Exact Semi-Geostrophic Flows in an Elliptical Ocean Basin

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Overview

- Nontrivial exact solutions of 2D (XY) Semi-Geostrophic equations in elliptical domain.
- Dual variable formulation: active scalar with Monge-Ampère equation for potential.
- Find exact quadratic solutions to Monge-Ampère 2nd BVP on ellipses.
- Take advantage of conserved quantities.
- Reduce the PDE (under quadratic ansatz) to Hamiltonian system of 2 ODE.
- Study parameter dependence of the Hamiltonian system: find bifurcations, give Mass Transport interpretation.
- Reinterpret results in the physical domain.

Motivation/Application

- Have an *explicit, nontrivial* solution. Few available in the S-G theory. Nonlinear analog for S-G of Kirchoff ellipses.
- S-G a useful model:
  - Physically valid approximation for small Rossby numbers.
  - Density stratification due to gravity makes flows 2D,
  - Can be used to model fully resolved Euler flows with coarsely resolved SG.
- Exact solution useful for validation purposes.
  - Numerical solution of Monge-Ampère/ Mass Transport & S-G still a challenge.
  - Can test solutions of PDE by solving a nonlinear ODE.

Euler, SG, and QG

2D Rotating Euler, stream function, Q. $\mathbf{v} = \mathtt{J}\nabla Q = \nabla^{\perp} Q$,

$$\nabla \frac{\partial Q}{\partial t} + \left(D^2 Q + \mathtt{I}\right) \mathtt{J}\nabla Q - \mathtt{J}\nabla P = 0 \qquad \text{(2D rotating Euler)}$$

Approximate: Acceleration Terms $\ll$ Rotation Term (Ro $\ll 1$).
Models:

$$Q = P \qquad \text{(Gestrophic Balance)}$$

Semi-geostrophic (Eliassen, Hoskins)

$$\nabla \frac{\partial P}{\partial t} + \left(D^2 P + \mathtt{I}\right) \mathtt{J}\nabla Q - \mathtt{J}\nabla P = 0, \qquad \text{(SG)}$$

Quasigeostrophic

$$\nabla \frac{\partial P}{\partial t} + D^2 P \mathtt{J}\nabla P + \mathtt{J}\nabla Q - \mathtt{J}\nabla P = 0. \qquad \text{(QG)}$$

Symmetries and Conservation Laws:

Potential Vorticity

The particle relabelling symmetry implies that the

$$\nu^2(t,\mathbf{y})/4 := \det(D^2P + \mathtt{I}) \qquad \text{(potential vorticity)}$$

remains a constant along each particle's Lagrangian trajectory.

Analogous to (ordinary) vorticity

$$\omega(t,\mathbf{y}) = \mathrm{Tr}\,(D^2Q + \mathtt{I}) \qquad \text{(vorticity)}$$

in Euler's equation.

Symmetries and Conservation Laws

Conservation of Energy

The time translation invariance symmetry, combines with the no-flux boundary condition to yield global conservation of the SG *energy*

$$H_{SG} := \frac{1}{2}\int_Y |\nabla P(t,\mathbf{y})|^2 \, d^2\mathbf{y}, \qquad \text{(Energy)}$$

Check:

$$\frac{d}{dt}H_{SG} = -\int_Y \operatorname{div}\left[\left(\frac{|\nabla P|^2}{2} + P\right) \mathtt{J}\nabla Q\right] d^2\mathbf{y} = 0,$$

Hamiltonian formulations of semigeostrophy based on such energies have been proposed by (Salmon '88 and Roulstone & Norbury '94)

Quadratic-potential ansatz and affine symmetry I

Assume physical domain an ellipse, area $\pi \mathtt{f}^2$, aspect ratio e^{φ}:

$$Y = \{\mathtt{F}\mathbf{z} \mid |\mathbf{z}| < 1\} \subset \mathbf{R}^2, \qquad \mathtt{F} = \begin{pmatrix} \mathtt{f}e^{\varphi/2} & 0 \\ 0 & \mathtt{f}e^{-\varphi/2} \end{pmatrix}$$

Affine invariance of convex functions (and ellipses) motivates *quadratic ansatz* for P, Q.

$$P(t, \mathbf{y}) = \mathbf{y}^T \mathtt{P}(t)\mathbf{y}/2 \qquad Q(t, \mathbf{y}) = \mathbf{y}^T \mathtt{Q}(t)\mathbf{y}/2$$

$\mathtt{P}(t), \mathtt{Q}(t)$ symmetric 2×2 matrices.

