The Fast Diffusion
Recent progress in the theory of Existence, Regularity and Asymptotics from a Functional Point of View

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Outline

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The Porous Medium - Fast Diffusion Equation

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

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- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
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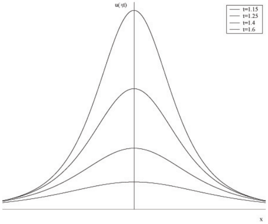
FDE profiles

- We have well-known explicit formulas for Selsfsmilar Barenblatt profiles with exponents less than one if $1 > m > (n-2)/n$:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$



The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

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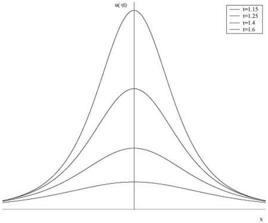

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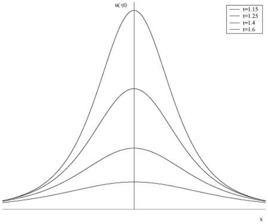

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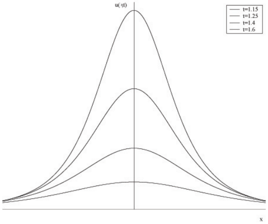

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- William J. Reed

  Many empirical distributions encountered in economics and other realms of inquiry exhibit power-law behaviour in the upper tail. In economics prime examples are the distributions of incomes (Pareto's law) and city sizes (Zipf's law or the rank-size property), as well as the standardized price returns on individual stocks or stock indices.
- Elsewhere, empirical size distributions for which power-law behaviour has been claimed include those of sand particle sizes; of meteor impacts on the moon; of numbers of species per genus in flowering plants; of frequencies of words in long sequences of text and of areas burnt in forest fires etc.
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- It seems unlikely that there is a single general theory that could explain all instances of power-law behaviour.

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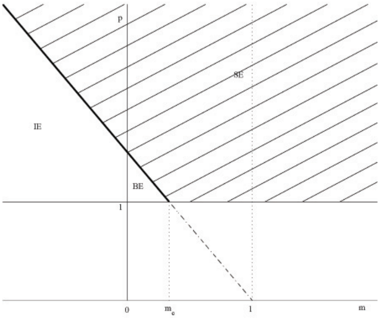

Figure 1. The (m, p) diagram for the PME/FDE in dimensions $n \geq 3$.
SE: smoothing effect, BE: backwards effect, IE: instantaneous extinction
Critical line $p = n(1 - m)/2$ (in boldface)

More exponents appear. One is $m = 0$. A third exponent $m = (n-2)/(n+2)$ (in dimensions $n \geq 3$), which is the inverse of the famous Sobolev exponent of the elliptic theory. The relation is clear by separation of variables. Exercise

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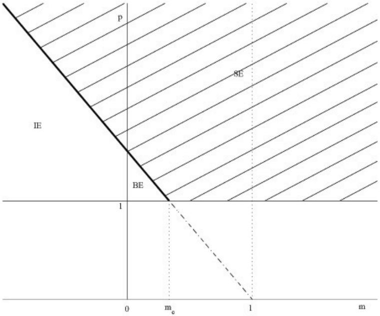

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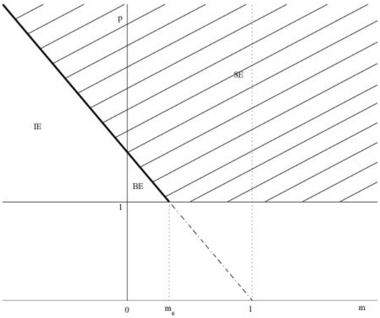

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- For $1 > m > m_c$ we can solve the initial value problem with any nonnegative measure, even a Borel Measure because there are very good local estimates Chasseigne-Vazquez 03. For $m < m_c$ Dirac masses cannot be initial data Brezis-Friedman 83, Pierre 87.
- Harnack holds for $m > m_c$ [1], but is very difficult for $m < m_c$ work just finished [2].

