Economics of Matching & Optimal Transport

Lecture 2: Markets with Money

Prepared as a Tutorial for IPAM

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Economic Version of Transportation Problem

- (X, μ),(Y, ν) finite measure spaces
- X, Y compact metric
- $\Pi : X \times Y \to \mathbb{R}_+$ continuous

Find $F : X \to Y$ (measure preserving) to maximize

$$\Gamma(F) = \int_X \Pi(x, F(x)) d\mu(x)$$

Interpretation

- X = workers, Y = firms
- $\Pi(x, y)$ = value of output if x works for y
- Γ = social gain

Problem of benevolent social planner

Expand/Recast to take account of

- unequal numbers of workers, firms
  - involuntary unemployment
  - firms not operating
- voluntary unemployment
- no optimal F

Recast

- (X, μ),(Y, ν) finite measure spaces

- X, Y compact metric; μ, ν full support

- $\Pi : X \times Y \to \mathbb{R}_+$ continuous

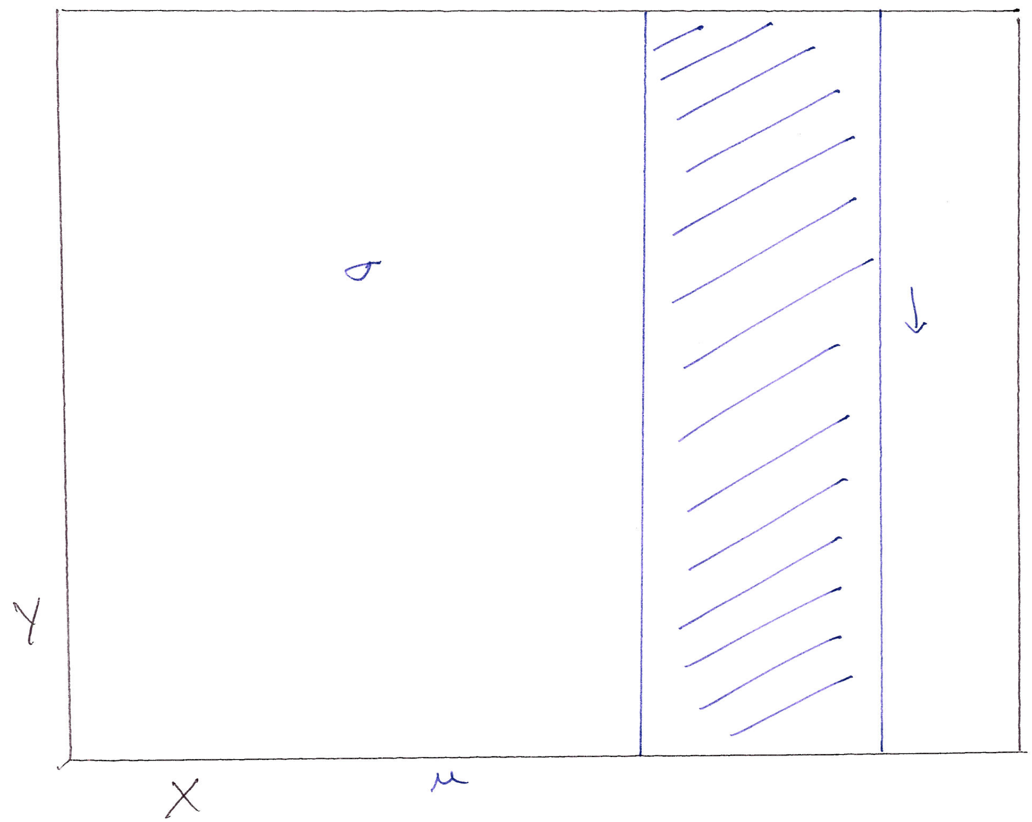
- $\rho : X \to \mathbb{R}_+$ continuous

- $V(x, y) = \Pi(x, y) - \rho(x)$

Fix population measures μ, ν.

