Perturbations of multivariate Christoffel-Darboux kernels

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joint work with

B. Beckermann (U. Lille)
E. Saff (Vanderbilt U.)

closely related to prior works with:

B. Gustafsson (KTH Stockholm)
M. Korda (LAAS Toulouse)
J.-B. Lasserre (LAAS Toulouse)
I. Mezic (UCSB)
E. Pauwels (U. Toulouse)
N. Stylianopoulos (U. Cyprus)

Contents

Spectral analysis of Koopman's operator

joint work with Korda and Mezic

In particular the presence of singular absolute continuous spectrum in the study and classification of some challenging dynamical systems.
Derived from Christoffel-Darboux analysis of data.

I. Complex orthogonal polynomials

$\mu \geq 0$ positive Borel measure, rapidly decreasing on $\mathbb{C}$, of infinite support, so that the *orthogonal polynomials* $P_n(z), \ n \geq 0$, are well defined by

$$P_n(z) = \gamma_n z^n + O(z^{n-1}), \quad \gamma_n > 0,$$

and

$$\langle P_n, P_k \rangle_{2,\mu} = \delta_{nk}.$$

Moment data (observables) and complex orthogonal polynomials are interchangeable via elementary matrix operations.

Christoffel-Darboux kernel

$$K_N(z,w) = \sum_{k=0}^{N-1} P_k(z)\overline{P_k(w)},$$

is the reproducing kernel in the space $\mathbb{C}_N[z]$ of polynomials of degree less than N:

$$\langle h, K_N(\cdot, w)\rangle_{2,\mu} = h(w), \quad \deg h < N.$$

E. B. Christoffel, *Über die Gaussische Quadratur und eine Verallgemeinerung derselben,* J. Reine Angew. Math. 55 (1858), 61-82.

G. Darboux, *Mémoire sur l'approximation des fonctions de trés-grands nombres, et sur une classe étendue de developpements en série*, Liouville J. (3) 4 (1878), 5-56; 377-416.

Christoffel function

$$|h(w)| \leq \|h\| \|K_N(\cdot, w)\|$$

with optimal solution $K_N(\cdot, w)$:

$$\max \frac{|h(w)|}{\|h\|} = \|K_N(\cdot, w)\|,$$

or equivalently

$$\Lambda_N(\mu, w) := \min \frac{\|h\|_{2,\mu}^2}{|h(w)|^2} = \frac{1}{K_N(w, w)}$$

where $\deg h < N$ and $h(w) \neq 0$.

The asymptotics of Christoffel's function $\Lambda_N(\mu, w)$ were and remain central for many problems of mathematical analysis.

Determinateness of 1-D moment problems

Note that the orthogonal polynomials $P_n(z)$ depend only on the moments of the underlying measure μ:

$$c_{kn} = \int_{\mathbb{C}} z^k \overline{z}^n d\mu(z), \quad k, n \geq 0.$$

Solving the moment problem (i.e. recovery of μ from $(c_{kn})_{k,n=0}^{\infty}$) encounters a natural and difficult obstacle: *is the measure μ unique?*

M. Riesz (1923), R. Nevanlinna (1924): *If $\mathrm{supp}\mu \subset \mathbb{R}$, then the moments determine the measure if and only if*

$$\lim_{N\to\infty} \Lambda_N(\mu, z) = 0,$$

for at least one $z \in \mathbb{C} \setminus \mathbb{R}$ (and hence for all). Following earlier ideas of H. Weyl.

Maximal point masses

Note that always $\Lambda_{N+1}(\mu, z) \leq \Lambda_N(\mu, z)$, so

$$\Lambda(\mu, z) = \lim_{N\to\infty} \Lambda_N(\mu, z)$$

exists.

In case $x \in \mathbb{R}$,

$$\mu(\{x\}) \leq \Lambda(\mu, x) = \min \int |h(y)|^2 d\mu(y), \quad h(x) = 1,$$

and the upper bound is attained (by extremal solutions of the moment problem).

The unit circle

$w(t) \geq 0$ integrable weight on $[-\pi, \pi)$. With geometric mean

$$G(w) = \exp(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln w(t) dt),$$

if w satisfies Szegö's condition

$$\int_{-\pi}^{\pi} \ln w(t) dt > -\infty,$$

and $G(w) = 0$ otherwise.

Szegö (1914) $\Lambda(w(t)dt, 0) = G(w)$.

Formulated equivalently as an extremum problem, with $z = e^{it}$:

$$\lim_{n \to \infty} \min_{A_j} \frac{1}{2\pi} \int_{-\pi}^{\pi} |z^n + A_1 z^{n-1} + \ldots + A_n|^2 w(t) dt = G(w).$$

Density of complex polynomials in Lebesgue space

$\mu \geq 0$ positive Boreal measure on $[-\pi, \pi)$ with Lebesgue decompsoition

$$\mu = w(t)dt + \mu_{sc} + \mu_d.$$

where $w(t)$ has bounded variation.

