The Maslov index and the spectrum of differential operators

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A short introduction: how to prove instability of pulses using the Maslov index

The motivating problem

A PDE with a steady state/ traveling wave/ rotating pattern ϕ

$$u_t = \Delta u - f(u),\ -\Delta\phi + f(\phi) = 0,\ \phi = \phi(x),\ x \in \mathbb{R}^d, f : \mathbb{R}^n \to \mathbb{R}^n,$$
$$u_t = \Delta u - c\partial_x u - f(u),\ -\Delta\phi + c\partial_x\phi + f(\phi) = 0,$$
$$\phi = \phi(x), (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Want to understand which geometric properties of ϕ lead to stability conclusions, whether ϕ is stable or unstable in the PDE

Linearized operator $Lu = -\Delta u + c\partial_x u + f'(\phi(\cdot))u$

Want to understand the spectral properties of L induced by geometric properties of ϕ

Several results: (1) In gradient systems, pulses are unstable;

(2) A very general formula equating the Morse and Maslov indices for PDE

(3) Hadamard-type formula and (Maslov) crossing form

The simplest old result

Theorem (An application of the Sturm Theorem)

Pulses are unstable in scalar reaction diffusion equations.

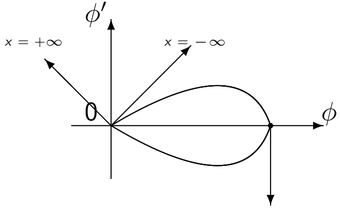
One dimension $d = 1$ scalar $n = 1$ steady state example

$$u_t = \partial_x^2 u - f(u),\ -\partial_x^2 \phi + f(\phi) = 0,$$
$$f : \mathbb{R} \to \mathbb{R},\ f(0) = 0,\ \phi = \phi(x),\ x \in \mathbb{R},$$
$$L\phi' = -\partial_x^2 \phi' + f'(\phi)\phi' = 0,$$

the pulse ϕ satisfies $\lim_{x\to-\infty} \phi(x) = \lim_{x\to+\infty} \phi(x) = 0$ exponentially.
Then $0 \in \mathrm{Sp}(L)$ and ϕ' is an eigenfunction.
There exists a *conjugate point* x_0 such that $\phi'(x_0) = 0$.
By Sturm Theorem there exists a negative (unstable) eigenvalue.

The simplest old result

The existence of a conjugate point implies the existence of a negative eigenvalue because the tangent vector to the homoclinic turns over and therefore must intersect the vertical line (the Dirichlet subspace)



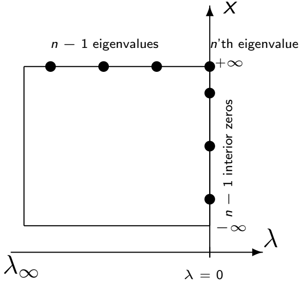


Figure: Sturm Oscillating Theorem: n-th eigenfunction has $(n-1)$ interior zeros.

The simplest new result

Theorem (Beck,Cox,Jones,YL,McQuighan, A.Sukhtayev)

Pulses are unstable in gradient systems of reaction diffusion equations

One dimension $d = 1$ system $n > 1$ steady state example

$$u_t = \partial_x^2 u - f(u),\ -\partial_x^2 \phi + f(\phi) = 0,$$
$$f : \mathbb{R}^n \to \mathbb{R}^n,\ f(0) = 0,\ \phi = \phi(x) \in \mathbb{R}^n,\ x \in \mathbb{R},$$
$$\text{assuming gradient structure} : f(u) = \nabla g(u),\ g : \mathbb{R}^n \to \mathbb{R},$$
$$L\phi' = -\partial_x^2 \phi' + \nabla^2 g(\phi)\phi' = 0,$$

the pulse ϕ satisfies $\lim_{x\to-\infty} \phi(x) = \lim_{x\to+\infty} \phi(x) = 0$ exponentially,

Theorem (Beck,Cox,Jones,YL,McQuighan, A.Sukhtayev)

provided ϕ is even symmetric about some point x_0, that is, $\phi(x_0 + x) = \phi(x_0 - x)$ for all $x \in \mathbb{R}$, so that $\phi'(x_0) = 0$.

The simplest new result (comments)

- This is called the *Gradient Conjecture* by Arnd Scheel.
- Also works for $u_t = D\partial_x^2 u - \nabla g(u)$ with a diagonal diffusion matrix D with positive entries.
- "Provided" part is generic as it holds as soon as $(\phi, \phi')^\top$ is the only solution to $(u, u')' = (u', f(u))$ contained in the intersection $\mathcal{W}^{\rm s}(x) \cap \mathcal{W}^{\rm u}(x)$ of the stable and unstable manifolds at zero for the ODE corresponding to the steady-state equation.
- There exists a *conjugate point* x_0 such that $\phi'(x_0) = 0$.
- Therefore, by the Morse-Maslov Square Theorem (Morse=Maslov, to be mentioned later) there exists a negative (unstable) eigenvalue.

Proof: The exponential dichotomy subspace $\mathbb{E}^{\rm u}_-(x,0) \subset \mathbb{R}^{2n}$ for the system $\partial_x \mathbf{p} = A(x,\lambda)\mathbf{p}$ (which is equivalent to $Lu = \lambda u$, $\mathbf{p} = (u, u')^\top$) at $\lambda = 0$ is $\mathbb{E}^{\rm u}_-(x,0) = \mathrm{span}\{(\phi', \phi'')^\top, \mathbf{p}_2, \dots, \mathbf{p}_n\}$, $\mathbf{p} = (p, q)^\top \in \mathbb{R}^{2n}$. $\mathbb{E}^{\rm u}_-(x,0)$ intersects the Dirichlet subspace $\mathcal{D} = \{(0,q)^\top : q \in \mathbb{R}^n\}$ since $\det(\phi' \,|\, p_2 \,|\, \dots \,|\, p_n) = 0$ at $x = x_0$. Hence, the Maslov index of the path $x \mapsto \mathbb{E}^{\rm u}_-(x,0)$ is nonzero, and thus the Morse index is nonzero.

The simplest new result (proofs)

The proof has two steps:

Maslov=Morse

Consider matrix Schrödinger operator $L = -\partial_x^2 + V(x)$ in $L^2(\mathbb{R})^n$ assuming $V(x) \to V_\pm$ as $x \to \pm\infty$. Then the Morse index of L (the number of negative isolated eigenvalues counting multiplicities) is equal to the Maslov index (the number of conjugate points counting multiplicities).

We prove this first for the operator L_a in $L^2((-\infty, a])^n$ on the half line and then let $a \to \infty$.

