Economics of Matching & Optimal Transport

Prepared as a Tutorial for IPAM

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Applications of ideas of Optimal Transport

- labor markets
- commodity markets

with and without money

Plan of two lectures

- history of a particular labor market without money
- what is the connection to optimal transport?
- structure of a market without money
  - stability, incentives
- structure of a market with money
  - stability, duality, incentives
- generalizations

Economic History: Medical Internships

- organization of medical training
  - formal classes: medical school
  - apprenticeships: internships/residencies
- decentralized market for residencies
- problems with the decentralized market
- success of centralization
- failure of centralization

Questions

1. Why was the market unstable early?

2. Why did the market become stable?

3. Why did the market become unstable again later?

4. What has any of this to do with optimal transport?

Classical Transportation Problem

Given

- $X, Y \subset \mathbb{R}^n$ with $\lambda(X) = \lambda(Y)$
- function $C : X \times Y \to \mathbb{R}$

Find $F : X \to Y$ (measure preserving) to minimize

$$\int_X C(x, F(x))\, d\lambda(x)$$

Mathematical focus: geometry of F

Economic Interpretation

- X = produced output of a firm, Y = customers
- Problem of a central manager of firm

Alternative Economic Version

- (X, μ), (Y, ν) abstract measure spaces
- $\mu(X) = \nu(Y)$
- function $\Pi : X \times Y \to \mathbb{R}$

Find $F : X \to Y$ (measure preserving) to maximize

$$\int_X \Pi(x, F(x)) d\mu(x)$$

Interpretation

- X = workers, Y = jobs/firms

  (each firm hires single worker)

- $\Pi(x, y)$ = profit generated if worker x occupies job y

  (works for firm y)

Problem of benevolent social planner

Of special interest

- X, Y finite
- μ, ν counting measure

F and F^{-1} **match** workers and jobs

Small issues

- $\mu(X) \neq \nu(Y)$?
  - unemployed workers
  - unfilled jobs
- workers prefer unemployment
- firms prefer not to operate

More careful about unemployed workers, unfilled jobs

Matching $M : X \cup Y \to X \cup Y$

- $M^2 =$ identity
- $M(x) \in Y \cup \{x\}$
- $M(y) \in X \cup \{y\}$

Convention : $\Pi(y, y) = 0$

Perspectives on matching problem

1. Social perspective: planner's problem

2. Individual perspective

3. Market perspective

Individual perspective

Decompose joint profit: worker wage, firm residual

$$\Pi(x, M(x)) = w_M(x) + r_M(M(x))$$

Matching M is **stable** if there exists a match-specific decomposition $w_M, r_M \geq 0$ and there does NOT exist $x_0 \in X, y_0 \in Y$:

$$M(x_0) \neq y_0 \text{ and } w_M(x_0) + r_M(y_0) < \Pi(x_0, y_0)$$

Market perspective

Worker/Job-specific wages $W : X \cup Y \to \mathbb{R}_+$

Matching M, wages W are **market equilibrium** if for all x_0, y_0:

$$y = M(x_0) \quad \text{maximizes} \quad W(x_0, y)$$

$$x = M(y_0) \quad \text{maximizes} \quad \Pi(x, y_0) - W(x, y_0)$$

(with obvious adjustments for unmatched workers, firms)

Theorem

Socially optimal matchings

$$\Updownarrow$$

Stable matchings

$$\Updownarrow$$

Market equilibrium matchings

Corollary (Adam Smith) The market is efficient.

Comment

- Social optimality makes no sense without money
- Market equilibrium makes no sense without money
- Stability does

Matching without money (ordinal problem)

- each $x \in X$: $\succ_x$ on $Y \cup \{x\}$

  (complete transitive strict) preference of worker x over jobs

  (or unemployment)

- each $y \in Y$: $\succ_y$ on $X \cup \{y\}$

  (complete transitive strict) preference of job y over workers

  (or not operating)

Matching M is **stable** if

$$z \in X \cup Y \Rightarrow M(z) \succeq_z z$$

and there do not exist $x_0 \in X, y_0 \in Y$ such that

- $M(x_0) \neq y_0$
- $y_0 \succ_{x_0} M(x_0)$
- $x_0 \succ_{y_0} M(y_0)$

That is: x_0, y_0 would prefer to be matched to each other rather than to their mates under F

M **not** stable $\rightarrow$ after-match unraveling

Group (partial) preferences over matchings:

$$M \succeq_X M' \;\;\Leftrightarrow\;\; M(x) \succeq_x M'(x) \text{ all } x \in X$$

$$M \preceq_Y M \;\;\Leftrightarrow\;\; M(y) \preceq_y M'(y) \text{ all } y \in Y$$

These are transitive relations.

Theorem

(i) Stable matchings exist

(ii) $M \succeq_X M' \;\Leftrightarrow\; M \preceq_Y M'$

(iii) Stable matchings form a lattice

(iv) for all M, M':

$$\begin{aligned}
\{x \in X : M(x) = x\} &= \{x \in X : M'(x) = x\} \\
\{x \in Y : M(y) = y\} &= \{y \in Y : M'(y) = y\}
\end{aligned}$$

Finding stable matches ?

- NIMP algorithm
- Gale-Shapley algorithm: deferred acceptance
  - each x proposes to favorite y
  - each y holds favorite proposer, rejects others
  - rejected x's propose to next favorite
  - each y holds favorite proposer, rejects others
  - repeat
  - stop when no rejections: X-optimal matching

Questions

1. Why was the market unstable?

   - individual behavior with time limits $\not\rightarrow$ stable matchings

2. Why did the market become stable?

   - NIMP algorithm $\rightarrow$ stable matchings

What if workers/firms lie?

$\mathcal{P}_X$ = all preference profiles for Workers

$\mathcal{P}_Y$ = all preference profiles for Firms

$\mathcal{M}$ = set of matchings

Mechanism $\mu : \mathcal{P}_X \times \mathcal{P}_Y \to \mathcal{M}$

μ is **stable mechanism** if for all $P_X \in \mathcal{P}_X$,

$$P_X \in \mathcal{P}_X, P_Y \in \mathcal{P}_Y \Rightarrow \mu(P_X, P_Y) \text{ is stable for } P_X, P_Y$$

Theorem The mechanism

$$(P_X, P_Y) \mapsto X\text{-optimal stable matching}$$

can never be manipulated by workers and is the only such.

Theorem The mechanism

$$(P_X, P_Y) \mapsto Y\text{-optimal stable matching}$$

can never be manipulated by firms and is the only such.

Theorem If the set of stable matchings for (P_X, P_Y) is not a singleton then every mechanism can be manipulated (by workers or firms or both).

What if workers/firms care about whole match?

Stable matchings may not exist.

Questions

3. Why did the market become unstable again?
   - because some interns were married to other interns