Robust FEM-based extraction of finite-time coherent sets using scattered, sparse, and incomplete trajectories

Oliver Junge
Department of Mathematics
TUM

joint work with Gary Froyland, UNSW
FINITE TIME COHERENT SETS

\[ \dot{x} = \omega(t, x) \]
FINITE TIME COHERENT SETS

\[ \dot{x} = \sigma(t, x) , \quad \phi_t^{t_0} \text{ flow} \]

coherent sets

non-coherent set
THE ISOPERIMETRIC PROBLEM

Find a set $A \subset \mathbb{R}^2$ with

$$|\partial A| = \text{fixed}$$

such that $|A| = \max$ $\iff \frac{|\partial A|}{|A|} = \min$

solution: $A = \text{disk}$
THE STATIC CASE [Federer, Fleming, 60]

\[
\inf_A \frac{|\partial A|}{\min \{ |A|, |A^c| \}} = \inf_u \frac{\|\nabla u\|_1}{\inf_a \|u - a\|_1} \approx \min_{u \perp 1} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} = \lambda_2 \text{ of } \Delta u = \lambda u + \text{b.c.}
\]
DYNAMIC ISOPERIMETRY

[Froyland, 15]

\[ A^t = \varphi_{t_0, t}^t(A) \]

coherent set: \[ \frac{1}{T} \int_0^T \frac{|\partial A^t|}{|A^t|} \, dt = \min \]
THE DYNAMIC CASE

\[ |\partial A| \hat{=} \| \nabla u \|_2^2 \]

\[ |\partial A^+| \hat{=} \| \nabla (\varphi^{t_{t_0}} u) \|_2^2 \]

with the transfer operator

\[ \varphi^{t_{t_0}}_{t_0} : u \mapsto u^\prime = u \circ \varphi^{t_{t_0}} \]

(for volume preserving flow maps)
DYNAMIC LAPLACE EIGENPROBLEM

\[ \min \frac{1}{2} \left( |\eta A|_1 + |\eta A^t|_1 \right) \]

\[ \approx \min_{\eta \perp 1} \frac{1}{2} \left( \| \nabla u \|^2_2 + \| \nabla \phi_x^t u \|^2_2 \right) \| u \|^2_2 \]

\[ = \lambda_2 \text{ of } \frac{1}{2} \left( \Delta + \phi_x^t \Delta \phi_x^t \right) u = \lambda u \]

\[ =: \Delta^d \text{ dynamic Laplace} \]

\[ \Leftrightarrow \Delta g(t) u = \lambda u \quad \text{[Karrasch, Keller, 16]} \]
\[ \Delta^d u = \lambda u + \text{ Neumann or Dirichlet } \text{ b.c.} \]

\[ \downarrow \text{ weak form} \]

\[ a(u,v) = \lambda \langle u,v \rangle_{L^2} \quad \text{for all } v \in H^1 \text{ or } H_0^1 \]

\[ \downarrow \text{ Ritz-} \text{Galerkin} \]

\[ a(u,v) = \lambda \langle u,v \rangle_{L^2} \quad \text{for all } v \in V_N, \dim < \infty \]

\[ \downarrow \text{ choose basis} \]

\[ A u = \lambda M u \]
FINITE ELEMENTS

\[ A_{ij} = \frac{1}{T} \sum_{t} \int_{\varphi^t(M)} \nabla(\varphi^t \psi_i) \cdot \nabla(\varphi^t \psi_j) \, dm \]
Computing $A_{ij}$

\[
\int_{\varphi^t(M)} \nabla (\varphi^*_t \nu_i) \cdot \nabla (\varphi^*_t \nu_j) \: dm
\]

\[
= \int_{M} \nabla (\varphi^*_t \nu_i) \circ \varphi^t \cdot \nabla (\varphi^*_t \nu_j) \circ \varphi^t \: dm
\]

\[
= \int_{M} (D\varphi^t)^{-T} \nabla \nu_i \cdot (D\varphi^t)^{-T} \nabla \nu_j \: dm
\]

\[
= \int_{M} \nabla \nu_i^T C_t^{-1} \nabla \nu_j \: dm
\]
COMPUTING $A_{ij}$

i.e.

$$A_{ij} = \int_{M} \nu \nu_{i}^T \vec{C}^{-1} \nu \nu_{j} \, dm$$

with

$$C = \frac{1}{t} \sum_{t} \Phi^{t} (\Phi^{t})^{T}$$

→ standard FEM quadrature

→ $\nu \nu_{i}$ constant for linear elements

→ often, low order quad suffices
A ROTATING DOUBLE-GYRE
A ROTATING DOUBLE-GYRE

625 nodes
integration: 2 s
assembly: 20 ms
eigenproblem: 50 ms
A ROTATING DOUBLE-GYRE