An application: Maslov index is positive

For the linearization $L = -D\partial_x^2 - \nabla^2 F(\phi(x))$ of $u_t = Du_{xx} + (\nabla F)(u)$, $u \in \mathbb{R}^n$ about a generic pulse ϕ the Maslov index is nonzero, and therefore the pulse is unstable.

The magic square

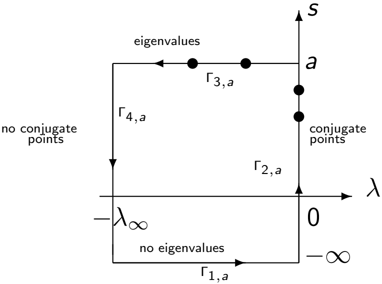


Figure: Illustrating the proof of Morse=Maslov Theorem: When λ_∞ is large enough, there are no crossings on $\Gamma_{1,a}$ and $\Gamma_{4,a}$, and the Morse index equals the number of crossings on $\Gamma_{3,a}$. By homotopy invariance, the Morse index is equal to the number of crossings on $\Gamma_{2,a}$; these are precisely the conjugate points in $(-\infty, a)$.

What is the Maslov index?

Let $\mathcal{G}$ be a Lagrangian subspace of an Hilbert space, $\mathcal{I} = [\alpha, \beta]$, and $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{G}))$, where $F\Lambda(\mathcal{G})$ is the set of Lagrangian subspaces that form with $\mathcal{G}$ a Fredholm pair.

(i) We call $s_* \in \mathcal{I}$ a conjugate point or crossing if $\Upsilon(s_*) \cap \mathcal{G} \neq \{0\}$.
There exists a neighbourhood $\mathcal{I}_0$ of s_* and a family $R_s \in C^1(\mathcal{I}_0, \mathcal{B}(\Upsilon(s_*), \Upsilon(s_*)^\perp))$, such that (a lemma)

$$\Upsilon(s) = \{u + R_s u \big| u \in \Upsilon(s_*)\}, \text{ for } s \in \mathcal{I}_0.$$

(ii) The finite dimensional form is called the crossing form at the crossing s_*:

$$\mathfrak{m}_{s_*,\mathcal{G}}(u,v) := \frac{d}{ds}\omega(u, R_s v)\Big|_{s=s_*} = \omega(u, \dot{R}_{s=s_*} v), \text{ for } u, v \in \Upsilon(s_*) \cap \mathcal{G}.$$

(iii) The crossing s_* is called regular if the form $\mathfrak{m}_{s_*,\mathcal{Z}}$ is non-degenerate, positive if $\mathfrak{m}_{s_*,\mathcal{Z}}$ is positive definite, and negative if $\mathfrak{m}_{s_*,\mathcal{Z}}$ is negative definite.
Definition. If all crossings are regular then they are isolated, and one defines the Maslov index as

$$\operatorname{Mas}(\Upsilon, \mathcal{G}) = -n_-(\mathfrak{m}_{a,\mathcal{G}}) + \sum_{a<s<b} \operatorname{sign}(\mathfrak{m}_{s,\mathcal{G}}) + n_+(\mathfrak{m}_{b,\mathcal{G}}),$$

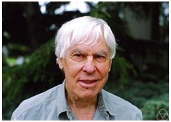
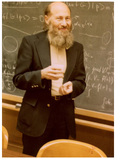
where n_+, n_- are the numbers of positive and negative squares, $\operatorname{sign} = n_+ - n_-$.

A Longer Introduction: the Evolution of the Sturm Theorem

My favorite evolution T-shirt

And evolution of my favorite theorem











Evolution of the Sturm Theorem: Sturm square (Dirichlet boundary conditions)

$L_s = -\partial_x^2 + V(x)$ in $L^2([0, s])$, Dirichlet boundary conditions, $s \in [\tau, 1]$.
When s changes the eigenvalues change.
Sturm Theorem: The number of negative eigenvalues (the Morse Index) is equal to the number of zeros of the eigenfunction corresponding to the zero eigenvalue (the conjugate points, the Maslov index).

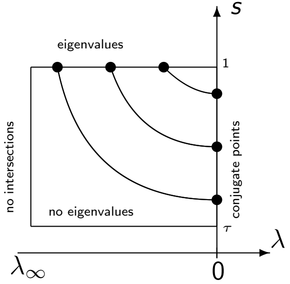


Figure: Dirichlet Sturm-Morse-Maslov square: - Morse index for $s = 1$ (3 eigenvalues) is equal to Maslov index for $\lambda = 0$ (3 conjugate points of negative signature)

Evolution of Sturm Theorem: Morse Square

Consider $L = -\partial_x^2 + V(x)$ in $L^2([0,1])^n$ with Dirichlet boundary conditions where $V(x)$ is a symmetric $(n \times n)$ matrix. Consider $L_s = -\partial_x^2 + V(x)$ in $L^2([0,s])^n$, Dirichlet boundary conditions, $s \in [\tau, 1]$.
Morse Index Theorem: The number of negative eigenvalues of L (the Morse index) is equal to the number of the *conjugate* points s (the Maslov index), that is, the points where the operator L_s has a nontrivial kernel.

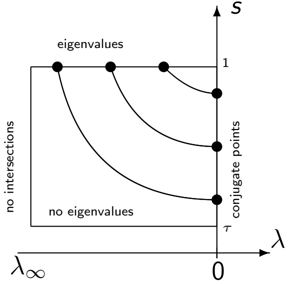
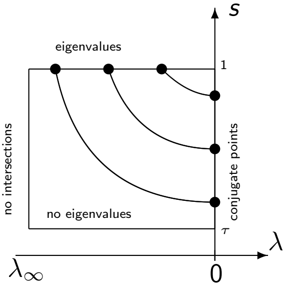


Figure: Dirichlet -Morse-Maslov square: - Morse index for $s = 1$ (3 eigenvalues) is equal to Maslov index for $\lambda = 0$ (3 conjugate points of negative signature)

Evolution of Sturm Theorem: Morse-Smale Square

Consider a family of elliptic second order differential operators L_s in $L^2(\Omega_s)$, where Ω_s is a family of manifolds, "shrinking" from the largest $\Omega = \Omega_1$ to the smallest Ω_τ, $s \in [\tau, 1]$.
Morse-Smale Index Theorem: The number of negative eigenvalues of L_1 (the Morse index) is equal to the number of the *conjugate* points s (the Maslov index), that is, the points where the operator L_s has a nontrivial kernel.



