A class of total variation minimization problems on the whole space

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Problem

Given F and G increasing on $[0,\infty)$, s.t. $F(0)=G(0)=0$ and $\lim_\infty G=\infty$, we study

$$(\mathcal{P}) \quad m_\pm := \inf\left\{\int_{\mathbb{R}^n} \mathrm{d}|\nabla u| \pm \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|)=1\right\}$$

• The motivation is the sharp L^1 Gagliardo-Nirenberg inequalities: If $1 \le q < s < 1^* := \frac{n}{n-1}$ then

$$\|u\|_{L^s(\mathbb{R}^n)} \le K_{opt} \|\nabla u\|^{\theta}_{\mathcal{M}(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \quad \forall u \in D^{1,q}(\mathbb{R}^n)$$

where $\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)}$ is the total variation of u, $n \ge 2$, and

$$D^{1,q}(\mathbb{R}^n) := \{u \in L^q(\mathbb{R}^n) : \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} < \infty\}.$$

Results

- By elementary arguments (symmetrization, Pòlya–Szegö inequality, and some change of function), $(\mathcal{P})$ is equivalent to a 1-D optimization problem.
- Under appropriate conditions on F and G, the infimum is attained and the minimizers are multiple of characteristic functions of balls.
- This leads to some sharp inequalities involving the total variation, as well as existence/nonexistence results for 1-Laplacian type PDEs.

Outline

- Motivation
- The Main Result, variants and extensions
- Applications
  - Sharp inequalities involving the total variation
  - Existence/nonexistence results for 1-Laplacian PDEs
- Optimal Transportation Approach (in particular cases).

Motivation

- Sharp L^1 Sobolev inequality:

$$\|u\|_{L^{1^*}(\mathbb{R}^n)} \leq (n\gamma_n^{1/n})^{-1}\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} \quad \forall u \in \mathrm{BV}(\mathbb{R}^n). \tag{1}$$

The best constant is $(n\gamma_n^{1/n})^{-1}$, and the extremals are characteristic functions of balls [Federer-Fleming, 60].

- Interpolation inequality: If $1 \leq q < s < 1^*$

$$\|u\|_{L^s(\mathbb{R}^n)} \leq \|u\|_{L^q(\mathbb{R}^n)}^{1-\theta} \|u\|_{L^{1^*}(\mathbb{R}^n)}^{\theta}, \tag{2}$$

where

$$\frac{1}{s} = \frac{(1-\theta)}{q} + \frac{\theta}{1^*} \text{ i.e. } \theta = \frac{n(s-q)}{s(n-q(n-1))}.$$

Motivation

• Combining L^1-Sobolev and interpolation inequalities, we obtain the L^1 Gagliardo-Nirenberg inequality:

$$\|u\|_{L^s(\mathbb{R}^n)} \leq \left(n\gamma_n^{1/n}\right)^{-\theta} \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)}^{\theta} \|u\|_{L^q(\mathbb{R}^n)}^{1-\theta} \qquad (3)$$

• Characteristic functions of balls are extremals in (3) as they are extremals in both (1) and (2). Then the best constant in (3) is $\left(n\gamma_n^{1/n}\right)^{-\theta}$.

• By a scaling argument, (3) is related to our variational problem $(\mathcal{P})$ when $F(t) = t^q/q$ and $G(t) = t^s$.

• Our goal is to generalize this fact (extremality of characteristic functions of balls) to more general variational problems involving the total variation.

Motivation

Proposition 1 *Assume* $1 \leq q < s < 1^*$*. If*

$$\inf\left\{E(u) := \int_{\mathbb{R}^n} d|\nabla u| + \frac{1}{q}\int_{\mathbb{R}^n} |u|^q : \ \|u\|_{L^s} = 1\right\}$$

has a minimizer u_∞*, then the* L^1 *Gagliardo-Nirenberg inequality holds, and the best constant and extremals are explicitly given in function of* u_∞*:*

$$K_{opt} = \left[K(n,q,s)/E(u_\infty)\right]^{\frac{n+s-nq}{s[n-q(n-1))}}, \text{ where}$$

$$K(n,q,s) = \frac{\alpha+\beta}{(q\alpha)^{\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}}, \quad \alpha = n - s(n-1),\ \beta = n(s-q),$$

and $u_{\sigma,x_0}(x) = C u_\infty\left(\sigma(x-x_0)\right)$ *are extremal functions.*

Proof of Proposition

- u_∞ is a minimizer:

$$E(u_\infty) \le E\left(\frac{u}{\|u\|_s}\right) = \frac{\|\nabla u\|_{\mathcal{M}}}{\|u\|_s} + \frac{\|u\|_q^q}{q\|u\|_s^q} \quad \forall u \in D^{1,q}(\mathbb{R}^n)$$

  with equality if $u = u_\infty$.