The no-flux condition forces $\mathtt{Q}(t) = q(t)\mathtt{F}^{-2}$ and $\mathbf{q}(t) = \mathbf{0}$.

Quadratic-potential ansatz and affine symmetry II

Our ansatz therefore constrains the fluid to slosh around on concentric ellipses in Y — with a velocity given by solving the 2×2 matrix ODE

$$\mathtt{P}'(t) + q(t)(\mathtt{P}(t) + \mathtt{I})\mathtt{J}\mathtt{F}^{-2} - \mathtt{J}\mathtt{P}(t) = 0 \qquad \text{(ODE physical vars.)}$$

for the variable speed $q(t)$ and the symmetric pressure matrix $\mathtt{P}(t)$.

Difficult to solve in Eulerian variables

- *decouple* the system: go to dual variables.
- *reduce dimensionality* of the system using variables consistent with the conserved quantities

Active scalar w. Nonlinear Law

Legendre trans. of convex $V(t,\mathbf{y}) = P(t,\mathbf{y}) + |\mathbf{y}|^2/2$,

$$U(t,\mathbf{x}) = \sup_{\mathbf{y}\in Y} \mathbf{y}\cdot\mathbf{x} - V(t,\mathbf{y}) \qquad \text{(Legendre transform)}$$

For $U(t,\mathbf{x}), V(t,\mathbf{y})$ smooth, strictly convex, SG, becomes [a]

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0 \qquad \text{(transport)}$$

$$\mathbf{u} = \mathtt{J}(\mathbf{x} - \nabla U) \qquad \text{(velocity)}$$

$$\det(D^2U) = \rho \qquad \text{(Monge-Ampère)}$$

$$\nabla U(t,\mathbf{R}^2) \subset \overline{Y} \qquad \text{(2nd BVP for M-A)}$$

in the time-dependent dual coordinates $\mathbf{x} = \nabla V(t,\mathbf{y})$.

[a] Hoskins. '75, Cullen-Norbury-Purser '91, Benamou-Brenier. '98

Compare with Vorticity for 2D Rotating Euler

Active Scalar with Monge-Ampère equation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \qquad \text{(transport)}$$

$$\mathbf{u} = \mathtt{J}(\mathbf{x} - \nabla U) \qquad \text{(velocity)}$$

$$\det(D^2 U) = \rho \qquad \text{(Monge-Ampère)}$$

Active Scalar with Laplace equation:

$$\frac{\partial \omega}{\partial t} + \operatorname{div}(\omega \mathbf{v}) = 0 \qquad \text{(transport)}$$

$$\mathbf{v} = \mathtt{J}(\mathbf{x} - \nabla V) \qquad \text{(velocity)}$$

$$\Delta V = \omega \qquad \text{(Euler)}$$

Related Works with Linear Potential Eqns

- self-gravitating fluid ellipsoids[a]
- elliptical vorticity patches in a shearing field[b]
- exact solutions for QG[c]

Most of these solutions built around the *quadratic* Newtonian potential generated inside a homogeneous ellipsoid.

Note: Our new solutions are built instead on *quadratic* solutions to the Monge-Ampère equation when the *domain and its gradient image* are both ellipses.

