The realizability problem for point processes: explicit constructions on the lattice.

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Operator Theoretic Methods in Dynamic Data Analysis and Control

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Outline

Motivations: analysis of complex systems

Nervous system in which neural spikes travel

Fluid composed of molecules

Plants growing in a certain region

Dealing with MANY-BODY SYSTEMS

Galaxy composed of stars

STRATEGY:

- focus on few characteristics of the system
- predict the selected characteristics

OUTPUT: Putative characteristics of the system

REALIZABILITY PROBLEM

Does there exist a real state of the original system which realizes the putative characteristics predicted?

General Framework: point processes on $\mathbb{R}^d$

Let $d \in \mathbb{N}$.

- **Point configuration** $\rightarrow \gamma \subset \mathbb{R}^d$: $\begin{cases} |\gamma \cap \Lambda| < \infty, \; \forall\, \Lambda \subset \mathbb{R}^d \text{ bounded} \\ |\gamma \cap \{\mathbf{x}\}| \in \{0,1\}, \; \forall\, \mathbf{x} \in \mathbb{R}^d. \end{cases}$

  Point configurations correspond to counting measures on $\mathbb{R}^d$ via:

  $$\gamma := (x_i)_{i\in I} \leftrightarrow \gamma(d\mathbf{r}) := \sum_{i\in I} \delta_{\mathbf{x}_i}(d\mathbf{r})$$

  where δ is the Dirac measure, $\mathbf{x}_i$ are distinct points in $\mathbb{R}^d$, either $I \subset \mathbb{N}$ is finite or $I = \mathbb{N}$ and $(x_i)_{i\in I}$ has no accumulation points.

- $\Gamma(\mathbb{R}^d)$:= set of all possible point configurations on $\mathbb{R}^d$.

  $$\Gamma(\mathbb{R}^d) \subset \mathcal{R}(\mathbb{R}^d)$$

  where $\mathcal{R}(\mathbb{R}^d)$ is the set of all Radon measures with the vague topology.

- **Point process μ on $\mathbb{R}^d$** := a probability measure on $\Gamma(\mathbb{R}^d)$.

Point configurations and complex systems

Points $\mathbf{x}_i$ of a configuration $\gamma \leftrightarrow$ Positions of the components of the system.

For example, x_i may represent the location of:

- molecules in a fluid
- stars in galaxies
- trains of neural spikes
- trees of a plant population

References

- Heterogenous materials: F. H. Stillinger, S. Torquato
- Fluid theory: J. K. Percus, C. Garrod
- Quantum chemistry: P. O. Lödwin, A. J. Coleman, J. K. Percus
- Spatial ecology: B. Bolker, R. Law, H. Metz

Low-order factorial moment measures (correlation functions)

Low-order factorial moment measures
- contain the most interesting properties
- are the most accessible to experiments

Let $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^3$, then the first two ***factorial moment measures*** can be defined as

- **one-particle density** $\rightarrow \rho_1^{\mu}(d\mathbf{r}_1) := \mathbb{E}_{\mu}(\gamma(d\mathbf{r}_1))$,
- **pair density** $\rightarrow \rho_2^{\mu}(d\mathbf{r}_1, d\mathbf{r}_2) := \mathbb{E}_{\mu}(\gamma^{\odot 2}(d\mathbf{r}_1, d\mathbf{r}_2))$,

Factorial n**−th power** of $\gamma = \sum_{i \in I} \delta_{x_i}$ $\rightarrow$ $\boxed{\gamma^{\odot n}(d\mathbf{r}_1, \ldots, d\mathbf{r}_n) := \sum_{i_1 \neq \ldots \neq i_n} \prod_{h=1}^{n} \delta_{x_{i_h}}(d\mathbf{r}_h)}$

In physical applications the factorial moment measures are usually known as ***correlation functions***

Realizability problem

The process μ is often unknown whereas the correlation functions can be derived via

- experimental measurements
- approximation schemes
- encoding of expected system properties
- statistical estimations

→ Correlation functions

Realizability Problem on $\mathcal{S} \subseteq \Gamma(\mathbb{R}^d)$ for correlations

Given a sequence $(\rho^{(n)})_{n=0}^N$ of symmetric Radon measures on $\mathbb{R}^{dn}$, find a point process μ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$\rho^{(n)} = \rho_\mu^{(n)} \quad \text{i.e.}$$

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$$\rho^{(n)} = \mathbb{E}_\mu(\gamma^{\odot n}) \quad \text{i.e.}$$

$$\langle f^{(n)}, \rho^{(n)} \rangle = \int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\odot n} \rangle \mu(d\gamma), \quad \forall f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{dn})$$

Analogy with the classical moment problem

Let $N \in \mathbb{N} \cup \{\infty\}$.