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The good and bad range II

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The good and bad range III

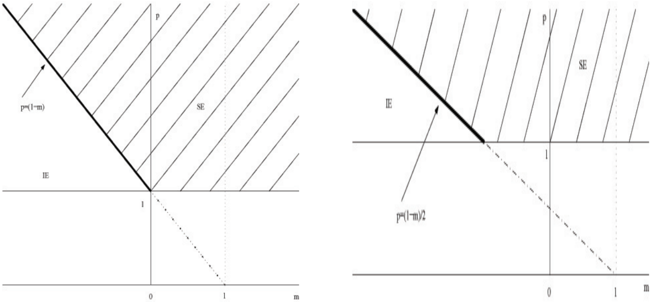

Figure 2. Left: (m, p) diagram for the PME/FDE in dimension $n = 2$
Right: (m, p) diagram for the PME/FDE in dimension $n = 1$

- There is existence and non-uniqueness if $n = 1$ and $-1 < m < 0$

Reading the classics

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The Aronson Caffarelli Estimate for PME

- CASE $m > 1$ Aronson-Caffarelli's result [3] is a positivity estimate for the PME, $m > 1$, valid for all nonnegative weak solutions defined in the whole space.

  We take a point x_0 and a ball $B_R(x_0)$ and try to see how positive is the solution at time $t0$ if there is a "mass" $M_R(x_0) = \int_{B_R(x_0)} u_0(x)\,dx$ at $t = 0$. It says

$$\frac{M_R(x_0)}{R^d} \le C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{\lambda/2}(t, x_0). \tag{1}$$

  with $\lambda = 2 + d(m-1)$. C_1 and C_2 given positive constants depending only on m and d. Looking at the three terms we discover that there is a time t_* where the second is already less than the first one. We can calculate this intrinsic positivity time as $t_* = C(m,d) R^\lambda / M^{m-1}$.

- For $t > t_*$ the third one is positive, hence $u(x_0, t) > 0$. Hence, for all large t we have $u = O(t^{d/\lambda})$. OK!

- We go on to prove that $u \in C^\alpha$ for some $\alpha > 0$. There is no way you can get positivity for small times because of finite propagation (free boundaries).

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Universal Pointwise Estimates for Good Fast Diffusion

- CASE $m_c < m < 1$ This range has wonderful a priori estimates of local type. We assume that $u \ge 0$.
- If $u_0 \in L^1_{loc}(\mathbb{R}^n)$ then for all $t > 0$ we have $u(\cdot, t) \in L^\infty(\mathbb{R}^n)$, cf. Herrero-Pierre, 1985.
- There is a universal constant $C > 0$ such that if $v = u^{m-1}$

$$t|\Delta v| \le C, \quad t|\frac{v_t}{v}| \le C, \quad t\frac{|\nabla v|^2}{v} \le C. \tag{2}$$

Estimates for the PME were original of Aronson, Crandall and Benilan. [4] [5]

Note that v satisfies the quadratic equation

$$v_t = v\Delta v - \gamma|\nabla v|^2, \qquad \gamma = 1/(1-m).$$

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Universal Estimates continued

- Universal estimates have been found in other problems.
- Some can can be found for the heat equation. They also work for the *p*-Laplacian equation (fast or slow) in similar exponent ranges[6]
- Similar estimates were discovered by Yau and Li [7] for flows on manifolds and they prove that they produce continuity.
- Later Hamilton for the Rcci flow. [8]
- For $m \le m_c$ the first estimate from below fails and the second also from below and the third from above.

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AC Type Estimate for Good Fast Diffusion

- In the paper Bonforte- Vazquez, Global positivity estimates and Harnack inequalities for the fast diffusion equation. J. Funct. Anal., 2006

  we take the approach to regularity through positivity inspired by the work of Di Benedetto and collaborators for PME, FDE and PLE using intrinsic versions of Harnack.