A **matching** is a measure $\sigma \in M_+(X \times Y)$ such that

$$\sigma_X \leq \mu|X \ , \sigma_Y \leq \mu|Y$$


σ
Y
X
μ

Problem: Find matching $\sigma \in M(X \times Y)$ to maximize

$$\Gamma(\sigma) = \int_{X \times Y} V(x,y) d\sigma(x,y)$$

Comments

- allows for unemployment, unfilled jobs
- avoids problem that maximal F might not exist
- allows "fractional matchings" –

  but if X, Y finite; μ, ν counting

  – matchings $\leftrightarrow$ doubly stochastic matrices

  – optimal matchings compact convex set

  – extreme optimal matchings are integral matchings

This is a Linear Programming problem

Primal Problem

Find $\sigma \in M_+(X \times Y)$ to maximize

$$\Gamma(\sigma) = \int_{X\times Y} V(x,y)d\sigma(x,y)$$

subject to $\sigma_X \leq \mu|X$, $\sigma_Y \leq \mu|Y$

Dual Problem

Find $q \in C_+(X \cup Y)$ to minimize

$$\gamma(q) = \int_X q(x)d\mu(x) + \int_Y q(y)d\mu(y)$$

subject to $q(x) + q(y) \geq V(x,y)$

Caution

- Duality: $\langle M, C \rangle$

- Abstract LP theory **does not work in this duality**

  Fundamental Theorem of Linear Programming may not hold

Theorem

(i) Primal problem has solutions $\mathcal{M}(\mu,\nu) \subset M_+(X \times Y)$

weak-* compact, convex

(ii) Dual problem has solutions $Q(\mu,\nu) \subset C_+(X \cup Y)$

norm compact, convex

(iii) Primal/Dual solutions have same value function

$$\begin{aligned} g(\mu,\sigma) &= \max_\sigma \left[\int_{X\times Y} V(x,y)d\sigma(x,y)\right] \\ &= \min_q \left[\int_X q(x)d\mu(x) + \int_Y q(y)d\mu(y)\right] \end{aligned}$$

Theorem

(i) g is weak* continuous, norm Lipschitz,

concave, homogeneous of degree 1

(ii) g is subdifferentiable

(iii) $\partial g(\mu, \nu) = Q(\mu, \nu)$

Can partially order $Q(\mu, \nu)$

- X-ordering:

$$q \geq_X q' \Leftrightarrow q(x) \geq q'(x) \text{ for all } x \in X$$

- Y-ordering:

$$q \leq_Y q' \Leftrightarrow q(y) \leq q'(y) \text{ for all } y \in Y$$

Theorem

(i) For $q, q' \in Q(\mu, \nu)$:

$$q \geq_X q' \Leftrightarrow q \leq_Y q'$$

(ii) $Q(\mu, \nu)$ is a complete lattice

Comments

- Social perspective: planner's problem
- Individualistic perspective
- Market perspective

Individualistic perspective

Matching σ is **stable** if there exist continuous

$$w_\sigma : X \to \mathbb{R}_+ \ , \ \mathrm{r}_\sigma : \mathrm{Y} \to \mathbb{R}_+$$

such that

- $\sigma_X\{x : w_\sigma(x) < \rho(x)\} = 0$
- $\sigma\{(x, y) : w_\sigma(x) + r_\sigma(y) \neq \Pi(x, y)\} = 0$
- there do not exist x_0, y_0 such that $w_\sigma(x_0) > \rho(x_0)$ and

$$w_\sigma(x_0) + r_\sigma(y_0) < \Pi(x_0, y_0)$$

Market perspective

Equilibrium: wage $w : X \to \mathbb{R}_+$, residual $r : Y \to \mathbb{R}_+$,

matching σ such that

- $\sigma_X\{x : w(x) < \rho(x)\} = 0$
- $\sigma_X\{x : w(x) > \rho(x)\} = \mu\{x : w_\sigma(x) > \rho(x)\}$
- $\sigma_Y\{y : r(y) < \sup_x[\Pi(x, y) - w(x)]\} = 0$
- $\sigma\{(x, y) : w(x) + r(y) \neq \Pi(x, y)\} = 0$

Theorem

Solutions to planner's problem

$$\Updownarrow$$

Stable matchings

$$\Updownarrow$$

Market equilibria

Where do wages/residuals come from?