[a] Dirichlet 1860, Dedekind 1860, and Riemann 1860,
[b] Kirchhoff 1876, Chandrasekhar '69, Moore & Saffman '71 and Kida '81
[c] Meacham, Pankratov, Shchepetkin & Zhmur '94

Main Theorem (Dual Vars)

For a fluid constrained to the ellipse $Y = \mathtt{F}(B_1)$, if the potential vorticity forms an elliptical patch $X(t) = \mathtt{E}(t)(B_1)$ in SG dual coordinates at $t = 0$ then it is an elliptical patch at $t > 0$.

There is an explicit ODE for the evolving ellipse, in terms of the physical ellipse. Two variables: orientation and aspect ratio of the potential ellipse. Two parameters: aspect ratio of physical ellipse, and total mass (Rossy number). (below)

Furthermore, the trajectories of the ODE are constrained to the level sets of the function $W_2^2 = W_2^2(X(t), Y)$ equal to the energy in the physical variables.

In addition, making the right choice of variables, the ODE can be expressed as a Hamiltonian dynamical system, with two degrees of freedom, and Hamiltonian W_2^2.

The ODE

Write

$$\rho(t,\mathbf{x}) = \frac{\mathtt{f}^2}{\epsilon^2} 1_{X(t)}(\mathbf{x}), \qquad U(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathtt{U}(t)\mathbf{x}$$

and

$$\mathtt{E}^2 = \epsilon^2 \mathtt{R}_\theta \mathtt{S}_\sigma \mathtt{R}_{-\theta}, \qquad \mathtt{F}^2 = \mathtt{f}^2 \mathtt{S}_\varphi$$

then

$$\mathtt{U}(t) = \frac{1}{Z(t)} \left(\mathtt{E}^{-2}(t) + \frac{\mathtt{F}^2}{\epsilon^2 \mathtt{f}^2} \right).$$

and, for $\sigma(t_0) \neq 0$, and $Z = Z(\sigma, \theta; \varphi)$ (from transport map)

$$\sigma' = -\frac{2}{Z\epsilon^2} \sinh(\varphi) \sin(2\theta)$$

$$\theta' = 1 - \frac{1}{Z\epsilon^2} \left(\cosh\varphi + \sinh(\varphi) \frac{\cosh\sigma}{\sinh\sigma} \cos(2\theta) \right)$$

The Energy = the Transport Cost

The trajectories of the ODE are constrained to the level sets of the function

$$W_2^2 = \frac{1}{2}\left(\epsilon^2 \cosh\sigma + \mathtt{f}^2 \cosh\varphi - Z\epsilon^2\mathtt{f}^2\right) \qquad \text{(Transport Cost)}$$

which is the Wasserstein distance between the two ellipses, and, equal, up to a constant, to the physical energy, H_{SG}.

Canonical coordinates

Changing variables to $r = (\sigma + 1/\sigma)/2$ and reexpressing the energy as a function of (r, θ) and the parameters λ and $s = \cosh\varphi \geq 1$, (parameters for the physical ellipse)

$$\bar{H}(r,\theta) = \lambda^2 s + r - \lambda\left(2 + 2rs + 2\cos(2\theta)\sqrt{(r^2-1)(s^2-1)}\right)^{1/2}$$

converts evolution to an autonomous Hamiltonian system governing the canonically conjugate variables $(r(t), \theta(t))$

$$\begin{aligned} \frac{dr}{dt} &= -\frac{\partial \bar{H}}{\partial \theta} \\ \frac{d\theta}{dt} &= \frac{\partial \bar{H}}{\partial r}. \end{aligned}$$

Strategy for solving

Active scalar w. nonlinear (M-A) equation for velocity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \mathbf{u} = \mathtt{J}(\mathbf{x} - \nabla U), \quad \begin{cases} \det(D^2 U) & = \rho \\ \nabla U(t, \mathbf{R}^2) & \subset \overline{Y} \end{cases}$$

Make quadratic ansatz. Observations (to follow):

- M-A quadratic when target and source ellipses
- Quadratic potential gives linear vector field
- Linear vector field takes ellipses to ellipses.

Difficult calculation. Reduce dimensionality by choose appropriate variables that capture symmetries & conservation laws.

