\epsilon |y|^2} \rho_1(y) \mathrm{d}y < \infty \ .$$

This implies that

$$(a,b) \mapsto \int_{\mathbb{R}^d} (\eta^*)' \left(\frac{a \cdot y + b|y|^2/2 + \varphi_0(y)}{h} \right) \mathrm{d}y$$

is a differentiable function of a and b in some neighborhood of (a_0, b_0). Assuming this for the moment, $\frac{\mathrm{d}}{\mathrm{d}b} J(a_0, b, \phi_0, \varphi_0)\big|_{b=b_0} = 0$, and from this we have that

$$1 = \int_{\mathbb{R}^d} \frac{|y|^2}{2} (\eta^*)' \left(\frac{a_0 \cdot w + b_0|y|^2/2 + \varphi_0(y)}{h} \right) \mathrm{d}y$$

which means that ρ_1 does indeed satisfy the variance constraint. In the same way, differentiating in a shows that ρ_1 does satisfy the mean constraint. Thus, $\rho_1 \in \mathcal{P}_{0,1}$.

3 Application to the constrained Navier–Stokes flow

The method used to prove the previous theorem appears to extend to prove the existence of an optimizer for

$$\omega_1 = \operatorname{argmin}\left\{ hS(\omega) + W_2^2(\omega, \omega_0) \ : \ \omega \in \mathcal{M}_{E,I} \ \right\} \ .$$

The extension is non–trivial, and I say “appears” as this has extension has not yet stood the tests of time and scrutiny.

Granted the correctness of this assertion, the result provides the basis for the construction of a JKO type scheme for the solution of the contrained Navier–Stokes evolution proposed by CPR.

There is an alternative route, but at present this seems problematic to extend, though interesting.

3.1 A free energy alternative

An alternate approach to the constrained variational problem considered by myself and Gangbo would be to consider instead the *unconstrained problem*

$$\rho_1^{(a)} = \text{argmin}\left\{[hS(\rho) - aI(\rho)] + W_2^2(\rho,\rho_0) \ : \ \rho \in \mathcal{P} \ \right\} \ ,$$

and then vary a so as to achieve the constraint.

It is clear that *increasing* a will *decrease* $I(\rho_1^{(a)})$. Thus as long as one can show that

$$I(\rho_0) < I(\rho_1^{(0)}) \ ,$$

one can expect to find a *negative* a such that $I(\rho_1^{(a)}) = I(\rho_0) = I$. This much can be easily justified.

The negativity is important, because then

$$\rho \mapsto S(\rho) - aI(\rho)$$

is convex, and the minimization problem is tractable.

3.2 And indeed, a should be negative

Since

$$\rho_1^{(0)} = \mathrm{argmin}\left\{hS(\rho) + W_2^2(\rho,\rho_0)\ :\ \rho \in \mathcal{P}\ \right\}$$

is what one gets form the first step of the JKO scheme for solving the heat equation, and since I steadily increases under the heat flow, one might expect that

$$I(\rho_1^{(0)}) \approx I(\rho_0) + dh > I(\rho_0)\ .$$

This raises the following question:

3.3 Monotonicity in discrete and continuous time

Suppose that for some continuous time gradient flow evolution, a certain functional $G(\rho)$ is monotone increasing or decreasing. Is this still the case for each step of the corresponding JKO scheme?

The answer depends on the properties of the functional $G(\rho)$. It is very easy to see that if $\rho \mapsto G(\rho)$ is displacement convex, and $G(\rho)$ is decreasing along the continuous flow, then it also is decreasing along the discrete flow. Likewise, if $\rho \mapsto G(\rho)$ is displacement concave, and $G(\rho)$ is increasing along the continuous flow, then it also is increasing along the discrete flow.

However, in the case of $I(\rho)$ and the heat flow, $I(\rho)$ is strictly displacement convex, and is increasing along the flow, so things are not so easy. Gangbo and I conjectured that this was the case, but did not succeed in proving it. A relatively simple proof was then devised by Adrian Tudorascu, but fortunately only after we developed the duality approach just described: Fortunately, for otherwise we would not have developed the duality approach, which seems to have some advantages in situations like the one described here.

3.4 Free energy approach for the constrained Navier Stokes flow

Suppose one tries to solve the unconsstrained free energy problem

$$\omega_1 = \text{argmin}\left\{h[S(\omega) - aI(\omega) - bE(\omega)] + W_2^2(\omega, \omega_0) \ : \ \omega \in \mathcal{M} \ \right\} \ ,$$

and then adjust a and b to achieve $\omega_1 \in \mathcal{M}_{E,I}$.

Recall that under the heat flow (coming from the unconstrained optimization), $I(\omega)$ is increasing, and $E(\omega)$ is decreasing. Thus we shall require a *negative* value for a to penalize increase in $I(\omega)$, and a *posative* value of b to penalize decrease in $E(\omega)$. Thus the functional

$$\omega \mapsto [S(\omega) - aI(\omega) - bE(\omega)]$$

will not be convex in the cases of interest, and the optimization problem becomes problematic. In fact, even the simpler problem of determining

$$\inf \left\{S(\omega) - aI(\omega) - bE(\omega) \ : \ \omega \in \mathcal{M} \ \right\}$$

becomes problematic for $b > 0$.

In fact, if $b \geq 8\pi$, a minimizer will not exist. To see this, write

$$\begin{aligned} S(\omega) - aI(\omega) - bE(\omega) &= \left(1 - \frac{b}{8\pi}\right)[S(\omega) + |a|I(\omega)] \\ &+ \frac{b}{8\pi}[S(\omega) - 8\pi E(\omega)] \end{aligned} \tag{2}$$

The last term on the right is positive by the sharp Logarithmic HLS inequality (C. and Loss, Beckner). It is also scale invariant. As b approaches 8π from below, there is less and less of a penalty against "bunching up" from the first term, and so the optimizer degenerates to point mass as b approaches 8π, and does not exist for larger values of b, as was shown by CLMP.