- population (μ, ν)
- σ stable (optimal) matching for (μ, ν)
- $q \in \partial g(\mu, \nu)$
- $(x, y) \in \text{support}(\sigma)$

$\Rightarrow w(x) = q(x) + \rho(x), r(y) = q(y)$

Example

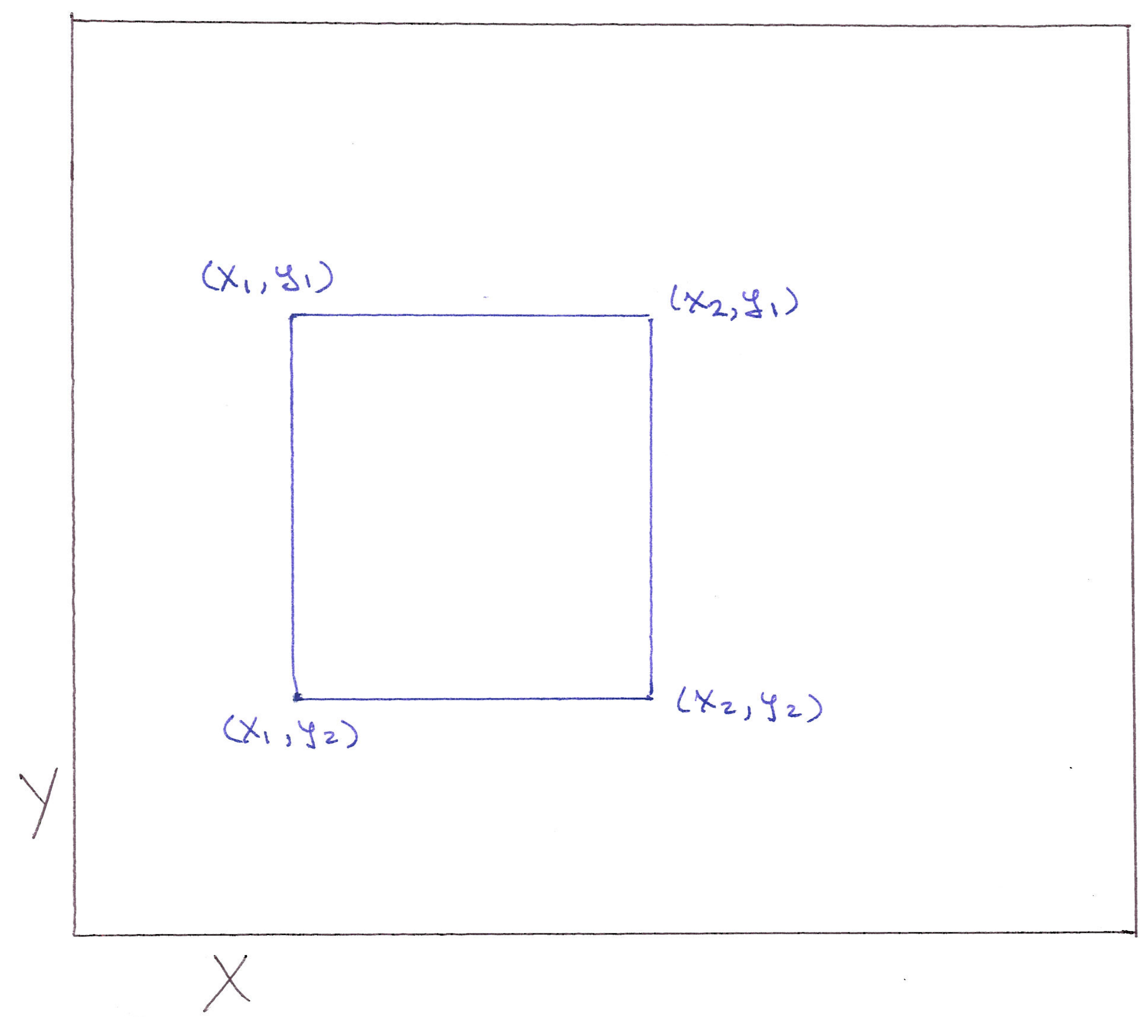
- $X = Y = [0,1]$
- $\mu = \nu = \lambda$
- $\Pi(x,y) = xy + \beta$, $\beta \geq 0$
- $\rho \equiv 0$