Moment Problem on $K \subseteq \mathbb{R}$

Given a sequence $(m_n)_{n=0}^N$ with $m_n \in \mathbb{R}$, find a finite nonnegative Borel measure μ concentrated on K s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \int_K x^n \mu(dx).$$

Realizability Problem on $\mathcal{S} \subseteq \Gamma(\mathbb{R}^d)$

Given a sequence $(\rho^{(n)})_{n=0}^N$ with $\rho^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})$ symmetric, find a point process μ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$\langle f^{(n)}, \rho^{(n)} \rangle = \int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\odot n} \rangle \mu(d\gamma), \quad \forall f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{dn})$$

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$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} .$$

Realizability Problem on $\mathcal{S} \subseteq \Gamma(\mathbb{R}^d)$

Given a sequence $(\rho^{(n)})_{n=0}^N$ with $\boxed{\rho^{(n)} \in \mathcal{R}(\mathbb{R}^{dn})}$ symmetric, find a point process μ concentrated on $\mathcal{S}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$\langle f^{(n)}, \rho^{(n)} \rangle = \underbrace{\int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\odot n} \rangle \mu(d\gamma)}_{n\text{-th correlation function of } \mu} , \quad \forall f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{dn})$$

Analogy with the classical moment problem

Let $N \in \mathbb{N} \cup \{\infty\}$. $N = \infty \rightsquigarrow$ Full MP/RP $N \in \mathbb{N} \rightsquigarrow$ Truncated MP/RP

Moment Problem on $K \subseteq \mathbb{R}$

Given a sequence $(m_n)_{n=0}^N$ with $\boxed{m_n \in \mathbb{R}}$, find a finite nonnegative Borel measure μ concentrated on K s.t. for any $n = 0, 1, \ldots, N$ we have

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$$\langle f^{(n)}, \rho^{(n)} \rangle = \underbrace{\int_{\mathcal{S}} \langle f^{(n)}, \gamma^{\odot n} \rangle \mu(d\gamma)}_{n\text{-th correlation function of } \mu} , \quad \forall f^{(n)} \in \mathcal{C}_c^\infty(\mathbb{R}^{dn})$$

New approaches to RP through the MP

using the interplay between finite & infinite dimensional MP theory to solve RP

1. shifting the techniques used in finite dimensions to get new results in infinite dimensions
2. projecting RP to finite dimensions and solve the corresponding classical MP (particular geometry of the supports) so to get a new set of realizability conditions in both cases.

exploiting special infinite dimensional features appearing in the instances of the RP as tools to determine new explicit solutions for both RP and MP

Our Framework: Point processes on $\mathbb{Z}^d$

Let $d \in \mathbb{N}$.

- **Point configuration** $\rightarrow \gamma \subset \mathbb{Z}^d \ : \ |\gamma \cap \{\mathbf{x}\}| \in \{0,1\}, \forall \mathbf{x} \in \mathbb{Z}^d \quad (*)$.

  Point configurations on $\mathbb{Z}^d$ correspond to counting functions via:

  $$\gamma := (x_i)_{i \in I} \leftrightarrow \gamma(\mathbf{r}) := \sum_{i \in I} \delta_{\mathbf{x}_i, \mathbf{r}}$$

  where $\delta_{\mathbf{x}_i,\mathbf{r}}$ is the Kronecker delta, $\mathbf{x}_i$ are distinct points in $\mathbb{Z}^d$ and either $I \subset \mathbb{N}$ is finite or $I = \mathbb{N}$.