- Scaling: $u_\lambda(x) = u\left(\frac{x}{\lambda}\right)$, $\lambda > 0$:

$$E(u_\infty) \le \lambda^{n-1-\frac{n}{s}} \frac{\|\nabla u\|_{\mathcal{M}}}{\|u\|_s} + \lambda^{n(1-\frac{q}{s})} \frac{\|u\|_q^q}{\|u\|_s^q} := f(\lambda)$$

- Optimization in λ:

$$E(u_\infty) \le \min_{\lambda>0} f(\lambda) = f(\lambda_{min})$$

Main result

Theorem 2 *Assume that F and G are continuous increasing functions $[0,\infty) \to [0,\infty)$, s.t. $F(0) = G(0) = 0$, $F, G \in C^1((0,\infty))$ and $\lim_\infty G = \infty$. Consider problem $(\mathcal{P})$:*

$$m_\pm := \inf\left\{E_\pm(u) = \int_{\mathbb{R}^n} d|\nabla u| \pm \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|) = 1\right\}$$

Then

$$m_\pm = \inf_{\alpha>0} H_\pm(\alpha)$$

where

$$H_\pm(\alpha) := n\gamma_n^{1/n} \frac{\alpha}{G(\alpha)^{(n-1)/n}} \pm \frac{F(\alpha)}{G(\alpha)}.$$

Continuation of Theorem 2

Moreover, if the infimum of $H_\pm(\alpha)$ is attained, i.e.

$$V_\pm := \{\alpha > 0,\ H_\pm(\alpha) = m_\pm\} \neq \emptyset,$$

then the unique minimizers of $(\mathcal{P})$ are:

$$\{u_\alpha(x_0 + .), x_0 \in \mathbb{R}^n, \alpha \in V_\pm\} \cup \{-u_\alpha(x_0 + .), x_0 \in \mathbb{R}^n, \alpha \in V_\pm\}$$

where

$$u_\alpha := \alpha \chi_{B_{\rho_\alpha}} \text{ and } \rho_\alpha := \frac{1}{\gamma_n^{1/n} G(\alpha)^{1/n}}.$$

But if the infimum of $H_\pm(\alpha)$ is not attained in $(0, \infty)$, then $(\mathcal{P})$ has no minimizers.

Proof of the Main Result

- **Step 1:** $(\mathcal{P})$ is equivalent to the minimization of convex function on a compact convex set.
  - **Symmetrization**: If u^* is the symmetric-decreasing rearrangement of u, then by Pòlya-Szegö inequality and equimesurability of rearrangements:
    u^* is admissible in $(\mathcal{P})$ and $E_\pm(u^\star) \leq E_\pm(u)$.
    Then we can restrict $(\mathcal{P})$ to nonnegative, radially-symmetric, lsc, and nonincreasing functions u s.t. $\int_{\mathbb{R}^n} G(u) = 1$.
  - For such u, $\exists \beta : [0,\infty) \to [0,\infty)$ nonincreasing s.t.

$$A_t := \{u > t\} = B_{\beta(t)} \tag{4}$$

is the ball in $\mathbb{R}^n$ centered at 0 with radius $\beta(t)$.

Proof of the Main Result

- **Step 1** continues:
  - **Change of function:** Set $v := \beta^n$.
    By the co-area formula and the layer cake representation:

$$
\begin{aligned}
E_\pm(u) &= \int_0^\infty P(A_t)dt \pm \int_0^\infty |\{F(u) > t\}|dt \\
&= n\gamma_n \int_0^\infty \beta^{n-1}(t)dt \pm \gamma_n \int_0^{F(\infty)} \beta^n(F^{-1}(t))dt \\
&= n\gamma_n \int_0^\infty v^{(n-1)/n}(t)dt \pm \gamma_n \int_0^\infty F'(t)v(t)dt \\
&=: J_\pm(v).
\end{aligned}
$$