- We study local solutions of the FDE in the good exponent range $m_c < m < 1$. The change in the sign of the exponent $m-1$ implies that we get good lower estimates for $0 < t \le t_*$ if the ideas of AC can be made to work. Moreover, we can continue these estimates for $t \ge t_*$ thanks to the fortunate circumstance that we have further differential inequalities, like $\partial_t u \ge -Cu/t$ in the case of the Cauchy problem. We get a continuation of the lower bounds with optimal decay rates in time. The final form is

$$u(t,x) \ge M_R(x_0)\, H(t/t_c), \quad M_R(x_0) = R^{-d} \int_{B_R(x_0)} u_0\, dx. \tag{3}$$

- The critical time is defined as before; the function $H(\eta)$ is defined as $K\eta^{1/(1-m)}$ for $\eta \le 1$ while $H(\eta) = K\eta^{-d\vartheta}$ for $\eta \ge 1$, with $K = K(m,d)$. Note that for $0 < t < t_c$ the lower bound means

$$u(t,x_0) \ge k(m,d)(t/R^2)^{1/(1-m)}$$

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The AC Estimate for Bad Fast Diffusion

We know that for m, m_c all kinds of functional disasters may happen. In particular, extinction in finite holds for all integrable data (and some more) so that positivity for long times must be excluded. Let u be a local solution with extinction time > 0. We prove this result in M Bonforte- JL Vazquez, *Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations,* Preprint.

Theorem

Let $0 < m < 1$ and let u be the solution to the FDE under the above assumptions. Let x_0 be a point in Ω and let $d(x_0, \partial\Omega) \ge 5R$. Then the following inequality holds for all $0 < t < T$

$$(4) \qquad R^{-d} \int_{B_R(x_0)} u_0(x)\,dx \le C_1\, R^{-2/(1-m)}\, t^{\frac{1}{1-m}} + C_2\, T^{\frac{1}{1-m}} R^{-2}\, t^{-\frac{m}{1-m}}\; u^m(t, x_0).$$

with C_1 and C_2 given positive constants depending only on d. This implies that there exists a time t_* such that for all $t \in (0, t_*]$

$$(5) \qquad u^m(t, x_0) \ge C_1'\, R^{2-d} \|u_0(x)\|_{L^1(B_R)} T^{-\frac{1}{1-m}}\; t^{\frac{m}{1-m}}.$$

where $C_1' > 0$ depends only on d; t_* depends on R and $\|u_0(x)\|_{L^1(B_R)}$ but not on T.

The local boundedness result for Fast Diffusion

- The main result of this part is the local upper bound that applies for the same type of solution and initial data, under different restrictions on p. Here is the precise formulation.

  We take $d \geq 3$. recall that $m_c = (d-2)/d$, that $p_c = d(1-m)/2$.

Theorem

Let $p \geq 1$ if $m > m_c$ or $p > p_c$ if $m \leq m_c$. Then there are positive constants $C_1, \mathcal{C}_2$ such that for any $0 < R_1 < R_0$ we have

$$\sup_{x \in B_{R_1}} u(t,x) \leq \frac{C_1}{t^{d\vartheta_p}} \left[\int_{B_{R_0}} |u_0(x)|^p \, dx \right]^{2\vartheta_p} + C_2 \left[\frac{t}{R_0^2} \right]^{\frac{1}{1-m}} . \tag{6}$$

- We recall that $\vartheta_p = 1/(2p - d(1-m)) = 1/2(p - p_c)$. The constants C_i depend on m, d and p, R_1 and R_0 and blow up when $R_1/R_0 \to 1$; an explicit formula for C_i can be found.

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Local Boundedness II

- The proof consists in two steps: (1) The norm $\|u(\cdot,t)\|_{L^p_{loc}}$ grows with time in a controlled way in terms of its value at $t=0$, if $p \ge 1$, $p > 1-m$. This uses Herrero-Pierre's approach.

  (2) Solutions in $L^p_{x,t}$ locally in space/time are in fact bounded in a smaller cylinder if $p > p_c$. This uses Moser iteration.

- Local Boundedness implies existence of Large Solucions having boundary data $u = +\infty$. Such solutions form the Maximal Semigroup. A reference is E Chasseigne, JL Vazquez, Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. Arch. Ration. Mech. Anal. 164 (2002), no. 2, 133–187.