  NOTE: As we assumed the exclusion principle (*), we have:

  $$\boxed{\gamma_{\mathbf{r}} := \gamma(\mathbf{r}) \in \{0,1\}, \ \forall \, \mathbf{r} \in \mathbb{Z}^d.}$$

- $\Gamma(\mathbb{Z}^d)$:= set of all possible point configurations on $\mathbb{Z}^d \cong \{0,1\}^{\mathbb{Z}^d}$.
- **Point process μ on $\mathbb{Z}^d$** := a probability measure on $\Gamma(\mathbb{Z}^d)$.

RP for point processes on the d−dimensional lattice

Let d be a positive integer. Given a point process μ on $\mathbb{Z}^d$, we then have that:

$$\begin{cases} \rho_1^\mu(\mathbf{i}) := \mathbb{E}_\mu(\gamma_\mathbf{i}) \\ \rho_2^\mu(\mathbf{i},\mathbf{j}) := \mathbb{E}_\mu(\gamma_\mathbf{i}\gamma_\mathbf{j}) - \mathbb{E}_\mu(\gamma_\mathbf{i})\delta_\mathbf{0}(\{\mathbf{i}-\mathbf{j}\}) \end{cases}, \quad \mathbf{i},\mathbf{j} \in \mathbb{Z}^d$$

NOTE: $\rho_2^\mu(\mathbf{i},\mathbf{j})$ vanishes whenever $\mathbf{i}=\mathbf{j}$.

$\boxed{\mu \text{ point process on } \mathbb{Z}^d} \;\underset{?}{\rightleftarrows}\; \boxed{\text{Correlation functions of } \mu}$

Realizability problem of order 2 for point process on $\mathbb{Z}^d$

Given two functions $\rho_1(\mathbf{i})$ and $\rho_2(\mathbf{i},\mathbf{j})$ non-negative and symmetric for all $\mathbf{i},\mathbf{j} \in \mathbb{Z}^d$, does there exist a point process μ s.t.

$$\rho_1(\mathbf{i}) = \rho_1^\mu(\mathbf{i}) \text{ and } \rho_2(\mathbf{i},\mathbf{j}) = \rho_2^\mu(\mathbf{i},\mathbf{j})?$$

If such a point process μ on $\mathbb{Z}^d$ does exist we say that ρ_1, ρ_2 are **realizable**.

RP for translation invariant point processes on $\mathbb{Z}^d$

If μ is a ***translation invariant point process*** on $\mathbb{Z}^d$, then also the associated correlation functions are translation invariant and:

$$\begin{cases} \rho_1^{\mu}(\mathbf{i}) = \rho \\ \rho_2^{\mu}(\mathbf{i},\mathbf{j}) = \rho^2 g(\mathbf{i}-\mathbf{j}), \quad \mathbf{i} \neq \mathbf{j} \end{cases}$$

for some $\rho \in \mathbb{R}^+$ and $g : \mathbb{Z}^d \to \mathbb{R}^+$ symmetric.
The function g is known in classic fluid theory as ***radial distribution***.

Realizability problem of order 2 for translation invariant point process on $\mathbb{Z}^d$

Given $\rho \in \mathbb{R}^+$ and a symmetric function $g : \mathbb{Z}^d \to \mathbb{R}^+$, does there exist a translation invariant point process μ s.t.

$$\begin{cases} \rho_1^{\mu}(\mathbf{i}) = \rho \\ \rho_2^{\mu}(\mathbf{i},\mathbf{j}) = \rho^2 g(\mathbf{i}-\mathbf{j}), \quad \mathbf{i} \neq \mathbf{j} \end{cases} ?$$

If such a point process μ on $\mathbb{Z}^d$ does exist, then it is said to be **realizing** and the pair (ρ, g) is called **realizable**.

RP of order 2 for translation invariant point process on $\mathbb{Z}^d$

$$\left(\begin{array}{c} \exists\, \mu \text{ point process on } \mathbb{Z}^d \\ \text{realizing the pair } (\rho, g) \end{array} \right) \Rightarrow \left(\begin{array}{c} \forall\, \rho' \in \mathbb{R}^+ \text{ with } 0 \leq \rho' \leq \rho, \\ \exists\, \mu' \text{point process on } \mathbb{Z}^d \text{realizing } (\rho', g) \end{array} \right)$$

Thinning the process μ

We construct a new point process μ' independently keeping or deleting each point of the original process μ with probability $\frac{\rho'}{\rho}$ and $1 - \frac{\rho'}{\rho}$, respectively.