Proof of the Main Result

- **Step 1** continues:
  - **Minimization of a concave function:** Then $(\mathcal{P})$ is equivalent to:

$$(\mathcal{P})_{\mathbf{v}}: \quad m_{\pm} = \inf\{J_{\pm}(v) : v \in K\}$$

where K is the set of nonincreasing, nonnegative functions $v : \mathbb{R}_+ \to \mathbb{R}_+$, that satisfy the linear constraint

$$\int_0^\infty G'(t)v(t)dt = \frac{1}{\gamma_n}.$$

• Note that $v \mapsto J_{\pm}(v)$ is strictly concave, and K is convex and compact in $L^1(\mathbb{R}, G'(t)dt)$.

Proof of the Main Result

- **Step 2:** $(\mathcal{P})_v$ is attained at some $w_\alpha := \frac{\chi_{[0,\alpha]}}{\gamma_n G(\alpha)}$, $\alpha > 0$.
  - $w_\alpha \in K$ and $J_\pm(w_\alpha) = H_\pm(\alpha)$ where

$$H_\pm(\alpha) := n\gamma_n^{1/n} \frac{\alpha}{G(\alpha)^{(n-1)/n}} \pm \frac{F(\alpha)}{G(\alpha)}.$$

    Then

$$m_\pm \leq \inf_{\alpha>0} J_\pm(w_\alpha) = \inf_{\alpha>0} H_\pm(\alpha). \qquad (5)$$

  - Conversely: Any $v \in K$ can be written as:

$$v(t) = \int_0^\infty w_\alpha(t) d\mu_v(\alpha) \text{ where } d\mu_v(\alpha) := -\gamma_n G(\alpha) v'(\alpha)$$

    is a probability measure on $[0, \infty)$.

Proof of the Main Result

- **Step 2** continues: (Reduction of $(\mathcal{P})$ to a 1-D problem).
  - By Jensen's inequality:

$$
\begin{aligned}
J_\pm(v) &= J_\pm\left(\int_0^\infty w_\alpha d\mu_v(\alpha)\right) \\
&\geq \int_0^\infty J_\pm(w_\alpha) d\mu_v(\alpha) \\
&= \int_0^\infty H_\pm(\alpha) d\mu_v(\alpha) \\
&\geq \inf_{\alpha>0} H_\pm(\alpha) \int_0^\infty d\mu_v(\alpha) = \inf_{\alpha>0} H_\pm(\alpha)
\end{aligned}
$$

Then $$m_\pm = \inf_{v\in K} J_\pm(v) \geq \inf_{\alpha>0} H_\pm(\alpha) \geq m_\pm. \qquad (6)$$

Proof of the Main Result

- **Step 3:** The unique minimizers of $(\mathcal{P})$ are $\pm u_\alpha(x_0 + .)$ with $H_\pm(\alpha) = m_\pm$, where $u_\alpha := \alpha \chi_{B_{\rho_\alpha}}$, and $\rho_\alpha := \frac{1}{\gamma_n^{1/n} G(\alpha)^{1/n}}$
  - If $H_\pm(\alpha) = m_\pm$, then the admissible u in $(\mathcal{P})$ associated with w_α is u_α, and so:

$$E_\pm(u_\alpha) = J_\pm(w_\alpha) = H_\pm(\alpha) = m_\pm.$$

  - **Uniqueness**: If $\bar{u}$ is another minimizer in $(\mathcal{P})$, define $u := \bar{u}^*$, and associate $v \in K$. By the concavity of $J_\pm$,

$$m_\pm = J_\pm\left(\int_0^\infty w_\alpha d\mu_v(\alpha)\right) \geq \int_0^\infty H_\pm(\alpha) d\mu_v(\alpha) \geq m_\pm$$

$$\Longrightarrow \int_0^\infty \left(H_\pm(\alpha) - m_\pm\right) d\mu_v(\alpha) = 0.$$