Good choice of variables give Hamiltonian system.

Optimal map between ellipses

Find the optimal map and transportation cost between two ellipses in the plane[a].

This amounts to solving

$$\begin{cases} \det(D^2U) & = \rho \\ \nabla U(t, \mathbf{R}^2) & \subset \overline{Y} \end{cases} \qquad \text{(Monge-Ampère)}$$

when $\rho(\mathbf{x}) = \lambda^2 1_X(\mathbf{x})$, where $X = \mathtt{E}(B_1), Y = \mathtt{F}(B_1)$ are ellipses.

The optimal map $m^{-1}(\mathbf{x})$ is given by

$$m(\mathbf{y}) = \mathtt{F}^{-1}\sqrt{(\mathtt{F}\mathtt{E}^2\mathtt{F})}\mathtt{F}^{-1}\mathbf{y}$$

(Many linear maps, only one symmetric and positive definite.)

[a] Dowson & Landau '82, Givens & Shortt '84 Knott & Smith '84 and Olkin & Pukelsheim '82

Explicit Optimal Map in 2D

Matrix square root is not convenient for our purposes.

The optimal map between the ellipses $1_Y/|Y|$ and $1_X/|X|$ is given by $\mathbf{x} = m(\mathbf{y}) = \mathtt{V}\mathbf{y}$, where $\mathtt{V} = \mathtt{U}^{-1}$ is the positive symmetric matrix inverse to

$$\mathtt{U} := Z^{-1}\left(\mathtt{E}^{-2} + \mathtt{F}^2/\det(\mathtt{EF})\right) \qquad \text{(Explicit Map)}$$

where Z normalizes determinant of $\mathtt{U}$:

$$Z := \operatorname{trace}\sqrt{(\mathtt{FE}^2\mathtt{F})^{-1}}$$

Proof. Since $\mathtt{E} : B_1 \longrightarrow X$ and $\mathtt{F} : B_1 \longrightarrow Y$, where B_1 is the unit ball, and $\mathtt{U}$ is *symmetric*, this is equivalent to showing $\mathtt{F}^{-1}\mathtt{UE} = \mathtt{R}_\theta$ coincides with a rotation $\mathtt{R}_\theta : B_1 \longrightarrow B_1$ by some angle θ.

The rest is an explicit calculation in coordinates. $\square$

Dynamics of the free boundary

Given $\mathbf{u}(t,\mathbf{x}) = \mathtt{W}(t)\mathbf{x}$ on $\mathbf{R}^2$, $\mathtt{W}(t)$ C^1, 2×2 matrix

$$\begin{aligned} \mathbf{X}'(t,\mathbf{x}) &= \mathtt{W}(t)\mathbf{X}(t,\mathbf{x}) \\ \mathbf{X}(0,\mathbf{x}) &= \mathbf{x}. \end{aligned} \qquad \text{(Lagrangian trajectories)}$$

An ellipse $X \subset \mathbf{R}^2$ convected by linear velocity field $\mathbf{u}(\mathbf{x},t) = \mathtt{W}(t)\mathbf{x}$ yields a family of ellipses $\mathbf{X}(t,X) = \{\mathtt{E}(t)z \mid |z| < 1\}$ governed by

$$\frac{d}{dt}\mathtt{E}^{-2} = -\mathtt{W}^T\mathtt{E}^{-2} - \mathtt{E}^{-2}\mathtt{W} \qquad \text{(Ellipse ODE)}$$

Proof. Let $\mathbf{x}(t)$, be a trajectory of the ode, $\mathbf{x}'(t) = \mathtt{W}(t)\mathbf{x}(t)$. Differentiate the equation

$$\mathbf{x}^T\mathtt{E}^{-2}\mathbf{x} = 1.$$

□

Rotation, reflection and shear matrices

Define the rotation matrices through the angle θ,

$$\mathtt{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \mathtt{J} = \mathtt{R}_{\pi/2} \qquad \text{(Rotation Matrices)}$$

reflection matrices (through the line with angle θ),

$$\mathtt{K} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathtt{K}_\theta := \mathtt{R}_\theta \mathtt{K} \mathtt{R}_{-\theta} \qquad \text{(Reflection Matrices)}$$

and the shear matrix,

$$\mathtt{S}_\sigma := \begin{pmatrix} e^{\sigma} & 0 \\ 0 & e^{-\sigma} \end{pmatrix} \qquad \text{(Shear Matrix)}$$

Choice of Basis for Matrices

Make a choice of variables to capture symmetries in the equation.