Sturm square (Neumann boundary conditions)

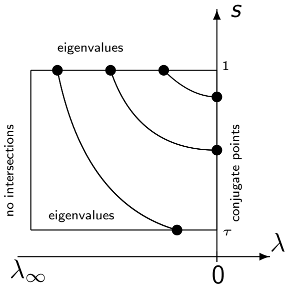


Figure: Neumann Sturm-Morse-Maslov diagramm: Morse index for $s = \tau$ (1 eigenvalue) - Morse index for $s = 1$ (3 eigenvalues) is equal to Maslov index for $\lambda = 0$ (2 conjugate points with negative signature)

Even more complicated Sturm squares

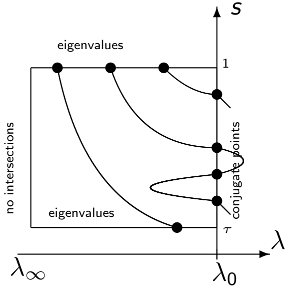


Figure: Neumann counting function-Maslov diagramm: number of eigenvalues smaller than λ_0 for $s = \tau$ (1 eigenvalue) - number of eigenvalues smaller than λ_0 for $s = 1$ (3 eigenvalue) is equal to Maslov index for $\lambda = \lambda_0$ (3 conjugate points with negative signature and 1 conjugate point with positive signature)

See Peter Howard and Alim Sukhtayev *JDE 2016* for more stunning pictures, they covered *all possible* boundary conditions on $[0, 1]$

Evolution of Sturm Theorem: Eigenvalue counting function and spectral flow

Let $\lambda_0 \in \mathbb{R}$, $s \in [\tau_1, \tau_2] \subset (0,1]$.

- Define the eigenvalue counting function

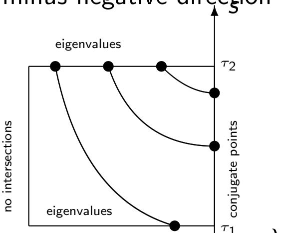
$$N(\lambda_0, s) = \sum_{\lambda \in \mathrm{Sp}(L_s), \lambda < \lambda_0} \dim\ker(L_s - \lambda) = \mathrm{card}\ \{\lambda \in \mathrm{Sp}(L_s) : \lambda < \lambda_0\}$$

  so that the Morse index $\mathrm{Mor}(L_s) = N(0, s)$

- Define the spectral flow through λ_0 of the family $\{L_s\}_{s\in[\tau_1,\tau_2]}$ by

$$\mathrm{sf}_{\lambda_0}(\{L_s\}_{s\in[\tau_1,\tau_2]}) = N(\lambda_0, \tau_1) - N(\lambda_0, \tau_2)$$

  as the net count of the eigenvalues of L_s passing through λ_0 in the positive minus negative direction when s changes from τ_1 to τ_2.



Evolution of Sturm Theorem: Morse, Maslov and the spectral flow

Spectral Flow Theorem

Maslov index = spectral flow= difference of Morse indices

[Booß-Bavnbek/Furutani, Cappell/Lee/Miller, Portaluri/Waterstraat,Robin/Salamon] and many more, [Arnold, Bott, Smale, Cox/Jones/Latushkin/A.Sukhtayev, Cox/Marzuola, Cox/Jones/Marzuola (2), Dalbono/Portaluri, Deng/Jones, Howard/A.Sukhtayev, Jones/Latushkin/Marangell, Jones/Latushkin/S.Sukhtaiev, Latushkin/A.Sukhtayev/S.Sukhtaiev] and many more.

Hadamard-type Formula

Derivative of the eigenvalue at the crossing point = the value of the Maslov crossing form

[Robin/Salamon,Cappell/Lee/Miller,Latushkin/A.Sukhtayev]

Morse and Maslov Indices for Partial Differential Operators

Multidimensional Partial Differential Operators

We will now formulate the Sturm-Morse-Maslov-Arnold-Smale-Robbin-Salamon Theorem for general multidimensional differential operators. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with smooth boundary. Assume that $a = \overline{a}, a_j, a_{jk} = \overline{a_{kj}}$ are contained in $C^\infty(\overline{\Omega})$, and that $\{a_{jk}\}_{j,k=1}^n$ is uniformly elliptic. Consider a formally self-adjoint differential expression

$$\mathcal{L} := -\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k + \sum_{j=1}^{n} a_j \partial_j - \partial_j \overline{a_j} + a. \tag{1}$$

Proposition: The linear operator defined by $Lf := \mathcal{L}f$, $f \in \mathrm{dom}(L) := C_0^\infty(\Omega)$, and considered in $L^2(\Omega)$ is closable. Its closure $\mathcal{L}_{min}$ is densely defined symmetric operator in $L^2(\Omega)$. Moreover, the linear operator acting in $L^2(\Omega)$ and given by $\mathcal{L}_{max} u := \mathcal{L}u$, $u \in \mathrm{dom}(\mathcal{L}_{max}) := \{u \in L^2(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$, is adjoint to $\mathcal{L}_{min}$, i.e., $(\mathcal{L}_{min})^* = \mathcal{L}_{max}$.

Assumption: We assume that the deficiency indices of $\mathcal{L}_{min}$ are equal:

$$\dim\ker(\mathcal{L}_{min} - i) = \dim\ker(\mathcal{L}_{min} + i).$$

Note: Can do for matrix a's and $Lip\ \Omega$.

Trace Operators, Green's Identity

The differential expression $\mathcal{L}$ is associated with two trace maps $\gamma_{D,\partial\Omega}$ and $\gamma^{\mathcal{L}}_{N,\partial\Omega}$ such that the second Green's identity holds

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = \overline{\langle \gamma^{\mathcal{L}}_{N,\partial\Omega} v, \gamma_{D,\partial\Omega} u\rangle}_{-1/2} - \langle \gamma^{\mathcal{L}}_{N,\partial\Omega} u, \gamma_{D,\partial\Omega} v\rangle_{-1/2}. \tag{2}$$

The Dirichlet trace map

$$\gamma_{D,\partial\Omega} \in \mathcal{B}(H^1(\Omega), H^{1/2}(\partial\Omega)), \quad \gamma_D u = u_{|_{\partial\Omega}}, u \in C(\overline{\Omega}). \tag{3}$$