Hence, the realizability problem reduces to:

RP of order 2 for translation invariant point process on $\mathbb{Z}^d$

Given $g : \mathbb{Z}^d \to \mathbb{R}^+$ symmetric, what is the least upper bound $\bar{\rho}_g(d)$ of all the densities ρ for which the pair (ρ, g) is realizable?

$$[0, \bar{\rho}_g(d)] \subseteq [0, 1].$$

Estimating the maximal realizable density $\bar{\rho}_g(d)$

General method to construct an upper bound

Upper bounds can be obtained by **necessary realizability conditions**.

$$
\begin{gathered}
(\rho, g) \text{ is realized by a point process on } \mathbb{Z}^d \\
\Downarrow \\
\rho > 0,\ g \geq 0 \text{ on } \mathbb{Z}^d \\
\Downarrow \\
\text{associated covariance matrix is p.s.d.} \\
S(\mathbf{i}, \mathbf{j}) = \mathbb{E}_\mu(\gamma_{\mathbf{i}}\gamma_{\mathbf{j}}) - \mathbb{E}_\mu(\gamma_{\mathbf{i}})\mathbb{E}_\mu(\gamma_{\mathbf{j}}) \\
\Downarrow \\
\text{infinite volume structure } \hat{S} \text{ is non-negative on } \mathbb{R}^d
\end{gathered}
$$

$$
\hat{S}(\mathbf{k}) = \rho + \rho^2 \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{k}\cdot\mathbf{x}}[g(\mathbf{x}) - 1] \geq 0,\ \forall\, \mathbf{k} \in \mathbb{R}^d
$$

Once the radial distribution g is specified, using the non-negativity of $\hat{S}$, one can derive an upper bound for all realizable ρ's and so for $\bar{\rho}_g(d)$.

Lower bounds for the maximal realizable density

Theorem I. (Kuna-Lebowitz-Speer 2007)

Let g be an even non-negative function on $\mathbb{Z}^d$ such that $C(g) := \sum_{\mathbf{i}\in\mathbb{Z}^d} |g(\mathbf{i}) - 1| < \infty$. Let b s.t. $1 \leq b < \infty$ and $\prod_{k=1}^{n} g(\mathbf{i}_k) \leq b$ whenever $\mathbf{i}_1, \ldots, \mathbf{i}_n \in \mathbb{Z}^d$ satisfy $\prod_{k<h} g(\mathbf{i}_k - \mathbf{i}_h) > 0$.

Then (ρ, g) is realizable for all $0 \leq \rho \leq (ebC(g))^{-1}$.

Theorem II. (Kuna-Lebowitz-Speer 2007)

Let g be an even non-negative function on $\mathbb{Z}^d$ such that $g(\mathbf{i} - \mathbf{j}) \geq 1$ for all $\mathbf{i} \neq \mathbf{j} \in \mathbb{Z}^d$ and $B := \sup_{\mathbf{i}\in\mathbb{Z}^d} \prod_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{i}\}} g(\mathbf{i} - \mathbf{j}) < \infty$.

Then (ρ, g) is realizable for all $0 \leq \rho \leq B^{-1}$.

A direct way of improving those lower bounds for the maximal realizable density is **explicitly constructing a realizable process** with a certain density in between the lower and upper bound estimated through the general methods.