Proof of the Main Result

- **Step 3** continues:
  Then
  $$\mu_v = \delta_\alpha,\ H(\alpha) = m_\pm \Longrightarrow v = v_\alpha,\ H(\alpha) = m_\pm$$
  Then $\bar{u}^* = u = u_\alpha = \alpha\chi_{B_{\rho_\alpha}}$. And $E_\pm(u) = E_\pm(\bar{u})$ implies that:
  $$|\bar{u}| = \alpha\chi_A \text{ with } |A| = |B_{\rho_\alpha}|,\ P(A) = P(B_{\rho_\alpha}).$$
  By the isoperimetric inequality, A is a ball with $|A| = |B_{\rho_\alpha}|$, i.e., $|\bar{u}| = u_\alpha(x_0 + .)$ or $\bar{u} = \pm u_\alpha(x_o + .)$
- **Step 4:** (nonexistence). If $H_\pm(\alpha)$ is not attained in $(0,\infty)$, then $\exists\, u_{\alpha_n}$ s.t. $\alpha_n \to 0$ (dispersion) or $\alpha_n \to \infty$ (concentration), and then $(\mathcal{P})$ has no minimizers.

Applications: sharp inequalities

- **Sharp L^1 logarithmic Sobolev inequality**: If $F(t) = t \ln t$ (or $F(t) = t^\beta \ln t$, $0 < \beta \le 1$) and $G(t) = t$, then $H_-(\alpha)$ is attained at $\alpha_\infty = 1/\gamma_n$, and $u_{\alpha_\infty} = \chi_{B_1}/\gamma_n$. We have

$$\int_{\mathbb{R}^n} |u| \ln \left(\frac{e^n \gamma_n |u|}{\|u\|_{L^1(\mathbb{R}^n)}} \right) \le \int_{\mathbb{R}^n} d\,|\nabla u|, \quad \forall u \in \mathrm{BV}(\mathbb{R}^n).$$

  It is sharp and the extremal functions are $\pm u_{\alpha_\infty}(x_o + .)$

- **Sharp L^1 Gagliardo-Nirenberg inequality**: Choose $F(t) = t^q/q$ and $G(t) = t^s$, with $1 < q < s < 1^*$. Then $H_+(\alpha)$ has a unique minimizer.

- **Sharp L^1 Sobolev inequality**: Choose $F = 0$ and $G(t) = t^{1^*}$, then $H_+(\alpha) = n\gamma^{1/n}$ has all $\alpha > 0$ as minimizers.

Applications: PDEs involving 1-Laplacian

- **A Nonexistence Theorem:** In addition to the previous assumptions, if F is convex and G is concave on $\mathbb{R}_+$, then for $\lambda \geq 0$, the PDE

$$\begin{cases} -\Delta_1 u = \lambda G'(u) - F'(u) \\ u \geq 0, \ \int_{\mathbb{R}^n} G(u) = 1 \end{cases} \tag{7}$$

has no solutions in $D^{1,q}(\mathbb{R}^n)$.

  - Formally, $\Delta_1 u = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$. A precised definition of solutions to 1-Laplacian PDEs can be found for e.g. in [Anzellotti, 83], [Demengel, 02].

Applications: PDEs involving 1-Laplacian

- **A Nonexistence Theorem (continuation)**
  - **Proof**: The convexity/concavity assumption implies that u solves (9) iff u is a minimizer of

$$\inf\{\int_{\mathbb{R}^n} d\,|\nabla u| + \int_{\mathbb{R}^n} F(|u|) : \int_{\mathbb{R}^n} G(|u|) = 1\}. \qquad (8)$$

    But (8) has no minimizers because the infimum in $H_+(\alpha)$ is not attained in $(0,\infty)$; in fact, it is 0, for $H_+(0^+) = 0 < H_+(\alpha)$.
  - For e.g., choosing $G(t) = t$ and $F(t) = t^q/q$ $(q > 1)$, and using a scaling argument, we have that the PDE $-\Delta_1 u = 1 - u^{q-1}$ has no solutions in $D^{1,q}(\mathbb{R}^n)$.

Applications: PDEs involving 1-Laplacian

- **Existence Theorem:** If $H_{\pm}(\alpha)$ has a minimizer in $(0,\infty)$, then the PDE (where $\lambda \in \mathbb{R}$ is part of the unknown)

$$\begin{cases} -\Delta_1 u = \lambda G'(u) \pm F'(u) \\ u \geq 0, \ \int_{\mathbb{R}^n} G(u) = 1 \end{cases} \tag{9}$$

has a nontrivial solution.