The conormal derivative is defined by

$$\gamma^{\mathcal{L}}_{N,\partial\Omega} u := \sum_{j,k=1}^{n} a^{jk}\nu_j \gamma_{D,\partial\Omega}(\partial_k u) + \sum_{j=1}^{n} \overline{a_j}\nu_j \gamma_{D,\partial\Omega} u, \;\; u \in H^2(\Omega), \tag{4}$$

with $\nu = (\nu_1, \cdots, \nu_n)$ denoting the outward unit normal on $\partial\Omega$. It is further extended to a bounded operator $\gamma^{\mathcal{L}}_{N,\partial\Omega} \in \mathcal{B}(D^1_{\mathcal{L}}(\Omega), H^{-1/2}(\partial\Omega))$.
Note: we need

$$D^1_{\mathcal{L}}(\Omega) := \{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega)\}.$$

to be dense in the domain $\mathrm{dom}(\mathcal{L}_{max}) = \{u \in L^2(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$

Symplectic Form

Let us introduce the following complex symplectic bilinear form

$$\begin{aligned}&\omega : [H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)]^2 \to \mathbb{C},\\ &\omega((f_1, g_1), (f_2, g_2)) = \overline{\langle g_2, f_1\rangle}_{-1/2} - \langle g_1, f_2\rangle_{-1/2},\\ &(f_1, g_1), (f_2, g_2) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega).\end{aligned}$$

Then the second Green's identity reads as follows

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = \omega\left((\gamma_{D,\partial\Omega} u, \gamma^{\mathcal{L}}_{N,\partial\Omega} u), (\gamma_{D,\partial\Omega} v, \gamma^{\mathcal{L}}_{N,\partial\Omega} v)\right),$$

for all $u, v \in \mathcal{D}^1_{\mathcal{L}}(\Omega)$. We denote $\mathrm{Tr}_{\mathcal{L}}\, u := (\gamma_{D,\partial\Omega} u, \gamma^{\mathcal{L}}_{N,\partial\Omega} u)$.
We need $\mathrm{ran}\,\mathrm{Tr}_{\mathcal{L}}$ to be dense in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.
The *annihilator* of $\mathcal{F} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ is defined by

$$\mathcal{F}^\circ := \{(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) | \\ \omega((f, g), (\phi, \psi)) = 0, \text{ for all } (\phi, \psi) \in \mathcal{F}\}.$$

The subspace $\mathcal{F}$ is called **Lagrangian** if $\mathcal{F}^\circ = \mathcal{F}$

One-to-one Correspondence Between Self-Adjoint Operators and Lagrangian Planes

Recall

$$D^1_{\mathcal{L}}(\Omega) := \{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega)\}.$$

Theorem [S. Sukhtaiev/YL]

The **self-adjoint extensions** of $\mathcal{L}_{min}$ whose domains are contained in $H^1(\Omega)$ are in one-to-one correspondence with **Lagrangian planes** in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, that is, the following two assertions hold.

1) Let $\mathscr{D} \subset \mathcal{D}^1_{\mathcal{L}}(\Omega)$, and let $\mathcal{L}_{\mathscr{D}}$ be the linear operator acting in $L^2(\Omega)$ and given by the formula

$$\mathcal{L}_{\mathscr{D}} f := \mathcal{L}_{max} f, \ \ f \in \mathrm{dom}(\mathcal{L}_{\mathscr{D}}) := \mathscr{D}.$$

If $\mathcal{L}_{\mathscr{D}}$ is **self-adjoint** then the set

$$\mathcal{G}_{\mathscr{D}} := \overline{\mathrm{Tr}_{\mathcal{L}}(\mathscr{D})}^{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}$$

is a **Lagrangian** plane in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, with respect to form ω.

One-to-one Correspondence Between Self-Adjoint Operators and Lagrangian Planes, contd.

Theorem [S. Sukhtaiev/YL], contd.

2) A **Lagrangian** plane $\mathcal{G} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ defines a **self-adjoint** extension of $\mathcal{L}_{min}$. Namely, the linear operator $\mathcal{L}_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})}$ acting in $L^2(\Omega)$ and given by the formula

$$\mathcal{L}_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})} f := \mathcal{L}_{max} f,\ f \in \mathrm{dom}\left(\mathcal{L}_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})}\right) := \mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G}),$$

is essentially **self-adjoint**; here $\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})$ denotes the preimage of $\mathcal{G}$.

Examples:

- The operator $-\Delta_{\mathbf{D},\Omega}$ is associated with the Lagrangian plane $\{0\} \times H^{-1/2}(\partial\Omega)$
- The operator $-\Delta_{\mathbf{N},\Omega}$ is associated with the Lagrangian plane $H^{1/2}(\partial\Omega) \times \{0\}$
- 1-D θ-periodic Laplacian is associated with

$$\{(e^{i\theta}\alpha, \alpha, e^{i\theta}\beta, \beta,)^\top : \alpha, \beta \in \mathbb{C}\}.$$

Examples, contd. (a popular way of defining PDE operators)

• Let $\mathcal{X}$ be a closed subspace in $H^1(\Omega)$. Assume $H^1_0(\Omega) \subset \mathcal{X} \subset H^1(\Omega)$. Suppose that the form $\mathfrak{l} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{C}$, $\mathrm{dom}(\mathfrak{l}) := \mathcal{X} \times \mathcal{X}$ is closed and bounded from below in $L^2(\Omega)$.

• Then there exists a unique self-adjoint operator $\mathcal{L}_{\mathcal{X}}$ acting in $L^2(\Omega)$ such that $\mathfrak{l}[u, v] = \langle \mathcal{L}_{\mathcal{X}} u, v\rangle_{L^2(\Omega)}$ for all $v \in \mathcal{X}$ and $u \in \mathrm{dom}(\mathcal{L}_{\mathcal{X}}) := \big\{u \in \mathcal{X} :$ there exists $w \in L^2(\Omega)$ such that $\langle w, v\rangle_{L^2(\Omega)} = \mathfrak{l}[u, v]$ for all $v \in \mathcal{X}\big\}$.

Proposition [S. Sukhtaiev/YL]

The subspace
$$\mathcal{G}_{\mathcal{X}} := \{(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : f \in \gamma_{D,\partial\Omega}(\mathcal{X}), \langle g, \gamma_{D,\partial\Omega} w\rangle_{-1/2} = 0 \text{ for all } w \in \mathcal{X}\}$$
is Lagrangian in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Moreover, $\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G}_{\mathcal{X}})$ is a core of $\mathcal{L}_{\mathcal{X}}$.