Kolmogorov (1941), Krein (1945) *The system* $1, z, z^2, \ldots$ *is dense in* $L^p(\mu), p \geq 1$, *if and only if*

$$\int_{-\pi}^{\pi} |\ln w(t)| dt = \infty.$$

Hint: By Szegö's Theorem, if $G(w) > 0$, then e^{-it} cannot be approximated by $1, e^{it}, e^{i2t}, \ldots$.

Accelerated convergence of Fourier series

Let $\alpha \geq 0$ be a positive measure supported on a compact set $I \subset \mathbb{R}$. For a continuous function $f \in C(I)$ we set

$$S_N(\alpha, f, x) = \sum_{k=0}^{N-1} \langle f, P_k \rangle P_k(x).$$

Then

$$\sup_{\|f\|_{\infty,I} \leq 1} |S_N(\alpha, f, x)|^2 \leq \|f\|_{2,\alpha}^2 K_N(\alpha; x, x) \leq \alpha(I) K_N(\alpha; x, x).$$

Consequently

$$|f(x) - S_N(\alpha, f, x)| = |f(x) - Q(x) - S_N(\alpha, f - Q, x)| \leq$$

$$\inf_{\deg Q < N} \|f - Q\|_{\infty,K} (1 + \alpha(I) \sqrt{K_N(\alpha : x, x)}).$$

Lebesgue (1905) was the first to use this scheme for studying the convergence of Fourier type developments.

The unit disk

$\mu = dA$ on $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$

$$K_N(z,w) \to K(z,w) = \frac{1}{\pi(1 - z\overline{w})^2}, \quad z, w \in \mathbb{D},$$

the Bergman kernel, and

$$K_N(z,z) = \frac{N+1}{\pi} \frac{1}{|z|^2 - 1} |z|^{2N+2}(1 + O(1/N)), \quad |z| > 1.$$

measures how fast a polynomial of degree less than N lifts outside the disk, keeping its square norm average on $\mathbb{D}$ bounded.

Classical asymptotics extended to more general domains by Carleman (1922) and Suetin (1969).

Paradigm

For a positive measure μ on the line or circle:

$$\lim_n \frac{1}{K_n^\mu(x,x)} = \mu(\{x\}),$$

while

$$\lim_n \frac{n}{K_n^\mu(x,x)} = \frac{\mu'(x)}{\omega_I(x)}, \quad a.e.,$$

on an interval I, where ω_I is the equilibrium measure.

The circle again

Simple example: $\mu = \frac{d\theta}{2\pi}$ on $[-\pi, \pi)$.

$$K_n(z, w) = \frac{1 - z^n \overline{w}^n}{1 - z\overline{w}},$$

implies for $z = e^{it}, w = e^{is}$:

$$K_n(z, z) = |K_n(z, z)| = \lim_{s \to t} \frac{\sin \frac{n(t-s)}{2}}{\sin \frac{t-s}{2}} = n.$$

Hence

$$\frac{n}{K_n(z, z)} = 1.$$

CD kernels, their level sets and their perturbations

Early work: V. B. Uvarov, *The connection between systems of polynomials that are orthogonal with respect to different distribution functions*, Zh. Vychisl. Mat. Mat. Fiz., 1969, Volume 9, Number 6, 1253-1262.

More recent kernel computations: D. Lubinsky, *Some recent methods for establishing universality limits*, J. Nonlinear Analysis 71(2009), 2750-2765.

Shape reconstruction: for instance of an archipelago, with islands and lakes (Gustafsson, Saff, P., Totik, Stahl, Stylianopoulos)

Statistical motivation: A "cloud" approximated by a measure with a positive density (possibly along a real algebraic variety) and "outliers" represented by a finite atomic measure (Lassrre, Pauwels, P.)

Multivariate setting: techniques from pluripotential theory, asymptotics of orthogonal polynomials, real algebraic geometry, matrix inequalities.

First mild complications

Let μ be a positive measure, rapidly decreasing at infinity in $\mathbb{C}^d$. The support of μ may be contained in a proper algebraic subset $S(\mu)$ (the Zariski closure of $\operatorname{supp}(\mu)$).

Along $S(\mu)$ not all monomials are independent, so an ordering of them, by an appropriate degree is necessary.