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- The adaptation of these two arguments to p-Laplacian flows is being done by Razvan Iagar with us.. It also works for p, 1 without the restriction p larger than a critical p_* (explained in the book Smoothing).

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Reading the classics II

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Outline

Asymptotic behaviour for PME

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2+n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

Asymptotic behaviour for PME

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- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

Asymptotic behaviour for PME

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

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- The rates. Carrillo-Toscani 2000. Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):
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- If we are only assuming $u_0 \geq 0$, u_0 integrable The rate of convergence cannot be improved without more information on u_0.
- m may be also less than 1 but supercritical ($\rightarrow$ with even better convergence called relative error convergence)
- For $m < (n-2)/n$ we do not have the original model. Big surprises;
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Calculations of the entropy rates

- We rescale the function as $\quad u(x,t) = R(t)^n \rho(y\,R(t), s)$
  where $R(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$. Equation becomes
  $$\rho_s = \mathrm{div}\,(\rho(\nabla\rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
  (this conserves mass $\int u(x,t)dx + \int \rho(y,\tau)\,dy$).
- Then define the entropy
  $$E(u)(t) = \int (\frac{\rho^m}{m-1} + \frac{c}{2}\rho y^2)\,dy$$
  The minimum of entropy is identified as the Barenblatt profile.
- Calculate
  $$\frac{dE}{ds} = -\int \rho|\nabla\rho^{m-1} + cy|^2\,dy = -D$$
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Outline

The Cauchy Problem for the FDE in $\mathbb{R}^d$

We consider the solutions $u(t,x)$ of

$$\begin{cases} \partial_t u = \Delta(u^m/m) = \nabla \cdot (u^{m-1}\nabla u) \\ u(0,\cdot) = u_0 \end{cases}$$

where $m \in (0,1)$ (fast diffusion) and $(t,x) \in Q_T = (0,T) \times \mathbb{R}^d$

- Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- For $1 > m > m_c$ *the mass* $\int_{\mathbb{R}^d} u(y,t)\,\mathrm{d}y$ *is preserved in time if* $u_0 \in L^1(\mathbb{R}^d)$. Non-negative solutions are positive and smooth for all $x \in \mathbb{R}^d$ and $t > 0$. Intermediate asymptotics, as $t \to +\infty$
- If $m < m_c$ mass is NOT preserved and solutions may extinguish in finite time.

$$u_0 \in L^{p_c}(\mathbb{R}^d), \quad p_c = \frac{d\,(1-m)}{2} \quad \Longrightarrow \quad \exists\ T = T(u_0)\ : \quad u(\tau,\cdot) \equiv 0 \quad \forall\ t \geq T$$

- With minor changes we can also do $m \leq 0$

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Barenblatt solutions and rescaling

The BS

$$U_{D,T}(t,x) := \frac{1}{R(t)^d}\left(D + \frac{1-m}{2}\left|\frac{x}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

- with $R(\tau) := \left[d\,(m-m_c)\,(t+T)\right]^{\frac{1}{d\,(m-m_c)}}$ if $m_c < m < 1$
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Time-dependent rescaling:

$\tau := \log\left(\frac{R(t)}{R(0)}\right)$ and $y := \frac{x}{R(t)}$. The function $v(\tau,y) := R(t)^d\,u(t,x)$ solves a nonlinear *Fokker-Planck type equation*

$$\begin{cases} \partial_\tau v = \Delta_y v^m + \nabla_y \cdot (y\,v) & (\tau,y) \in (0,+\infty)\times\mathbb{R}^d \\ v(0,y) = v_0(y) = R(0)^d\,u_0(R(0)\,y) & y \in \mathbb{R}^d \end{cases}$$

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Assumptions

(H1) u_0 is a non-negative function in $L^1_{loc}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,x) \le u_0(x) \le U_{D_1,T}(0,x) \quad \forall\, x \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(x) = U_{D_*,T}(0,x) + f(x) \quad \forall\, x \in \mathbb{R}^d$$

(H1') v_0 is a non-negative function in $L^1_{loc}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \le v_0(x) \le V_{D_1}(y) \quad \forall\, y \in \mathbb{R}^d$$

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$$v_0(y) = V_{D_*}(y) + f(y) \quad \forall\, x \in \mathbb{R}^d$$

A New Critical Exponent m_* appears

$$m_* = \frac{d-4}{d-2} < m_c = \frac{d-2}{d}$$

it has a meaning in the linearized analysis in terms of spectral properties of some self-adjoint operator.