• Thus $\mathcal{L}_{\mathcal{X}}$ above is associated with $\mathcal{G}_{\mathcal{X}}$ as indicated in the theorem; e.g., $\mathcal{X} = H^1_0(\Omega)$ and $\mathcal{G}_{\mathcal{X}} = \{0\} \times H^{-1/2}(\partial\Omega)$ produce the operator with the Dirichlet while $\mathcal{X} = H^1(\Omega)$ and $\mathcal{G}_{\mathcal{X}} = H^{-1/2}(\partial\Omega) \times 0$ with Neumann boundary conditions

Lagrangian planes and self-adjoint operators: proofs/ideas

• If $\mathcal{L}_{\mathcal{D}}$ is self-adjoint then $\mathcal{G}_{\mathcal{D}} \subset \mathcal{G}_{\mathcal{D}}^{\circ}$ is isotropic by Green's identity
$\langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = \omega\Big((\gamma_{D,\partial\Omega}u, \gamma_{N,\partial\Omega}^{\mathcal{L}}u), (\gamma_{D,\partial\Omega}v, \gamma_{N,\partial\Omega}^{\mathcal{L}}v)\Big)$.
Co-isotropic $\mathcal{G}_{\mathcal{D}}^{\circ} \subset \mathcal{G}_{\mathcal{D}}$ because $\mathcal{G}_{\mathcal{D}}^{\circ} \cap \mathrm{Tr}_{\mathcal{L}}(\mathcal{D}_{\mathcal{L}}^1(\Omega)) \subset \mathcal{G}_{\mathcal{D}}$ using Green's identity and because $\mathrm{Tr}_{\mathcal{L}}(\mathcal{D}_{\mathcal{L}}^1(\Omega))$ is dense in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$; here
$D_{\mathcal{L}}^1(\Omega) := \{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$.

• If $\mathcal{G}$ is Lagrangian then the following lemma holds (inspired by [Booss-Bavnbek/Furutani]):

Lemma. The graph-norm closure of the subspace
$[\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})] = \big\{[x] : x \in \mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})\big\}$ is Lagrangian in $\mathrm{dom}\,\mathcal{L}_{max}/\,\mathrm{dom}\,\mathcal{L}_{min}$ with $[\cdot]$ being the quotient map and $\widetilde{\omega}([x],[y]) = \langle \mathcal{L}_{max}x, y\rangle_{L^2(\Omega)} - \langle x, \mathcal{L}_{max}y\rangle_{L^2(\Omega)}$.

• Then the closure of $\mathcal{L}_{max}\big|_{\mathrm{Tr}_{\mathcal{L}}^{-1}(\mathcal{G})}$ is self-adjoint by the classical theory [Birman-Krein-Vishik-Alonso-Simon]

Family of Differential Operators

Let us consider a family of differential expressions

$$\mathcal{L}^t := -\sum_{j,k=1}^{n} \partial_j a_{jk}^t \partial_k + \sum_{j=1}^{n} a_j^t \partial_j - \partial_j \overline{a_j^t} + a^t,\ t \in \mathcal{I} := [\alpha, \beta],$$

- $a_{jk} : t \mapsto a_{jk}^t,\ a_{jk} \in C^1(\mathcal{I}, L^\infty(\Omega)),\ a_{jk}^t(x) = \overline{a_{kj}^t(x)},\ 1 \le j \le n, x \in \overline{\Omega},$
- $a_{jk}^t(x)\xi_k\overline{\xi_j} \ge c \sum_{j=1}^{n} |\xi_j|^2$ for all $x \in \overline{\Omega}, \xi = (\xi_j)_{j=1}^n \in \mathbb{C}^n, t \in \mathcal{I}$; for some $c > 0$,
- $a_j : t \mapsto a_j^t,\ a_j \in C^1(\mathcal{I}, L^\infty(\Omega)),\ 1 \le j \le n,$
- $a : t \mapsto a^t,\ a \in C^1(\mathcal{I}, L^\infty(\Omega)),\ a^t(x) \in \mathbb{R},\ x \in \Omega,\ t \in \mathcal{I}.$

Minimal and Maximal Operators, Weak Solutions

- The minimal and maximal operators are defined as follows

$$\mathcal{L}^t_{min} f = \mathcal{L}^t f,\ f \in \mathrm{dom}(\mathcal{L}^t_{min}) := H^2_0(\Omega).$$

$$\mathcal{L}^t_{max} u := \mathcal{L}^t u,\ u \in \mathrm{dom}(\mathcal{L}^t_{max}) := \{u \in L^2(\Omega) : \mathcal{L}^t u \in L^2(\Omega)\}.$$

- $\mathcal{K}_{\lambda,t} := (\gamma_D, \gamma_N^{\mathcal{L}^t})\Big(\big\{w \in H^1(\Omega) : \mathcal{L}^t w - \lambda w = 0 \text{ weakly}\big\}\Big)$, is Lagrangian in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$,
- the path $s \mapsto \mathcal{K}_{\lambda(s),t(s)}$ is continuous (we pick some $s \mapsto \lambda(s), t(s)$)
- Define $\mathcal{L}^t_{\mathscr{D}_t} u := \mathcal{L}^t u,\ u \in \mathrm{dom}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) := \mathscr{D}_t$,. If $\mathcal{L}^t_{\mathscr{D}_t} - \lambda$ is Fredholm operator then the pair $(\mathcal{K}_{\lambda,t}, \mathcal{G}_t)$ is Fredholm, where $\mathcal{G}_t := \overline{\mathrm{Tr}_{\mathcal{L}^t}(\mathscr{D}_t)}$
- $\dim\left(\mathcal{K}_{\lambda,t} \cap \mathcal{G}_t\right) = \dim\ker\left(\mathcal{L}^t_{\mathscr{D}_t} - \lambda\right)$

Main Result

Theorem (YL, S. Sukhtaiev)

Let $\mathscr{D}_t \subset \mathcal{D}^1_{\mathcal{L}^t}(\Omega)$, $t \in \mathcal{I}$, and assume that the linear operator $\mathcal{L}^t_{\mathscr{D}_t}$ acting in $L^2(\Omega)$ and given by

$$\mathcal{L}^t_{\mathscr{D}_t} u := \mathcal{L}^t u,\ \ u \in \text{dom}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) := \mathscr{D}_t,$$

is self-adjoint with the property $\text{Sp}_{ess}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) \cap (-\infty, 0] = \emptyset,$ for all $t \in \mathcal{I}$. Assume further that there exists $\lambda_\infty < 0$, such that

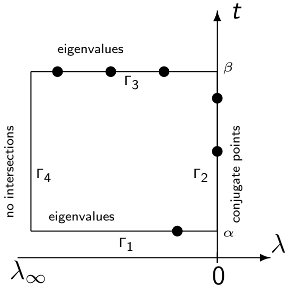
$$\ker\left(\mathcal{L}^t_{\mathscr{D}_t} - \lambda\right) = \{0\},\ \text{for all } \lambda \leq \lambda_\infty, t \in \mathcal{I}.$$