The usual definition of a CD kernel makes sense only along $S(\mu)$:

$$\frac{1}{\sqrt{K_n^{\mu}(z,z)}} = \min\{\|p\|_{2,\mu} : \deg p \leq k, p(z) = 1\}.$$

A related object to consider is the cosine function

$$C_n^{\mu}(z,w) := \frac{K_n^{\mu}(z,w)}{\sqrt{K_n^{\mu}(z,z)}\sqrt{K_n^{\mu}(w,w)}}.$$

Statistics interlude

The mean $m(\mu)$ and positive definite covariance matrix $\mathcal{C}(\mu)$ of a random variable with values in $\mathbb{C}^d$ with law described by some probability measure μ have the entries

$$m(\mu)_j = \int z_j d\mu(z), \quad \mathcal{C}(\mu)_{j,k} = \int \overline{(z_j - m(\mu)_j)}(z_k - m(\mu)_k)d\mu(z).$$

The *Mahalanobis distance* between a point $z \in \mathbb{C}^d$ and the probability measure μ is given by

$$\Delta(z,\mu) = \sqrt{(z - m(\mu))\mathcal{C}(\mu)^{-1}(z - m(\mu))^*}.$$

Relation to the CD kernel

In terms of the moment matrices and tautological vector $v_d(z) = (1, z_1, ..., z_d)$,

$$V^* \widetilde{M}_d(\mu) V = \begin{bmatrix} \langle 1, 1 \rangle_{2,\mu} & 0 \\ 0 & \mathcal{C}(\mu) \end{bmatrix}, \quad V := \begin{bmatrix} 1 & -m(\mu) \\ 0 & I \end{bmatrix},$$

where $\langle 1, 1 \rangle_{2,\mu} = \mu(\mathbb{C}^d) = 1$ and $p_0^\mu(z) = K_0^\mu(z, z) = 1$ for all $z \in \mathbb{C}^d$. Therefore

$$K_d^\mu(z, z) = v_d(z) M_d(\mu)^{-1} v_d(z)^*$$

$$= \mathrm{v}_d(z) V \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{C}(\mu)^{-1} \end{bmatrix} V^* v_d(z)^* = 1 + \Delta(z, \mu)^2.$$

Thus, up to some additive constant, $\sqrt{K_n^\mu(z, z)}$ can be considered as a natural generalization of the Mahalanobis distance between a point $z \in \mathbb{C}^d$ and the probability measure μ.

Small perturbations

The proximity of the two measures is imposed by the following condition: there exists a sufficiently small $\epsilon \in (0, 1)$ such that

$$(1 - \epsilon) \|p\|^2_{2,\mu} \leq \|p\|^2_{2,\nu} \leq (1 + \epsilon) \|p\|^2_{2,\mu} \tag{1}$$

for polynomials of a prescribed "degree".
Then

$$(1 - \epsilon) K^\nu_n(z, z) \leq K^\mu_n(z, z) \leq (1 + \epsilon) K^\nu_n(z, z), \tag{2}$$

$$|K^\mu_n(z, w) - K^\nu_n(z, w)| \leq \epsilon \sqrt{K^\nu_n(z, z)} \sqrt{K^\nu_n(w, w)}, \tag{3}$$

$$|C^\mu_n(z, w) - C^\nu_n(z, w)| \leq 2\epsilon. \tag{4}$$

Additive perturbations of measures

Let $\mathrm{supp}(\mu)$ be an infinite set such that there exists an infinite number of orthogonal polynomials p_j^{μ}, and consider the case of adding ℓ disjoint point masses in the Zariski closure $\mathcal{S}(\mu)$ of the support of the original measure μ (and later outside (μ)). That is,

$$\sigma = \sum_{j=1}^{\ell} t_j \delta_{z_j}, \quad t_j > 0, \quad z_1, ..., z_\ell \in \mathcal{S}(\mu) \text{ disjoint.} \tag{5}$$

Main result

Consider the following matrices (depending on n)

$$C := C_n^{\mu}(z_1, \dots, z_\ell; z_1, \dots, z_\ell),$$
$$\widetilde{C} := C_n^{\mu}(z_1, \dots, z_\ell, z; z_1, \dots, z_\ell, z) = \begin{bmatrix} C & b^* \\ b & 1 \end{bmatrix},$$
$$D := \operatorname{diag}\left(\frac{1}{\sqrt{t_j K_n^{\mu}(z_j, z_j)}}\right)_{j=1,\dots,\ell},$$

and the constants

$$\Sigma_m := 1 - \sum_{j=0}^{m-1} (-1)^j b C^{-1} (D^2 C^{-1})^j b^*, \ \ m \geq 1.$$

Alternating inequalities

Then, for all $z \in \mathbb{C}^d$,

$$\frac{K_n^{\mu+\sigma}(z,z)}{K_n^{\mu}(z,z)} = 1 - b(D^2 + C)^{-1} b^*, \tag{6}$$

and

$$\Sigma_1 \le \Sigma_3 \le \Sigma_5 \le ... \le \frac{K_n^{\mu+\sigma}(z,z)}{K_n^{\mu}(z,z)} \le ... \le \Sigma_4 \le \Sigma_2 \le \Sigma_0. \tag{7}$$