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Conv. to the asymptotic profile (without rate)

Theorem

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(i) For any $m > m_*$, there exists a unique D_* such that $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = 0$ for any $t > 0$. Moreover, for any $p \in (q(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,dx = 0$

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$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m) + d\,(1-m)}$$

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- We translate the result as power decay in the original problem

Corollary

Let $d \geq 3$, $m \in (0,1)$, $m \neq m_*$. Consider a solution u with initial data satisfying (H1)-(H2). For τ large enough, for any $q \in (q_*, \infty]$, there exists a positive constant C such that

$$\|u(t) - U_{D_*}(t)\|_q \leq C\, R(t)^{-\alpha}$$

where $\alpha = \lambda_{m,d} + d(q-1)/q$ and large means $T - t > 0$, small, if $m < m_c$, and $t \to \infty$ if $m \geq m_c$

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For any $p \in (d/2, \infty]$, there exists a positive constant $\mathcal{C}$ and $\gamma > 0$ such that

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Idea of the proof

Passing to the quotient: the function $w(t,x) := \dfrac{v(t,x)}{V_{D_*}(x)}$ solves

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{1}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

with

$$0 < \inf_{x \in \mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \le w(t,x) \le \sup_{x \in \mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

We then get a number of preliminary estimates on w like conservation of the relative mass and the Harnack Principle

$$\|w(t)\|_{C^k(\mathbb{R}^d)} \le \overline{H}_k < +\infty \quad \forall\, t \ge t_0$$

$\exists\ t_0 \ge 0$ s.t. (H1) holds if $\exists\ R > 0$, $\sup_{|y|>R} u_0(y)\, |y|^{\frac{2}{1-m}} < \infty$, and $m > m_c$

Heuristics: linearization and weights

Take $w(t,x) = 1 + \varepsilon\, \frac{g(t,x)}{V_{D_*}^{m-1}(x)}$ and formally consider the limit $\varepsilon \to 0$ in

$$\begin{cases} w_\tau = \dfrac{1}{V_{D_*}}\, \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{1}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

Then g solves

$$g_\tau = V_{D_*}^{m-2}(x)\, \nabla \cdot \left[V_{D_*}(x)\, \nabla g(t,x) \right]$$

and the entropy and Fisher information functionals

$$\mathsf{F}(g) := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2\, V_{D_*}^{2-m}\, dx \quad \text{and} \quad \mathsf{I}(g) := \int_{\mathbb{R}^d} |\nabla g|^2\, V_{D_*}\, dx$$

consistently verify

$$\frac{\mathrm{d}}{d\tau} \mathsf{F}(g(\tau)) = -\, \mathsf{I}(g(\tau)). \tag{7}$$

It is interesting to write $d\nu = V\, dy$ and $d\mu = V^{2-m} dy = d\nu/(D + cy^2)$.

Linear weighted problem

Consider the following linear equation for g,

$$\frac{\partial g}{\partial \tau} = A_m\, g$$

where

$$A_m\, g := m\, V_{D_*}^{m-2}(y)\, \nabla \cdot [V_{D_*}\, \nabla\, g] \ .$$

The linear operator $A_m : L^2(\mathbb{R}^d, d\mu) \to L^2(\mathbb{R}^d, V_{D_*}^{2-m}\, dx)$ is the positive self-adjoint operator associated to the closure of the quadratic form - the *Dirichlet Form* - defined for $\phi \in C_c^\infty(\mathbb{R}^d)$ by