Suppose, finally, that the path $t \mapsto \mathcal{G}_t := \overline{\text{Tr}_{\mathcal{L}^t}\left(\mathscr{D}_t\right)},\ t \in \mathcal{I}$, is contained in $C\left(\mathcal{I}, \Lambda\left(H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\right)\right)$.
Then

$$\text{Mor}\left(\mathcal{L}^\alpha_{\mathscr{D}_\alpha}\right) - \text{Mor}\left(\mathcal{L}^\beta_{\mathscr{D}_\beta}\right) = \text{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right),$$

where $\mathcal{K}_{\lambda,t} := (\gamma_D, \gamma_N^{\mathcal{L}^t})\Big(\left\{w \in H^1(\Omega) : \mathcal{L}^t w - \lambda w = 0 \text{ weakly}\right\}\Big),\ t \in \mathcal{I}.$

Illustrations



- 1D $\theta-$periodic Schrödinger operator with matrix-valued potential on a bounded interval. Variation of θ.
- multidimensional $\vec{\theta}-$periodic Schrödinger operator on a cube. Scaling the cube.
- Schrödinger operator with Robin-type boundary conditions on one-parameter family of domains $\Omega_t \subset \mathbb{R}^n$. Variation of t.
- Maslov index for operators on graphs.

Maslov=Morse: ideas of the proofs

• The main new point: [BB/F, BB/Zhu] used abstract traces in $\operatorname{dom}\mathcal{L}_{max}/\operatorname{dom}\mathcal{L}_{min}$ of strong solutions; needed $\operatorname{dom}\mathcal{L}^t_{\mathscr{D}_t}\subset H^2(\Omega)$.
We are using PDE traces of weak solutions; need $\operatorname{dom}\mathcal{L}^t_{\mathscr{D}_t}\subset H^1(\Omega)$.
• The main part of the proof: $\operatorname{Mas}(\mathcal{K}_{\alpha,\lambda}\big|_{\lambda\in\Gamma_1},\mathcal{G}_\alpha)=-\operatorname{Mor}\mathcal{L}^\alpha_{\mathscr{D}_\alpha}$ and similarly for Γ_3 because there are no conjugate points on Γ_4.
• To prove the main part, fix a conjugate point $\lambda(s_*)\in(\lambda_\infty,0)$ such that $\mathcal{K}_{\lambda(s_*),\alpha}\cap\mathcal{G}_\alpha\neq\{0\}$. We know [Cox/Jones/Marzuola] that $s\mapsto\mathcal{K}_{\lambda(s),\alpha}$ is smooth; pick a smooth $R_{s+s_*}:\mathcal{K}_{\lambda(s_*),\alpha}\to(\mathcal{K}_{\lambda(s_*),\alpha})^\perp$ for small s with $R_{s_*}=0$ so that $\mathcal{K}_{\lambda(s),\alpha}=\big\{(\phi,\psi)+R_{s+s_*}(\phi,\psi):(\phi,\psi)\in\mathcal{K}_{\lambda(s_*),\alpha}\big\}$. Let u_0 be the eigenfunction of $\mathcal{L}^\alpha_{\mathscr{D}_\alpha}$ so that $\mathcal{L}^\alpha_{\mathscr{D}_\alpha}u_0=\lambda(s_*)u_0$ and let $(\phi_s,\psi_s)=\operatorname{Tr}_{\mathcal{L}^\alpha}u_0+R_{s+s_*}\operatorname{Tr}_{\mathcal{L}^\alpha}u_0$
Lemma. There is a continuous $s\mapsto u_s\in H^1(\Omega)$ such that $\mathcal{L}^\alpha_{\mathscr{D}_\alpha}u_s=\lambda(s)u_s$ weakly, $\operatorname{Tr}_{\mathcal{L}^\alpha}u_s=(\phi_s,\psi_s)$ for small $|s|$ and

$$\|u_s-u_0\|_{H^1(\Omega)}\le c\|\operatorname{Tr}_{\mathcal{L}^\alpha}(u_s-u_0)\|_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}.$$

• Using Green's identity: $\omega((\phi_0,\psi_0),R_{s+s_*}(\phi_0,\psi_0))=-\langle u_0,su_s\rangle_{L^2(\Omega)}$
Then (Maslov) crossing form $\mathfrak{m}_{s_*}((\phi_0,\psi_0),(\phi_0,\psi_0))=$
$\frac{d}{ds}\big|_{s=0}\omega((\phi_0,\psi_0),R_{s+s_*}(\phi_0,\psi_0))=\lim -s^{-1}\langle u_0,su_s\rangle_{L^2(\Omega)}=-\|u_0\|^2_{L^2(\Omega)}$. BINGO!

Hadmard-type Formulas for the Derivatives of Eigenvalues

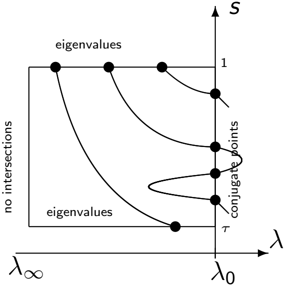
Derivatives of the eigenvalues

A general question: Given a family of differential operators, how would the eigenvalues change with s? How do they cross through a fixed point λ_0 when the parameter changes?

• Given a family $\{L_s\}_{s\in[a,b]}$ of differential operators with an eigenvalue $\lambda_0 = \lambda(s_0)$ at a conjugate point $s_0 \in [a, b]$ we compute $\frac{d\lambda(s_0)}{ds}$.

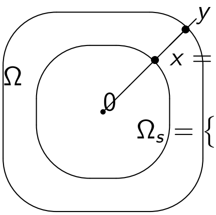
• If $\frac{d\lambda(s_0)}{ds} > 0$, respectively, $\frac{d\lambda(s_0)}{ds} < 0$ then the eigenvalue crosses λ_0 in the positive, respectively, negative direction as s changes from a to b.

• One can relate $\frac{d\lambda(s_0)}{ds}$ and the value of the (Maslov) crossing form. This eventually leads to the formula "Maslov Index=Spectral Flow".



Hadamard's formula for the eigenvalues

• L_s are elliptic multidimensional operators on domains $\Omega_s \subset \mathbb{R}^d$ depending on a parameter. If $\lambda_1(s), \lambda_2(s), \ldots$ are the eigenvalues, how to compute $\frac{d\lambda_j}{ds}$?
• Finding $\frac{d\lambda_j}{ds}$ is a classical problem: [Rayleigh1894], [Hadamard1908], [Garabedian/Schiffer1952], [Henry2005], [Burenkov/Lamberti/deCristotoris], [Grinfeld].
• We consider the case of star-shaped domains centered at zero.