One point mass

$\sigma = t_1 \delta_{z_1}$ adds one point mass to μ:

$$C = 1, \quad b = C_n^{\mu}(z, z_1), \quad D^2 = \frac{1}{t_1 K_n^{\mu}(z_1, z_1)}.$$

and

$$\frac{K_n^{\mu+\sigma}(z,z)}{K_n^{\mu}(z,z)} = 1 - b(1+D^2)^{-1} b^* = 1 - \frac{|C_n^{\mu}(z,z_1)|^2}{1 + \frac{1}{t_1 K_n^{\mu}(z_1,z_1)}}.$$

Green function estimate

Consider an ordering with respect to total degree, and suppose that $\mathrm{supp}(\mu)$ is infinite. Denote by Ω be the unbounded connected component of $\mathbb{C}^d \setminus \mathrm{supp}(\mu)$, and let $g_\Omega : \mathbb{C}^d \mapsto \mathbb{R}$ be the plurisubharmonic Green function with pole at ∞ which is strictly positive on Ω. Then, for $z \in \mathcal{S}(\mu) \cap \Omega$,

$$\liminf_{m\to\infty} K^{\mu}_{n_{tot}(m)}(z,z)^{1/m} \geq e^{2g_\Omega(z)}. \tag{8}$$

If, in addition, Ω is L-regular and μ belongs to the multivariate generalization of the class **Reg**, then the limit

$$\lim_{m\to\infty} K^{\mu}_{n_{tot}(m)}(z,z)^{1/m} = e^{2g_\Omega(z)} \tag{9}$$

exists for all $z \in \mathcal{S}(\mu)$, and equals zero for $z \in \mathcal{S}(\mu) \setminus \Omega$.

Function theory of a complex variable

Ratio asymptotics for complex orthogonal polynomials:

Let Ω denote a subdomain of the unbounded component of $\mathbb{C} \setminus (\mu)$, and suppose that there is a function g analytic and different from zero in Ω such that

$$\lim_{n\to\infty} \frac{p_n^\mu(z)}{p_{n+1}^\mu(z)} = g(z) \tag{10}$$

uniformly on any compact subset of Ω. Let F be a compact subset of Ω. Then $1/K_n^\mu(z,z) \to 0$ uniformly for $z \in F$, and

$$\lim_{n\to\infty} C_n^\mu(z,w) \frac{|p_n^\mu(z)|}{p_n^\mu(z)} \frac{p_n^\mu(w)}{|p_n^\mu(w)|} = \frac{\sqrt{(1-|g(z)|^2)(1-|g(w)|^2)}}{1-g(z)\overline{g(w)}} \tag{11}$$

uniformly for $z, w \in F$.

CD kernel perturbation estimate

Under the preceding assumptions, and in particular $z_1, ..., z_\ell \in \Omega$ with distinct $g(z_1), ..., g(z_\ell)$, we have uniformly on compact subsets of Ω

$$\lim_{n\to\infty} \frac{K_n^{\mu+\sigma}(z,z)}{K_n^{\mu}(z,z)} = \left|\prod_{j=1}^{\ell} \frac{g(z)-g(z_j)}{1-g(z)\overline{g(z_j)}}\right|^2, \tag{12}$$

and, at a point mass $z = z_m$,

$$K_n^{\mu+\sigma}(z_m,z_m) = \frac{1}{t_m}\left(1+\mathcal{O}\left(\frac{1}{K_n^{\mu}(z_m,z_m)}\right)\right)_{n\to\infty}. \tag{13}$$

Bergman space asymptotics

Let μ area measure on a domain with piece-wise boundary, and

$$\nu = \mu + \sigma, \quad \sigma = \sum_{j=1}^{\ell} t_j \delta_{z_j}, \quad t_j > 0, \quad z_1, ..., z_\ell \in \mathbb{C} \setminus (\mu) \text{ distinct.}$$

Then the Hausdorff distance between (ν) and the level set $S_n := \{z \in \mathbb{C} : K_n^\nu(z, z) \leq \gamma_n\}$ with γ_n as in Theorem **??**(b) tends to zero as $n \to \infty$.

Bergman space asymtotics

Let μ be area measure on G, and suppose that Γ is a Jordan curve which is either piecewise analytic without cusps, or otherwise possesses an arc length parametrization with a derivative being $1/2$-Hölder continuous.