Important: Π is supermodular

$$\frac{\partial^2 \Pi}{\partial x \partial y} > 0$$

Π supermodular $\Rightarrow$ stable matching is assortative:

higher x matched with higher y


(X1, Y1)
(X2, Y1)
(X1, Y2)
(X2, Y2)
Y
X

Assortative matching

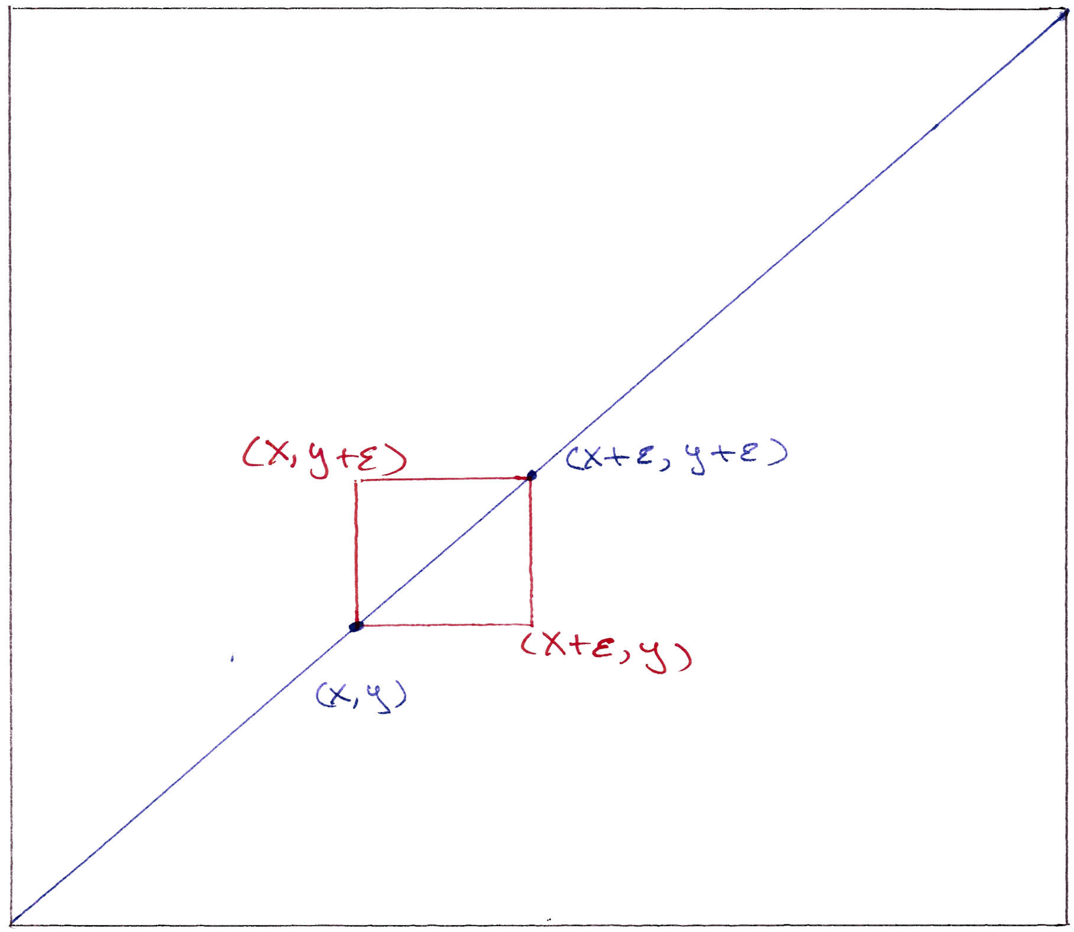
$$+ \quad \mu = \nu = \lambda$$

$$+ \quad \lambda(X) = \lambda(Y) = 1$$

$\Longrightarrow$ matching is diagonal

Diagonal matching + subdifferential inequality

$\rightarrow$ differential equation for $w =$ wages



(X, y+ε)
(x+ε, y+ε)
(X+ε, y)
(x, y)