The iso-$g^{(\alpha)}$ realizability problem

Let us consider now radial distributions of the following form:

$$g^{(\alpha)}(\mathbf{x}) := \begin{cases} 0 & \text{if } \ \mathbf{x} = 0 \\ \alpha & \text{if } |\mathbf{x}| = 1 \\ 1 & \text{if } |\mathbf{x}| > 1 \end{cases} .$$

for some $\alpha \geq 0$. Then taking $g = g^{(\alpha)}$ we get the following:

The iso-$g^{(\alpha)}$ realizability problem of order 2 on $\mathbb{Z}^d$

Given $\rho, \alpha \in \mathbb{R}^+$, does there exist a translation invariant point process μ on $\mathbb{Z}^d$ such that its first two correlation functions are given by

$$\begin{cases} \rho_1^\mu(\mathbf{i}) = \rho \\ \rho_2^\mu(\mathbf{i}, \mathbf{j}) = \rho^2 g^{(\alpha)}(\mathbf{i} - \mathbf{j}) \end{cases} ,$$

i.e. $\mathbb{E}_\mu(\gamma_\mathbf{i}) = \rho$ and $\mathbb{E}_\mu(\gamma_\mathbf{i}\gamma_j) = \begin{cases} \rho & \text{if } \mathbf{i} = \mathbf{j} \\ \alpha\rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| = 1 \\ \rho^2 & \text{if } |\mathbf{i} - \mathbf{j}| > 1 \end{cases}$?

General bounds for $\bar{\rho}_\alpha(d)$ in the iso-$g^{(\alpha)}$ problem

Applying the general methods in Kuna-Lebowitz-Speer 2007 to the iso-$g^{(\alpha)}$ problem, we get the following bounds $\bar{\rho}_\alpha(d)$ for any $d \in \mathbb{N}$

$$\bar{\rho}_\alpha(d) \leq R_F(\alpha, d) := \frac{1}{1 + 2d|1-\alpha|}, \ \forall\, \alpha \geq 0$$

$$\bar{\rho}_\alpha(d) \geq r_A(\alpha, d) := \begin{cases} \frac{1}{e(1+2d(1-\alpha))}, & \text{if } 0 \leq \alpha < 1, \\ \frac{1}{\alpha^{2d}}, & \text{if } \alpha \geq 1. \end{cases}$$

Lower bounds improvements for $d = 1$ by explicitly constructing a translation invariant realizing process on $\mathbb{Z}$ at some value of ρ and α BUT...

Problem

No explicit construction is available for any $d \geq 2$!

Explicit construction I: $d = 1$ and $\alpha \geq \frac{1}{2}$

Fix $\alpha \geq \frac{1}{2}$. Construct a point process μ by superposition of two point processes on $\mathbb{Z}$.

1. Choose with probability $\frac{1}{2}$ one of the following two partitions of $\mathbb{Z}$:

   $\boxed{\ldots \cup \{-2,-1\} \cup \{0,1\} \cup \{2,3\} \cup \ldots}$ $\boxed{\ldots \cup \{-1,0\} \cup \{1,2\} \cup \{3,4\} \cup \ldots}$

2. Assign a configuration (γ_i, γ_{i+1}) to each pair $(i, i+1)$ of sites in the chosen partition independently, by taking:

$$(\gamma_i, \gamma_{i+1}) = \begin{cases} (1,0) & \text{with probability} \quad p \\ (0,1) & \text{with probability} \quad p \\ (0,0) & \text{with probability} \quad q \\ (1,1) & \text{with probability} \quad 1-2p-q \end{cases}$$

   Hence: $\rho_1^\mu(i) = \mathbb{E}_\mu(\gamma_i) = p + 1 - 2p - q = 1 - p - q$.

3. The optimal choices of p, q for which μ solves the iso-$g^{(\alpha)}$ problem with maximal density are:
   - $q = 0,\ p = \frac{\sqrt{2-2\alpha}}{(1+\sqrt{2-2\alpha})}$ for $\frac{1}{2} \leq \alpha \leq 1 \Rightarrow \boxed{\rho_1^\mu(i) = \frac{1}{1+\sqrt{2-2\alpha}}}$
   - $p = 0$ and $q = \frac{(2\alpha-2)}{(2\alpha-1)}$ for $\alpha \geq 1 \Rightarrow \boxed{\rho_1^\mu(i) = \frac{1}{2\alpha-1}}$.

Explicit construction II: $d = 1$ and $0 \leq \alpha \leq \frac{1}{2}$

Fix $0 \leq \alpha \leq 1$.