(a) For all $n \geq 0$ and $z \in G$:

$$K_n^\mu(z,z) \leq \frac{1}{\pi(z,\Gamma)^2}.$$

(b) With $r(n) := \sqrt{1 + \frac{1}{n+1}}$ the estimates

$$\begin{aligned} \frac{1}{e} \max_{z \in G_{r(n)}} K_n^\mu(z,z) &\leq \max_{z \in \mathrm{supp}(\mu)} K_n^\mu(z,z) \leq \max_{z \in G_{r(n)}} K_n^\mu(z,z) \\ &\leq \gamma_n := \frac{c_1}{(\Gamma, \partial G_{r(n)})^2}. \end{aligned}$$

hold for all $n \geq 0$.

(c) We have

$$K_n^\mu(z,z) = \frac{n+1}{\pi}\frac{|\Phi'(z)|^2}{|\Phi(z)|^2-1}|\Phi(z)|^{2n+2}(1+\mathcal{O}(\frac{1}{n})_{n\to\infty})$$

uniformly on compact subsets of $\mathbb{C}\setminus \mathrm{supp}(\mu)$.

(d) The asymptotics

$$\frac{p_n^\mu(z)}{p_{n+1}^\mu(z)} = \frac{1}{\Phi(z)}(1+\mathcal{O}((1/n)_{n\to\infty})$$

is valid uniformly on compact subsets of $\mathbb{C}\setminus \mathrm{supp}(\mu)$.

Discretization of Bergman space scenario

Consider the discrete measures

$$\widetilde{\nu} = \widetilde{\mu} + \sigma, \quad \widetilde{\mu} = \sum_{j=\ell+1}^{N} t_j \delta_{z_j}, \quad t_j > 0, \quad z_{\ell+1}, ..., z_N \in (\mu) \text{ distinct},$$

where we assume that $\widetilde{\mu}$ is sufficiently close to μ such that $\|M_n(\widetilde{\mu}, \mu) - I\| \leq 1/2$. Then there exist $c > 0$ and $q \in (0, 1)$ depending only on μ, σ but not on $\widetilde{\nu}$ such that, for the outliers,

$$j = 1, 2, ..., \ell : \quad 1 - t_j K_n^{\widetilde{\nu}}(z_j, z_j) \leq cq^n,$$

whereas for the other mass points of $\widetilde{\nu}$

$$j = \ell + 1, ..., N : \quad t_j K_n^{\widetilde{\nu}}(z_j, z_j) \leq \frac{3}{2} \min\left\{ t_j \gamma_n, \frac{t_j}{\pi(z_j, \Gamma)^2} \right\},$$

where γ_n depends only on n and the geometry of the Green function level lines.

Conclusion for a regular cloud measure

μ with finitely many external point masses $\sigma = \sum_j t_j \delta_{z_j}$ yields asymptotics

$$K_n^{\mu+\sigma}(z_j, z_j) = \frac{1}{t_j}[1 + \mathcal{O}(\frac{1}{K^{\mu}(z_j, z_j)})], \quad n \to \infty.$$

The *leverage score*

$$1 - t_j K_n^{\mu+\sigma}(z_j, z_j)$$

becomes small.

With effective bounds in the case of Bergman space.

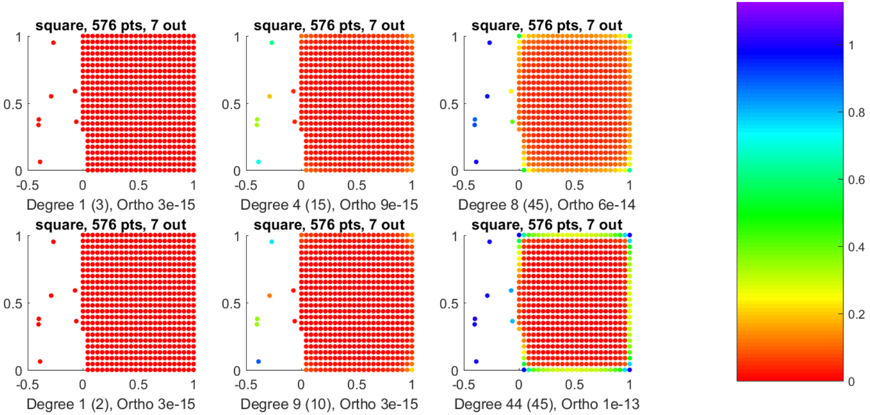


Figure: A cloud of $N = 576$ points, with 7 random outliers, the others obtained by discretizing normalized Lebesgue measure on the unit square by a regular grid.