625 scattered nodes

20000 nodes

200 nodes
The Bickley Jet
The Bickley Jet

\[ \lambda_k \]

\[ (x, y) \]

\[ \text{gap} \]
EXAMPLE: OCEAN FLOW

AVISO data / domain of Agulhas leakage parameters from [Itadjighasem, Haller, 16]
EXAMPLE: OCEAN FLOW

here: zero Dirichlet boundary condition
OCEAN FLOW: COHERENT SETS

obtained by k-means with $u_1, u_2, u_3$
OCEAN FLOW: COHERENT SETS
3D EXAMPLE: UNSTEADY ABC-FLOW

\[ \begin{align*}
\dot{x} &= (A + \frac{1}{2} t \sin(\pi t)) \sin z + C \cos y \\
\dot{y} &= B \sin x + (A + \frac{1}{2} t \sin(\pi t)) \cos z \\
\dot{z} &= C \sin y + B \cos x
\end{align*} \]
3D EXAMPLE: RAYLEIGH–BÉNARD CONVECTION
3D EXAMPLE: RAYLEIGH–BÉNARD CONVECTION

data by J. Schumacher, U Ilmenau
CONVERGENCE ORDER

A true eigenvalue \( \lambda \), an approximation \( \overline{\lambda} \)

A true eigenvector \( \overline{u} \), \( \overline{v}_h \)

If \( u \in H^{s+1} \), then, using \( P^s \) Lagrange elements

\[
\left| \frac{\lambda - \overline{\lambda} h}{\lambda} \right| = O(h^{2s})
\]

\[
\| u - \overline{v}_h \|_{H^1} = O(h^s)
\]

Under certain conditions

\[
\| u - \overline{v}_h \|_{L^2} = O(h^{s+1})
\]
Figure 1: Standard map: Errors in the first nontrivial eigenvalue (left) and corresponding 1-dimensional eigenspace (right) of the dynamic Laplace operator. Slope of corresponding line in brackets.

Figure 2: Errors in numerical approximation of first nontrivial eigenvalue (left) of the dynamic Laplace and corresponding eigenspace (right) for rotating double gyre to $t = 1.0$.

Slope of corresponding line in brackets.

1.2 Rotating Double Gyre

As a second example, we consider the flow map introduced in [11]. This is a Hamiltonian system that produces two rotating gyres. The Hamiltonian $H$ is derived from a streamfunction given by

$$ (x, y, t) = s(t) P(x, y) + s(t) F(x, y) $$

with

$$ P(x, y) = \sin(2\pi x) \sin(\pi y) $$

$$ F(x, y) = \sin(\pi x) \sin(2\pi y) $$

$$ s(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^2 (3 - 2t) & \text{for } t \in [0, 1] \\ 1 & \text{for } t > 1 \end{cases} $$

Test influence of quadrature order here

The reference solution was computed on a regular mesh on 513 $\times$ 513 nodes with triangular $P^2$-Lagrange elements. We used quadrature order 5 (since we did not observe the same rates for smaller orders). For the eigenvalue error, we do not observe a clear agreement of the experimentally obtained order with that predicted in (4). On the other hand, we do get a reasonable agreement with (5) for the eigenfunction. As mentioned above, this may be due to a lack of regularity of the eigenfunctions.
TEST: BICKLEY JET

$P^1$
6000 triangles
18000 quad points

$P^2$
240 triangles
720 quad points
TEST: OCEAN FLOW

\( P^1 \)
7000 elements
20,000 quad points

\( P^2 \)
1000 elements
3000 quad points
Computing $A$

by approximating the transfer operator:

$$
\phi^t \approx \sum_k \alpha_{jk} v_k^t
$$

↑

basis at $t=0$

↑

basis at $t$

Related:

[Froyland, Padberg, 15]

[Hadjighasem et al, 16]

[Banisch, Koltai, 17]
COMPUTING $A$

by approximating the transfer operator using collocation:

$$q_\ast^t v_j^0 (x_i^t) = \sum_k \alpha_{jk} v_k^t (x_i^t) \quad \forall i$$

\[\text{nodes}\]

$$\sum_k \alpha_{jk} \delta_{ki}$$

$$v_j^0 (\varphi^{-t} (x_i^t)) = \alpha_{ji}$$
Computing A by approximating the transfer operator using collocation

**Special case:** choose \( x^t_i := \varphi^t(x_i) \)

Then \( a_{ji} = v_j^0 (\bar{\varphi}^t (\varphi^t(x_i))) \)

\[ = v_j^0 (x_i) \]

\[ = \delta_{ji}, \quad \text{(nodal basis again)} \]

\[ \therefore A = I \quad \text{in these bases} \]
ROTATING DOUBLE GYRE

diffusion tensor

\sim 2 \text{ secs}

transfer operator

\sim 0.2 \text{ secs}
MISSING DATA

For each time $t_k$:
- triangulate remaining points
- compute $A_M$ as usual

Trade missing points for more timesteps.
Rotating Double Gyre

625 points
2 time steps

~ 250 points
5 time steps
BICKLEY JET

3000 nodes
2 time steps

600 nodes
10 time steps
CONCLUSION

+ works on data (scattered, sparse)
+ robust (data can be incomplete)
+ yields info on entire domain
+ preserves symmetry
+ no free parameter
+ no artificial noise
+ standard convergence theory

• higher order elements $\rightarrow$ significant mesh size reduction ($\sim 5x$)
REFERENCES

Froyland, Junge:  
Robust FEM-based extraction of finite-time coherent sets using scattered, sparse, and incomplete trajectories, SIADS, 2018.


**CoherentStructures.jl**, Julia package, work in progress.

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[DFG](https://www.dfg.de)  
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