$x = sy \in \partial\Omega_s$, L_s is $-\Delta + V$ restricted to Ω_s

$$\Omega_s = \left\{x = \sigma y \in \Omega : \sigma \in [0, s), y \in \partial\Omega\right\},\ s \in [\tau, 1], \tau \in (0, 1)$$

Maslov versus Hadamard

We computed the Hadamard derivatives $\frac{d\lambda_j}{ds}$ via the (Maslov) crossing form. It implies the "Maslov index=Spectral Flow" formula as follows.

• Suppose $\lambda_0 = \lambda(s_0)$ is an eigenvalues of the differential operator L_{s_0} of multiplicity $m = m(s_0)$. Then $\lambda_0 \in \mathrm{Sp}(L_{s_0})$ if and only if $\Upsilon(s_0) \cap \mathcal{G} \neq \{0\}$ for a given Lagrangian subspace $\mathcal{G}$ responsible for the boundary conditions and a C^1-path $\Upsilon : [a, b] \to F\Lambda(\mathcal{G})$ taking values in the set of Lagrangian planes forming with $\mathcal{G}$ Fredholm pairs; $\Upsilon(s) = \mathrm{Tr}_s\left(\{u : (L_s - \lambda(s))u = 0\}\right)\left(= \mathrm{Tr}_s\left(\mathcal{K}_{\lambda,s}\right)\right)$.

• One considers the (Maslov) crossing form $\mathfrak{m}_{s_0}$ on the finite dimensional subspace $\Upsilon(s_0) \cap \mathcal{G}$. Pick a basis $\{q_j\}_{j=1}^m$ in $\Upsilon(s_0) \cap \mathcal{G}$.

• Let $\lambda_j(s)$, $j = 1, \ldots, m$, be the eigenvalues of L_s for s near s_0.

[A. Sukhtayev/YL] proved the "Hadamard vs Maslov" derivative formula

$$\frac{d\lambda_j(s_0)}{ds} = \mathfrak{m}_{s_0}(q_j, q_j), \quad j = 1, \ldots, m.$$

• If s_0 is the only conjugate point in $[a, b]$ and the form $\mathfrak{m}_{s_0}$ is non degenerate and has $n_+(\mathfrak{m}_{s_0})$ positive and $n_-(\mathfrak{m}_{s_0})$ negative squares, then the spectral flow is

$$sf_{\lambda_0}(\{L_s\}_{s\in[a,b]}) = n_+(\mathfrak{m}_{s_0}) - n_-(\mathfrak{m}_{s_0}) = \mathrm{Mas}(\Upsilon(\cdot)|_{[a,b]}, \mathcal{G})$$

since n_+ eigenvalues move through λ_0 to the right and n_- to the left as s increases

Maslov vs Hadamard: simple Dirichlet eigenvalues

Ω is a star-shaped domain;

$\Omega_s = \big\{x = \sigma y \in \Omega : \sigma \in [0, s), y \in \partial\Omega\big\}$, $s \in [\tau, 1], \tau \in (0, 1)$

$L := -\Delta + V$ in $L^2(\Omega)$, Dirichlet boundary conditions;

$L_s := -\Delta + V$ in $L^2(\Omega_s)$, $s \in [\tau, 1], \tau \in (0, 1)$, Dirichlet boundary conditions;

$\widehat{L}_s := -\Delta + V^s$ in $L^2(\Omega)$, $V^s(x) := s^2 V(sx)$, $x \in \Omega$, Dirichlet boundary conditions; thus $\widehat{L}_s$ is L_s pulled back to $L^2(\Omega)$.

Define: $\mathcal{K}_{\lambda,s} := \big\{u \in H^1(\Omega) : -\Delta u + V^s u - s^2\lambda u = 0\big\}$ for $\lambda \in \mathbb{R}, s \in [\tau, 1]$;

$\mathcal{G} := \big\{(0, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : g \in H^{-1/2}(\partial\Omega)\big\}$, the Dirichlet subspace,

$\mathrm{Tr}_s = (\gamma_D, \frac{1}{s}\gamma_N) : H^1(\Omega) \to H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ the rescaled trace

Then $\lambda_0 \in \mathrm{Sp}\left(L_{s_0}; L^2(\Omega_{s_0})\right)$ if and only if $s_0^2\lambda_0 \in \mathrm{Sp}\left(\widehat{L}_{s_0}; L^2(\Omega)\right)$ if and only if $\mathrm{Tr}_{s_0}\left(\mathcal{K}_{\lambda_0,s_0}\right) \cap \mathcal{G} \neq \{0\}$

Assume: $\lambda_0 = \lambda(s_0)$ is a simple eigenvalue of L_{s_0}, u_{s_0} is the eigenfunction of $\widehat{L}_{s_0}$. Take $\lambda(s)$, simple eigenvalues of L_s with the eigenfunctions u_s of $\widehat{L}_s$ for s near s_0.

Maslov vs Hadamard: simple Dirichlet eigenvalues

Claim: $\frac{d\lambda(s_0)}{ds} = \frac{1}{s_0}\mathfrak{m}_{s_0}\big(\operatorname{Tr}_{s_0} u_{s_0}, \operatorname{Tr}_{s_0} u_{s_0}\big)$,

where $\mathfrak{m}_{s_0}(p,q) = \omega(p, \frac{dR(s_0)}{ds}q)$ is the (Maslov) crossing form on $\operatorname{Tr}_{s_0}\big(\mathcal{K}_{\lambda_0,s_0}\big) \cap \mathcal{G}$ for the flow $\Upsilon : s \mapsto \operatorname{Tr}_s\big(\mathcal{K}_{\lambda_0,s}\big)$ in $F\Lambda(\mathcal{G})$ of $\mathcal{H} := H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ with the symplectic form $\omega\big((f_1,g_1),(f_2,g_2)\big) = \langle g_2, f_1\rangle_{1/2} - \langle g_1, f_2\rangle_{1/2}$ and $\Upsilon(s) = \operatorname{graph} R(s)$ for $R(s) : \operatorname{ran} P_{s_0} \to \ker P_{s_0}$ where P_{s_0} is the orthogonal projection in $\mathcal{H}$ onto $\Upsilon(s_0)$. Fix $q = \operatorname{Tr}_{s_0} u_{s_0} \in \Upsilon(s_0)$.