1. We distribute particles on $\mathbb{Z}$ with a Bernoulli measure such that each site is occupied with probability λ
2. If in this initial configuration a site is occupied we delete the particle occupying its left neighbour, if it exists, with probability κ. Hence:

$$\rho_1^\mu(i) = \mathbb{E}_\mu(\gamma_i) = \lambda^2(1-\kappa) + \lambda(1-\lambda) = \lambda(1-\lambda\kappa).$$

3. The optimal choices of λ, κ for which such a process μ solves the iso-$g^{(\alpha)}$ problem with maximal density are:

$$\lambda = 1/(1+\sqrt{1-\alpha}) \text{ and } \kappa = \sqrt{1-\alpha} \Rightarrow \boxed{\rho_1^\mu(i) = \frac{1}{(1+\sqrt{1-\alpha})^2}}$$

N.B. This improves the bound given by the previous construction only for $0 \leq \alpha \leq \frac{1}{2}$.

Improved lower bounds for $\bar{\rho}_\alpha(1)$ in the iso-$g^{(\alpha)}$ problem

Let us summarize the lower bounds coming from these two constructions:

$$\bar{\rho}_\alpha(1) \geq r_L(\alpha, 1) := \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^2}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{1+\sqrt{2-2\alpha}}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{2\alpha-1}, & \text{if } \alpha \geq 1. \end{cases}$$

Case $\alpha = 0$:

- The bound $r_L(0, 1) = \frac{1}{4} = 0.25$ strongly improves $r_A(0, 1) = \frac{1}{3e} \approx 0.1226$.
- Further explicit constructions provided only a slightly improvement of $r_L(0, 1) = 0.25$ to $\bar{\rho}_0(1) \geq 0.265$.

The upper bound $R_F(0, 1) = \frac{1}{3} \approx 0.3333$ was also slightly improved to

$$\bar{\rho}_0(1) < (326 - \sqrt{3115})/822 \approx 0.3287.$$

Open Problem

Reduce the gap between lower and upper bounds for $\bar{\rho}_\alpha(1)$ for any $\alpha \geq 0$!

An explicit construction for any $d = 2$: Step I

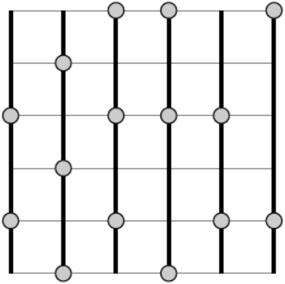
Fix $\alpha \geq 0$ and $\eta > 0$.

$\boxed{BP\eta}$:=***basic process of density*** η:= a point process on $\mathbb{Z}$ realizing $(\eta, g^{(\alpha)})$

Define a process $B^{(1)}$ on $\mathbb{Z}^2$ s.t.:

- for a fixed $i_1 \in \mathbb{Z}$, the process $B^{(1)}(i_1, i_2)$ is a $BP\eta$ in i_2
- for any fixed $i_1, j_1 \in \mathbb{Z}$ with $i_1 \neq j_1$, the processes $B^{(1)}(i_1, i_2)$ and $B^{(1)}(j_1, j_2)$ are independent $BP\eta$ processes in i_2 and j_2 respectively.

The process $B^{(1)}$ is a sequence of vertical $BP\eta$'s independent one from each other

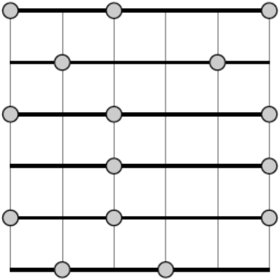


Example of process $B^{(1)}$ (with $\alpha = 0$)

An explicit construction for any $d = 2$: Step II

Similarly, the process $B^{(2)}$ is defined as a sequence of horizontal $BP\eta$'s independent one from each other, i.e.

$$\mathbb{E}_{B^{(2)}}(\gamma_{i_1,i_2}\gamma_{j_1,j_2}) = \begin{cases} \eta^2 & \text{if } i_2 \neq j_2 \\ \eta & \text{if } i_2 = j_2 \text{ and } i_1 = j_1 \\ \alpha\eta^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| = 1 \\ \eta^2 & \text{if } i_2 = j_2 \text{ and } |i_1 - j_1| > 1 \end{cases} .$$