$$\mathrm{I}[\phi] := m \int_{\mathbb{R}^d} |\nabla \phi|^2\, V_{D_*}\, dx\ .$$

If we can relate $\mathrm{I}[\phi]$ to $\mathrm{F}(\phi)$ by a functional inequality of the form

$$\lambda \mathrm{F}(\phi) \le \mathrm{I}[\phi],$$

combining this with $d\mathrm{F}(g(\tau))/d\tau) = -\,\mathrm{I}(g(\tau))$ to get after one integration

$$\mathrm{F}(g(\tau) \le \mathrm{F}(g(0))\, \mathrm{e}^{-\lambda\tau}.$$

Theorem - Spectral Gap: Hardy-Poincaré Inequalities

Let $d \geq 1$ and $D > 0$. If $m \in (0,1)$ and $1 \leq d \leq 4$, or $m \in (m_*, 1)$ and $d \geq 5$, then there exists a positive constant $\mathcal{C}_{m,d}$, which does not depend on D, such that

(8)
$$\int_{\mathbb{R}^d} |g - \overline{g}|^2 \ V_D^{2-m} \, dx \leq \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \ V_D \, dx \ , \quad \forall \, g \in \mathcal{D}(\mathbb{R}^d) \ ,$$
$$\overline{g} = \int_{\mathbb{R}^d} g \ V_D^{2-m} \, dx \ .$$

In case $d \geq 5$ and $m \in (0, m_*)$, we have

(9)
$$\int_{\mathbb{R}^d} g^2 \ V_D^{2-m} \, dx \leq \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \ V_D \, dx \ , \quad \forall \, g \in \mathcal{D}(\mathbb{R}^d)$$

with *optimal constant*

$$\mathcal{C}_{m,d} = \frac{8 \, m \, (1-m)}{[(d-2)\,(m - m_*)]^2}$$

Estimates of the optimal constant $\mathcal{C}_{m,d}$ when $m > m_*$ are needed

Recall that $m_* = (d-4)/(d-2)$ and $V_D = (D + \frac{1-m}{2m}|x|^2)^{-\frac{1}{1-m}}$.

Weighted Hardy Inequalities

We now consider the limit $D \to 0^+$, in the Spectral Gap Theorem . Letting $\alpha := 1/(m-1)$, we obtain the *Weighted Hardy inequality,*

$$\int_{\mathbb{R}^d} \frac{|g|^2}{|x|^2}\,|x|^\alpha\,dx \le \mathcal{H}_\alpha \int_{\mathbb{R}^d} |\nabla g|^2\,|x|^\alpha\,dx\,, \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d)\,.$$

with the optimal constant

$$\mathcal{H}_\alpha := \frac{4}{[2\,\alpha + d - 2]^2} = \frac{8\,m\,(1-m)}{[(d-2)\,(m-m_*)]^2} \cdot \frac{1-m}{2\,m}\,.$$

SKETCH OF PROOF. Such an inequality is easy to establish by the "completing the square method" as follows.

$$\begin{aligned} 0 \quad &\le \quad \int_{\mathbb{R}^d} \left|\nabla g + \lambda\,\frac{x}{|x|^2}\,g\right|^2 |x|^{2\alpha}\,dx \\ &= \int_{\mathbb{R}^d} |\nabla g|^2\,|x|^{2\alpha}\,dx + \left[\lambda^2 - \lambda\,(2\,\alpha + d - 2)\right] \int_{\mathbb{R}^d} \frac{|g|^2}{|x|^2}\,|x|^{2\alpha}\,dx\,. \end{aligned}$$

An optimization of the right hand side with respect to λ gives the desired inequality. □

The nonlinear functionals

Relative entropy

$$\mathcal{F}(w) := \frac{1}{1-m}\int_{\mathbb{R}^d}\left[(w-1)-\frac{1}{m}(w^m-1)\right] V_{D_*}^m\,dx$$

Relative Fisher information

$$\mathcal{I}(w) := \frac{1}{(m-1)^2}\int_{\mathbb{R}^d}\left|\nabla\left[\left(w^{m-1}-1\right)V_{D_*}^{m-1}\right]\right|^2 w\,V_{D_*}\,dx$$

These functionals are the linear counterpart of the nonlinear functionals that will be used in the nonlinear analysis:

Relative Entropy $\mathcal{F}$, linearized gives the $L^2(\mathbb{R}^d, V_{D_*}^{2-m}\,dx)$-norm
Relative Fisher Information $\mathcal{J}$, linearized gives the Dirichlet Form.