Subdifferential inequality $\rightarrow$

$$\begin{aligned} w(x) + r(y) &= \Pi(x,y) \\ w(x+\varepsilon) + r(y) &\geq \Pi(x+\varepsilon,y) \\ w(x+\varepsilon) + \Pi(x,y) - w(x) &\geq \Pi(x+\varepsilon,y) \\ w(x+\varepsilon) - w(x) &\geq \Pi(x+\varepsilon,y) - \Pi(x,y) \end{aligned}$$

Similarly

$$\begin{aligned} w(x+\varepsilon) + r(y+\varepsilon) &= \Pi(x+\varepsilon,y+\varepsilon) \\ w(x) + r(y+\varepsilon) &\geq \Pi(x,y+\varepsilon) \\ w(x+\varepsilon) + \Pi(x+\varepsilon,y+\varepsilon) - w(x+\varepsilon) &\geq \Pi(x+\varepsilon,y) \\ w(x+\varepsilon) - w(x) &\leq \Pi(x+\varepsilon,y+\varepsilon) - \Pi(x,y+\varepsilon) \end{aligned}$$

Hence

$$\Pi(x+\varepsilon,y)-\Pi(x,y) \leq w(x+\varepsilon)-w(x) \leq \Pi(x+\varepsilon,y+\varepsilon)-\Pi(x,y+\varepsilon)$$

Divide by ε, send $\varepsilon \to 0$, remember that $x = y$, Π smooth

$$\Rightarrow \quad w'(x) = \frac{\partial \Pi}{\partial x}(x,x) = x$$

$$\Rightarrow \quad w(x) = \frac{x^2}{2} + C$$

What is C ?

Recall $\Pi(x, y) = xy + \beta$

- $\beta = 0 \Rightarrow w(0) = 0 \Rightarrow C = 0$

  determinate

- $\beta > 0 \Rightarrow 0 \leq w(0) \leq \beta \Rightarrow 0 \leq C \leq \beta$

  indeterminate – but $w(0)$ determines whole wage structure

Manipulation: how?

- Workers: misrepresent ρ

- Firms: misprepresent Π

Manipulation: by whom?

- finite case: individuals or groups on one side

- infinite case: infinitesimal subsets (proxy individuals)

  large groups can always manipulate (with transfers)

If $\beta > 0$

- $C < \beta \rightarrow$ low quality workers manipulate

  claim to have reservation values $= \beta$

- $C > 0 \rightarrow$ low quality firms manipulate

  claim to have $\Pi(x, y) = xy$ (higher cost)

No manipulation $\leftrightarrow \beta = 0 \leftrightarrow \{w\}$ is a singleton

Theorem

No manipulation

$$\Updownarrow$$

$\partial g(\mu, \nu)$ is a singleton

$$\Updownarrow$$

g is Gateaux differentiable

$$\Updownarrow$$

g is Frechet differentiable

Theorem g is generically differentiable

Real economy is finite but large $\rightarrow$ asymptotics?

Theorem

- $\partial g : M_+(X \cup Y) \rightarrow C(X \cup Y)$ is USC

- ∂g is continuous at (μ, ν) if $g(\mu, \nu) =$ singleton

- $(\mu_n, \nu_n) \rightarrow (\mu, \nu)$, $\partial g(\mu, \nu)$ is a singleton

  $\Rightarrow \quad \partial g(\mu_n, \nu_n)$ is small for n large

  $\Rightarrow \quad$ for n large: no one can manipulate very much