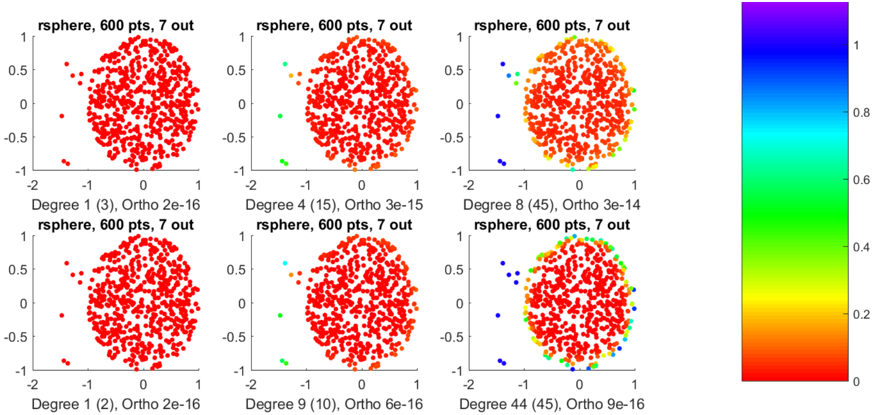
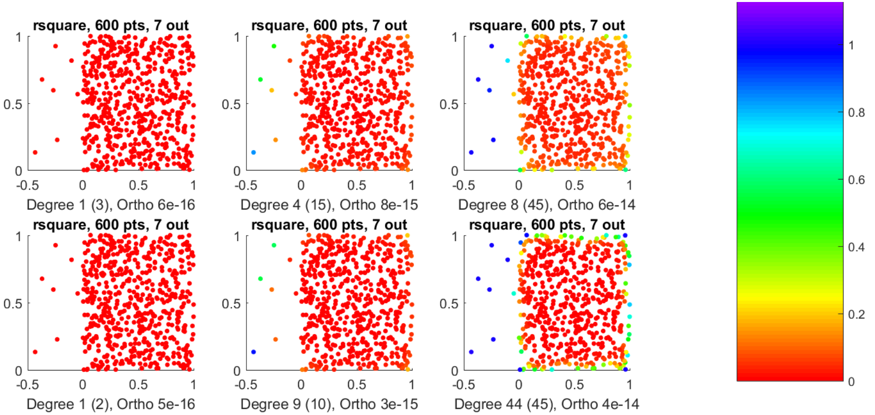


Figure: A cloud of $N = 600$ points, with $\ell = 7$ random outliers. The other points are random samplings of the unit disk.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Degree 1 (3), Ortho 6e-16
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Degree 4 (15), Ortho 8e-15
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Degree 8 (45), Ortho 6e-14
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Degree 1 (2), Ortho 5e-16
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Degree 9 (10), Ortho 3e-15
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Degree 44 (45), Ortho 4e-14

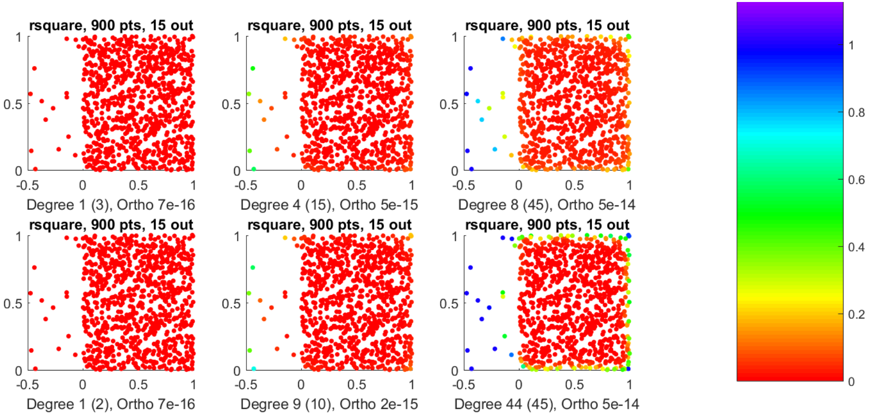


Figure: A first cloud of $N = 600$ points, with $\ell = 7$ random outliers, and a second one with $N = 900$ and 15 outliers. The other points are random samplings.

Real or complex setting?

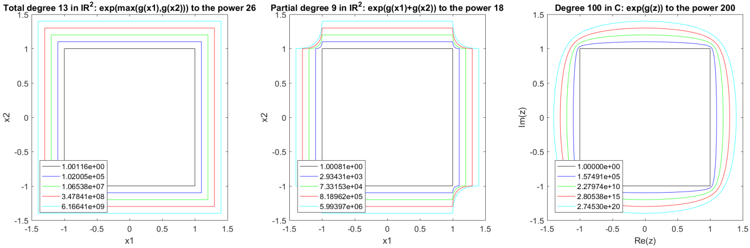


Figure: Comparison of level lines of Green functions for the unit square and three different families of orthogonal polynomials: for the left-hand image we utilized the plurisubharmonic Green function of $[-1, 1]^2 \subset \mathbb{R}^2$, which corresponds to bivariate OP with graded lexicograpghical ordering (i.e., total degree); the middle image is generated by using the tensor Green function corresponding to bivariate OP and partial degree is used; and for the right-most image, the complex Green function corresponding to Bergman OP of one complex variable is used.