Write $\mathtt{E}$, symmetric, 2×2 matrix, with determinant ϵ^2 as

$$\mathtt{E} = \epsilon \mathtt{R}_\theta \mathtt{S}_{\sigma/2} \mathtt{R}_{-\theta}. \qquad \text{(Canonical form)}$$

Then, if $\theta = \theta(t)$ and $\sigma = \sigma(t)$ evolve smoothly

$$\frac{d}{dt}\mathtt{E}^{-2} = \theta'(\mathtt{J}\mathtt{E}_t^{-2} - \mathtt{E}_t^{-2}\mathtt{J}) - \sigma'\mathtt{K}_\theta \mathtt{E}_t^{-2}$$

Proof. This follows from the matrix identities

$$\mathtt{R}_\theta \mathtt{K} = \mathtt{K}_\theta \mathtt{R}_\theta, \qquad \frac{d}{d\theta}\mathtt{R}_\theta = \mathtt{J}\mathtt{R}_\theta, \qquad \frac{d}{d\sigma}\mathtt{S}_\sigma = \mathtt{K}\mathtt{S}_\sigma.$$

□

Solving SG with quadratic ansatz

Idea of proof/calculation

$$\frac{d}{dt}\mathtt{E}^{-2} = -\mathtt{W}^T\mathtt{E}^{-2} - \mathtt{E}^{-2}\mathtt{W} \qquad \text{(Ellipse ODE)}$$

$$\mathtt{W}(t) = \mathtt{J}(\mathtt{I} - \mathtt{U}(t)) \qquad \text{(velocity from map)}$$

$$\mathtt{U} := Z^{-1}\left(\mathtt{E}^{-2} + \mathtt{F}^2/\det(\mathtt{EF})\right) \qquad \text{(Map from M-A)}$$

$$\frac{d}{dt}\mathtt{E}^{-2} = \theta'(\mathtt{J}\mathtt{E}_t^{-2} - \mathtt{E}_t^{-2}\mathtt{J}) - \sigma'\mathtt{K}_\theta\mathtt{E}_t^{-2} \qquad (\mathtt{E}\text{ derivative})$$

Classification of the Solutions

Two parameter system: Ro (rotation vs transport), asymmetry of Y. What are possible dynamical behaviors?

- ellipse fixed
- ellipse rotates at constant speed (occurs if Y is a circle)
- ellipse wiggles back and forth (with finite, or infinite period)
- ellipse stretches (can't stretch unboundedly, by cons. law)
- combination of stretching and wiggling/rotation

Classify Solutions: Non-generic: Fixed points. Aperiodic hetero and homo clinic orbits which link saddles at $\pm\sigma_2$ on the vertical axis.

Generic: the aspect ratio $a(t)$ and inclination ($\theta(t) \mod \pi$) of the dual potential vorticity ellipse evolve periodically. The principal axes of this ellipse (and of the isobars) either oscillate about the coordinate axes (*wobbling*) or precess eternally (*rotating pulsations*).

Theorem: Bifurcations of the energy landscape

If $\lambda, \varphi > 0$, $\tilde{H} \in C(\mathbf{R}^2)$ is smooth on $\mathbf{R}^2 \setminus \{\mathbf{0}\}$, has global minima placed symmetrically on the horizontal axis at $(\pm\sigma_-, 0)$, and a topological saddle point at $\sigma = 0$.