Proposition

Under assumptions (H1)-(H2),

$$\frac{d}{d\tau}\mathcal{F}(w(\tau)) = -\mathcal{I}(w(\tau))$$

Comparison of the functionals

- Let $m \in (0,1)$ and assume that u_0 satisfies (H1)-(H2) we get for the Relative entropy]

$$C_1 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx \le \mathcal{F}[w] \le C_2 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx$$

- For the Fisher information there is only approximate equivalence

$$\mathrm{I}[g] \le \beta_1 \, \mathcal{J}[w] + \beta_2 \, \mathrm{F}[g] \quad \text{with} \quad g := (w-1) \, V_{D_*}^{m-1}$$

- Use the spectral gap estimate, with $\mathcal{C}_{m,d} = m/\lambda_{m,d}$, to obtain

$$2\,\mathrm{F}[g] \le \mathcal{C}_{m,d} \, \mathrm{I}[g] \,,$$

which gives, for the solution of the linear problem $g_t = A_m \, g$, the *exponential decay of the weighted L^2-norm*

$$\mathrm{F}[g(t)] \le \mathrm{e}^{-2\lambda_{m,d} t} \, \mathrm{F}[g(0)] \quad \forall \, t \ge 0 \,.$$

- We prove that $\beta_2(\tau)$ goes to zero with large τ and then we easily get a similar exponential decay of the weighted L^2-norm

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Limit when m tends to 1

In the limit $m \to 1$, we observe that

$$\lim_{m\to 1^-} D_*^{1/(1-m)}\, V_{D_*} = (2\,\pi\, D_*)^{d/2}\,\mu \quad \text{with} \quad \mu(x) = \frac{e^{-\frac{|x|^2}{2\,D_*}}}{(2\,\pi\, D_*)^{d/2}}\ .$$

so that the equation *formally converges to the Ornstein-Uhlenbeck equation*,

$$g_t \;=\; m\, V_{D_*}^{m-2}(x)\,\nabla\cdot\left[V_{D_*}\,\nabla g\right] \quad \longrightarrow \quad g_t = \mu^{-1}\,\nabla\cdot\left(\,\mu\,\nabla g\,\right)\ .$$

Also the spectral gap inequality

$$\int_{\mathbb{R}^d} |g|^2\ V_D^{2-m}\,dx \le \mathcal{C}_{m,d}\int_{\mathbb{R}^d} |\nabla g|^2\ V_D\,dx \quad \forall\, g\in C^\infty(\mathbb{R}^d) \text{ such that } \int_{\mathbb{R}^d} g\ V_D^{2-m}\,dx = 0$$

formally converges to the Gaussian-Poincaré inequality

$$\int_{\mathbb{R}^d} |\phi|^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\nabla\phi|^2\ \mathrm{d}\mu \quad \forall\,\phi\in C^\infty(\mathbb{R}^d)\ \text{ such that } \int_{\mathbb{R}^d}\phi\,\mathrm{d}\mu = 0\ ,$$

where $\mathrm{d}\mu := \mu\,dx$. In the Gaussian case, a logarithmic Sobolev inequality holds, [Gross]

$$\int_{\mathbb{R}^d} |\phi|^2\ \log\left(\frac{|\phi|^2}{\int_{\mathbb{R}^d}|\phi|^2\,\mathrm{d}\mu}\right)\ \mathrm{d}\mu \;\le\; 2\int_{\mathbb{R}^d} |\nabla\phi|^2\ \mathrm{d}\mu\ ,$$

which is stronger than the Gaussian Poincaré inequality. With the measure $V_{D_*}\,dx$. Although the spectral gap inequality holds true, there is no corresponding logarithmic Sobolev inequality.

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End

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