III. Shape reconstruction in 2D

joint work with Gustafsson, Saff and Stylianopoulos

CD kernel at work with methods of one complex variable

$G = \cup G_j$ an *archipelago*, i.e. a finite union of simply connected domains, with real analytic boundary $\Gamma = \partial G$ and $\mu = \chi_G d\mathrm{Area}$.

"Observables" are finitely many moments

$$a_{mn} = \int z^m \overline{z}^n \; d\mu(z)$$

such as derived from geometric tomography.
They determine the complex orthogonal polynomials $P_n(z),\ 0 \le n \le N$, and the CD-kernel

$$K_n(z, w) = \sum_{j=0}^{n-1} P_j(z)\overline{P_j(w)}.$$

Christoffel function as defining function of the archipelago

Normalized Christoffel function

$$\gamma_n(z) = [K_n(z,z)]^{-1/2}$$

satisfies:

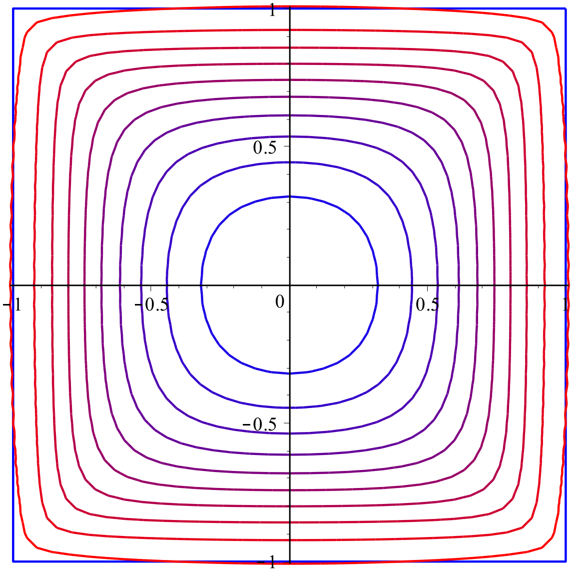
$$\sqrt{\pi}\mathrm{dist}(z,\Gamma) \leq \gamma_n(z) \leq C\mathrm{dist}(z,\Gamma)$$

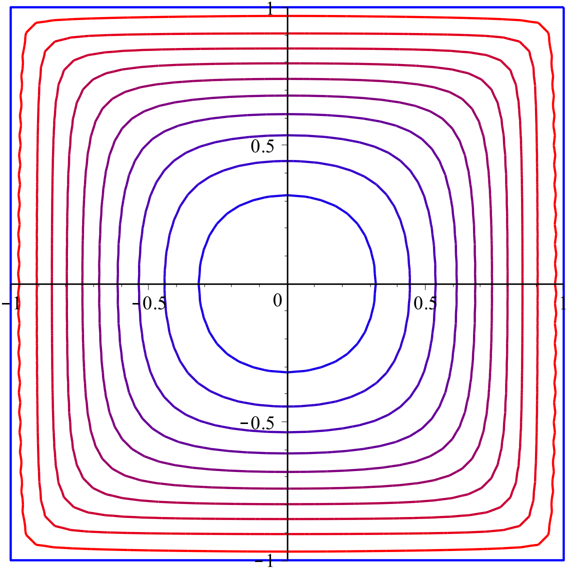
for $z \in G$, close to Γ. Moreover:

$$\gamma_n(z) = O(\frac{1}{n}), \quad z \in \Gamma.$$

Outside $\overline{G}$ this function decreases exponentially to zero. On analytic boundaries one has sharp estimates.