These are the only critical points unless $\lambda > 2$ and $\varphi \leq \varphi_{cr}(\lambda)$, where $\sinh \frac{\varphi_{cr}(\lambda)}{2} = \frac{1}{\lambda\sqrt{2}} \left(-1 + \sqrt{\frac{\lambda^2 - 1}{3}} \right)^{3/2}$.

If $\varphi < \varphi_{cr}(\lambda)$ there are two additional non-degenerate critical points $(\pm\sigma_+, \pi/2)$ and $(\pm\sigma_2, \pi/2)$ on the vertical axis: a local maximum at $\sigma_+ > 0$ and a saddle point at $\sigma_2 > \sigma_+$ these coalesce into a single degenerate critical inflection point $\sigma_+ = \sigma_2 > 0$ at the bifurcation eccentricity $\varphi = \varphi_{cr}(\lambda)$. The critical points are the solutions to

$$\sinh \sigma_\pm = \lambda \sinh \left(\frac{\sigma_\pm \mp \varphi}{2} \right)$$

and the corresponding critical values are ordered by

$$\begin{array}{rcccl} \tilde{H}(\sigma_-,0) =: h_- & < & h_1 := \tilde{H}(0,0) & < & h_+ := \tilde{H}(\sigma_+,\pi/2) \\ h_- & < & h_2 := \tilde{H}(\sigma_2,\pi/2) & < & h_+. \end{array}$$

There are no further critical points. A continuous increasing curve $\varphi_{bi} :]2,\infty[\longrightarrow]0,\infty[$ strictly less that $\varphi_{cr}(\lambda)$ separates the region $0 < \varphi < \varphi_{bi}(\lambda)$ of the parameter space where $h_1 > h_2$ from the region $\varphi_{bi}(\lambda) < \varphi < \varphi_{cr}(\lambda)$ where $h_1 < h_2$. The curve $\lambda \longrightarrow (\varphi_{bi}(\lambda), \sigma_2(\lambda, \varphi_{bi}(\lambda)))$ simultaneously solves

$$\cosh(\frac{\sigma-\varphi}{2}) = \cosh\frac{\varphi}{2} + \frac{\cosh\sigma - 1}{2\lambda}.$$

If $\varphi \neq \varphi_{bi}(\lambda)$, one connected component of the level set $\{(\sigma\cos\theta, \sigma\sin\theta) \in \mathbf{R}^2 \mid \tilde{H}(\sigma,\theta) = h_1\}$ consists of a smoothly immersed figure-eight curve with orthogonal self-intersection at the origin.

Show pictures

- Bifurcation Diagram for topology of H

- level sets of H in each of the 5 parameter regimes. Which depend on size λ and eccentricity, ϕ of the physical domain.

In the figures, we fix $\lambda = 3$, increase ϕ, to pass through the regions.

Symmetry breaking bifurcation:

optimal transport problem

Transportation problem: *Among all ellipses $X \subset \mathbf{R}^2$ of fixed area π/λ^2, find the one nearest the unit disc $Y = B_1$ in Wasserstein distance.*

If $\lambda \leq 2$, solution: a circle $X = B_{1/\lambda}$,

If $\lambda > 2$, (meaning X is less than half as big as the unit disc), more efficient to stretch the ellipse out so that some of the particles near the boundary of B_1 need not be transported so far.

Solution: aspect ratio $a_- := \exp[2\cosh^{-1}(\lambda/2)]$, arbitrary orientation θ;

Note: If Y ellipse, instead of the unit ball, get unique solution. (orientation must be same, by theorem).

Back to physical variables

By performing the Legendre transform of a quadratic function, recover the pressure

$$\mathtt{P}(t) = \frac{1}{\lambda z(t)}(\mathtt{S}_{-\varphi} + \mathtt{R}_{\theta(t)}\mathtt{S}_{\log\sigma(t)}\mathtt{R}_{-\theta(t)}) - \mathtt{I}.$$

where we have the ode's for the variables. φ, σ.

End