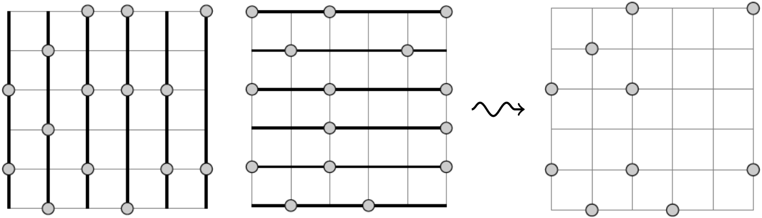


Example of process $B^{(2)}$ (with $\alpha = 0$)

An explicit construction for any $d = 2$: Step III

Let us define now the process P on $\mathbb{Z}^2$ as

$$P(i_1, i_2) := B^{(1)}(i_1, i_2)B^{(2)}(i_1, i_2),$$



Using the independence of the processes $B^{(1)}$ and $B^{(2)}$ and the fact that they are made out of basic processes with the same density η we get that:

$$P \text{ realizes } (\eta^2, g^{(\alpha)}).$$

New lower bounds on $\bar{\rho}_\alpha(d)$

Let $\alpha \geq 0$.

OUR CONSTRUCTION applied to a BP with density $\bar{\rho}_\alpha(1)$

$\downarrow$

a point process on $\mathbb{Z}^d$ which realizes the pair $((\bar{\rho}_\alpha(1))^d, g^{(\alpha)})$

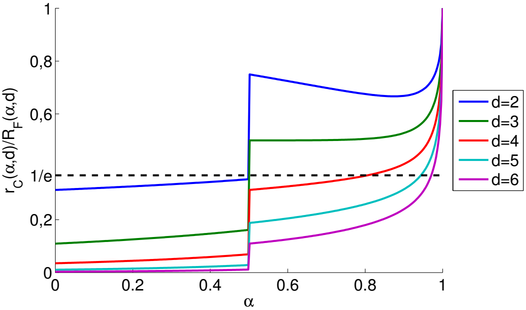
$$\bar{\rho}_\alpha(d) \geq (\bar{\rho}_\alpha(1))^d,\ \forall \alpha \geq 0 \qquad \bar{\rho}_\alpha(1) \geq \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^2}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{1+\sqrt{2-2\alpha}}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{2\alpha-1}, & \text{if } \alpha \geq 1. \end{cases}$$

$$\boxed{\bar{\rho}_\alpha(d) \geq r_C(\alpha, d) := \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^{2d}}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{(1+\sqrt{2-2\alpha})^d}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{(2\alpha-1)^d}, & \text{if } \alpha \geq 1. \end{cases}}$$

Comparison with the known lower bounds

$$r_C(\alpha, d) := \begin{cases} \frac{1}{(1+\sqrt{1-\alpha})^{2d}}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \frac{1}{(1+\sqrt{2-2\alpha})^{d}}, & \text{if } \frac{1}{2} \leq \alpha \leq 1, \\ \frac{1}{(2\alpha-1)^{d}}, & \text{if } \alpha \geq 1. \end{cases} \quad r_A(\alpha, d) := \begin{cases} \frac{1}{e(2d(1-\alpha)+1)}, & \text{if } 0 \leq \alpha < 1, \\ \frac{1}{\alpha^{2d}}, & \text{if } \alpha \geq 1. \end{cases}$$

NOTE: if $\alpha \geq 1$ then $r_A(\alpha, d) \leq r_C(\alpha, d)$



Coloured lines $= \frac{r_C(\alpha,d)}{R_F(\alpha,d)}$ for $d = 2, \ldots, 6$

Dotted line $= \frac{r_A(\alpha,d)}{R_F(\alpha,d)}$ for any dimension d

where $R_F(\alpha, d) = \frac{1}{(2d(1-\alpha)+1)}$.

Thank you for your attention

For more details see:

E. Caglioti, M. Infusino, T. Kuna, **Translation invariant realizability problem on the $d-$dimensional lattice: an explicit construction**, Electronic Communications in Probability 21 (2016), n.1, 1–9.