Reconstruction experiments

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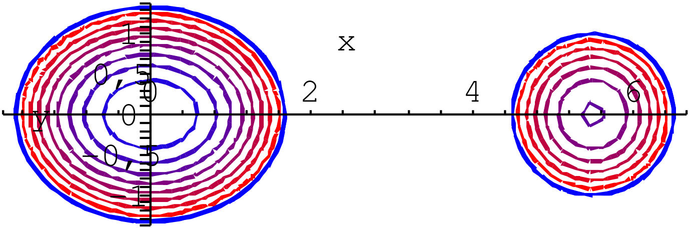
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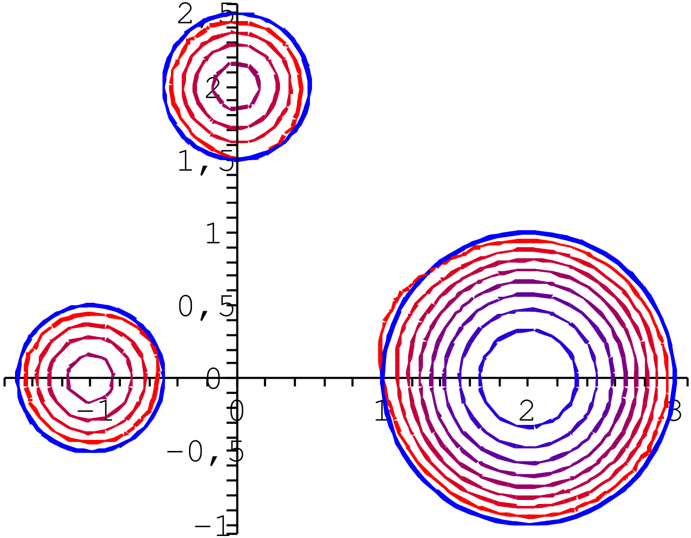
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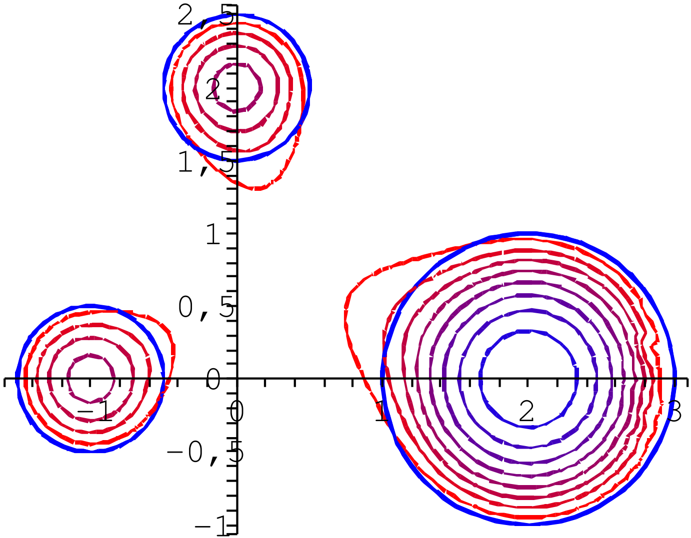
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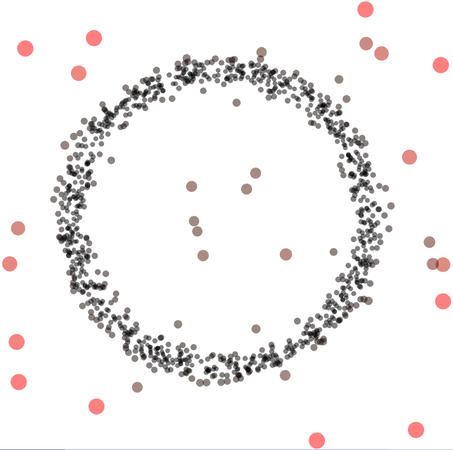
Supports on real algebraic varieties

and identifying the density of a measure with singular support
CD kernel at work in the context of only real variables
joint work with Lasserre and Pauwels

Detecting a circle:



2D torus

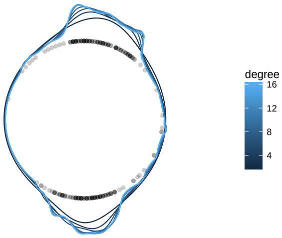


Figure: Dragon fly orientation with respect to the sun, on the torus. The curves represent the empirical Christoffel function for different values of the degree.

2D sphere

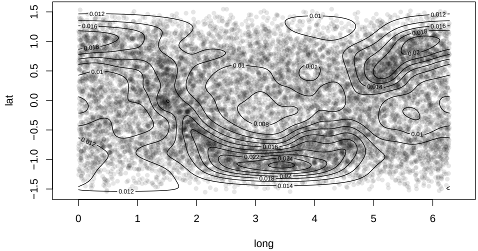


Figure: Each point represent the observation of a double star in the sky. They live on the sphere and are associated to their longitude and latitude. The level sets are those of the empirical Christoffel functions evaluated on the sphere in $\mathbb{R}^3$. The degree is 8. The band which is highlighted by the level sets corresponds to the Milky Way.

2D torus lying on 3D sphere

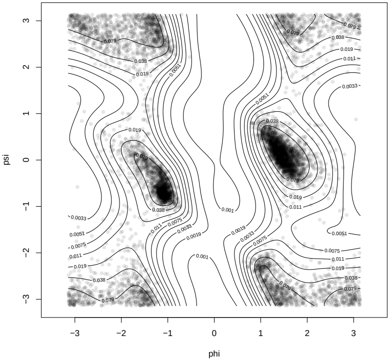


Figure: Each point two dihedral angles for a Glycine amino acid. These angles are used to describe the global three dimensional shape of a protein. They live on the bitorus. The level sets are those of the empirical Christoffel functions evaluated on the sphere in $\mathbb{R}^4$. The degree is 4.

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