  - For e.g., choosing $F(t) = t^q/q$ and $G(t) = t^s$ with $1^* < s < q$ or $1 < q < s < 1^*$, and using a scaling argument, we have that $-\Delta_1 u = u^{s-1} - u^{q-1}$ has nontrivial nonnegaive solutions in $D^{1,q}(\mathbb{R}^n)$.
  - Similarly, if $s < q < 1 + s/n$, then $-\Delta_1 u = u^{s-1} + u^{q-1}$ has nontriivial nonnegative solutions in $\mathrm{BV}(\mathbb{R}^n)$.

Optimal Transportation Approach

- Following similar arguments as in [Cordero-Nazaret-Villani, 04] and [A, Ghoussoub, Kang, 04], we can recover our result in some special cases (e.g. $F(t) = t$ and $G(t) = t^s$ with $s < 1^*$, that is, the special case $q = 1$ of the sharp L^1 Gagliardo-Nirenberg inequalities) – without using the isoperimetric inequality.
- This is not surprising because $q = 1$ is the limit case of $q = 1 + s(p-1)/p$ as $p \to 1$, where the sharp L^p Gagliardo-Nirenberg inequality are recovered using OT.
- This result as well as previous works in [A, 06] and [A, 08], suggest that the proof of the remaining L^p GN (i.e. $q \not\in \{1 + s(p-1)/p,\, p(s-1)/(p-1)\}$) should be searched for by other means than OT. Then, possibly, use their link with OT to derive new results in OT.

Optimal Transportation Approach

- **Duality theorem:** If $\psi : [0,\infty) \to \mathbb{R}$ is s.t. $\psi(0) = 0$ and $x \mapsto x^n \psi(x^{-n})$ is convex and non-increasing, then

$$\sup\{-H^{\psi}(f_1) : \ f \in \mathcal{P}(B_1)\}$$
$$= \inf\{-H^{\psi+nP_\psi}(f_0) + \int_{\mathbb{R}^n} d|\nabla\left(P_\psi(f_0)\right)| : f_0 \in \mathcal{P}(\mathbb{R}^n)\}$$

and any function $f_\infty \in \mathcal{P}(B_1)$ satisfying

$$\|\nabla\left(P_\psi(f_\infty)\right)\|_{\mathcal{M}(\mathbb{R}^n)} = n\|P_\psi(f_\infty)\|_{L^1(\mathbb{R}^n)}. \tag{10}$$

solves both variational problems.
Here $H^{\psi}(f) := \int_{\mathbb{R}^n} \psi(f)$ and $P_\psi(x) := x\psi'(x) - \psi(x)$.

Optimal Transportation Approach

• The duality theorem follows from the displacement convexity [McCann, 94] of H^ψ which leads to [A, 02]:

$$-H^\psi(f_1) \leq -H^{\psi+nP_\psi}(f_0) + \int_{\mathbb{R}^n} P_\psi(f_0)\text{div}(T),$$

and the relations: If $|T(x)| \leq 1$, then

$$\int_{\mathbb{R}^n} P_\psi(f_0)\text{div}(T) \leq \|\nabla\left(P_\psi(f_0)\right)\|_{\mathcal{M}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} d|\nabla\left(P_\psi(f_0)\right)|,$$

and if $f_0 = f_1 \in \mathcal{P}(B)$ (i.e. $T(x) = x$), then

$$\|\nabla\left(P_\psi(f_0)\right)\|_{\mathcal{M}(\mathbb{R}^n)} = \int_{B_1} P_\psi(f_0)\text{div}(x) = n\|P_\psi(f_0)\|_1.$$

Optimal Transportation Approach

- For e.g., if we choose If $\psi = \frac{x^\gamma}{\gamma - 1}$, $\gamma = 1/s$, with $s < 1^*$, and $f_0 = u^s$ with $\|u\|_s = 1$, then the duality theorem shows that functions u s.t.

$$\mathrm{spt}(u) \subset B_1, \; \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} = n\|u\|_1, \; \|u\|_s = 1$$

are minimizers of

$$\inf_{u \in BV(\mathbb{R}^n)} \left\{ \left(\frac{s}{s-1} - n \right) \int_{\mathbb{R}^n} |u| + \int_{\mathbb{R}^n} \mathrm{d}|\nabla u| : \int_{\mathbb{R}^n} |u|^s = 1 \right\}.$$

In particular, $u = \frac{\chi_{B_1}}{|B_1|^{1/s}}$ is a minimizer because $\|\nabla \chi_{B_1}\|_{\mathcal{M}(\mathbb{R}^n)} = P(B_1) = n|B_1|$.