(A little lemma:) There exists a smooth family $s \mapsto w_s \in \mathcal{K}_{\lambda_0,s}$ such that $q + R(s)q = \operatorname{Tr}_s w_s$ and $w_{s_0} = u_{s_0}$. Therefore (a little calculation), $\mathfrak{m}_{s_0}(q,q) = \omega\big(\operatorname{Tr}_{s_0} w_{s_0}, \frac{d}{ds}\big(\operatorname{Tr}_s w_s\big)\big|_{s=s_0}\big) = -\frac{1}{s_0}\langle \gamma_{N,\partial\Omega} u_{s_0}, \gamma_{D,\partial\Omega}\big(\frac{dw_s}{ds}\big|_{s=s_0}\big)\rangle_{1/2}$

Since $w_s \in \mathcal{K}_{\lambda_0,s}$, we have $-\Delta w_s + V^s w_s - s^2\lambda_0 w_s = 0$. We s-differentiate, $L^2(\Omega)$-multiply by w_s, use Green's formula to get $\langle \gamma_{N,\partial\Omega} w_{s_0}, \gamma_{D,\partial\Omega}\big(\frac{dw_s}{ds}\big|_{s=s_0}\big)\rangle_{1/2} + \langle \frac{dV^s}{ds}\big|_{s=s_0} w_{s_0}, w_{s_0}\rangle_{L^2(\Omega)} - 2s_0\lambda_0 = 0.$

Since u_s is the normalized eigenfunction we have $-\Delta u_s + V^s u_s = s^2\lambda(s) u_s$. We s-differentiate, $L^2(\Omega)$-multiply by u_s, use that $\widehat{L_s}$ is self-adjoint to get $\langle \frac{dV^s}{ds}\big|_{s=s_0} u_{s_0}, u_{s_0}\rangle_{L^2(\Omega)} = 2s_0\lambda_0 + s_0^2\frac{d\lambda(s_0)}{ds}$. Combining the above proves the claim.

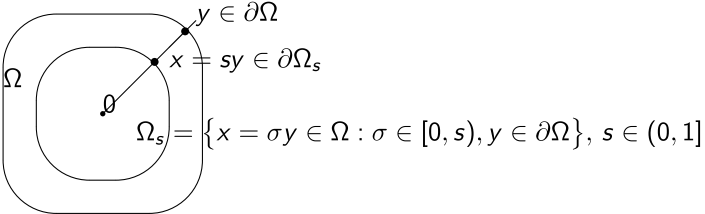
Schrödinger operator: Robbin BCs, star-shaped domains

• $\Omega \subset \mathbb{R}^d$ is a Lipshitz star-shaped domain centered at zero, $\Omega_s = \{x = \sigma y \in \Omega : \sigma \in [0, s), y \in \partial\Omega\}$, $s \in (0, 1]$, are subdomains in $\Omega = \Omega_1$, and $V(\cdot) = V(\cdot)^\top$ is a continuous $(n \times n)$ matrix function.

• $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is a given compact selfadjoint operator. Need more assumptions on Θ, very technical, will skip this.

• Consider $L = -\Delta + V$ on $L^2(\Omega)$ with $\operatorname{dom} L = \{u \in H^1(\Omega) :$ u satisfies general Robbin boundary conditions $\gamma_{N,\partial\Omega} u = \Theta\gamma_{D,\partial\Omega} u$ on $\partial\Omega\}$

• $L_s u = -\Delta u + V(x)u$, $x \in \Omega_s$, $s \in [\tau, 1]$, $\tau \in (0, 1)$, acts in $L^2(\Omega_s)$, satisfies Robbin boundary conditions on $\partial\Omega_s$.



Schrödinger operator: star-shaped domains

• Denote $(U_s^{\partial}h)(x) = s^{(d-1)/2}h(sy)$, $y \in \partial\Omega$,
$(U_{1/s}^{\partial}f)(z) = s^{-(d-1)/2}f(s^{-1}z)$, $z \in \partial\Omega_s$,
$(U_s w)(x) = s^{d/2}w(sx)$, $x \in \Omega$; $\Theta_D = \gamma_{D,\partial\Omega}^{*}\Theta\gamma_{D,\partial\Omega} \in \mathcal{B}(H^1(\Omega), (H^1(\Omega))^*)$.
• Rescaling $\widehat{L}_s$ of L_s from $L^2(\Omega_s)$ onto $L^2(\Omega)$ is given by
$\widehat{L}_s v = -\Delta v + V^s(x)v(x)$, $V^s(x) = s^2 V(sx)$, $x \in \Omega$, with Robbing boundary conditions $\gamma_{N,\partial\Omega}u = \Theta\gamma_{D,\partial\Omega}u$ on $\partial\Omega$.
• $\mathcal{K}_{\lambda,s}$, $\lambda \in \mathbb{R}$, $s \in (0,1]$, the subspace of $H^1(\Omega)$ of the weak solutions of the rescaled eigenvalue equation $-\Delta v + V_{\lambda,s}(x)v = 0$, where $V_{\lambda,s}(x) = V^s(x) - s^2\lambda$, $x \in \Omega$.
• Flow $\Upsilon_\lambda : s \mapsto \mathrm{tr}_s(\mathcal{K}_{\lambda,s})$ of Lagrangian subspaces in the boundary space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ with $\omega\big((f_1, g_1), (f_2, g_2)\big) = \langle g_2, f_1\rangle_{1/2} - \langle g_1, f_2\rangle_{1/2}$, where $\mathrm{tr}_s(v) = (\gamma_{D,\partial\Omega}v, \frac{1}{s}\gamma_{N,\partial\Omega}v)$ is the rescaled trace.
• A Lagrangian subspace $\mathcal{G} = \{(f, \Theta f) : f \in H^{1/2}(\partial\Omega)\}$ corresponds to the Robbin boundary conditions.
• Conclusion: $\lambda_0 \in \mathrm{Sp}(L_{s_0})$ if and only if $\mathrm{tr}_{s_0}(\mathcal{K}_{\lambda_0,s_0}) \cap \mathcal{G} \neq \{0\}$.

Perturbation theory for Schrödinger operators on star-shaped domains

A general finite dimensional reminder from [Kato]:
Assume $T(s) = \lambda^{s_0} I_{m\times m} + (s-s_0)T^{(1)} + (s-s_0)^2 T^{(2)} + o(s-s_0)^2$ as $s \to s_0$ is a family of m-dimensional operators where $\lambda^{s_0} \in \mathrm{Sp}(T_{s_0})$ has multiplicity m.
Let $\lambda_j^{(1)}$, $j = 1, \dots, m$, denote m eigenvalues of $T^{(1)}$ (may be repeated). Then $\lambda_j^s \in \mathrm{Sp}(T(s))$ are given as follows:

$$\lambda_j^s = \lambda^{s_0} + (s-s_0)\lambda_j^{(1)} + o(s-s_0) \text{ as } s \to s_0$$

Back to Schrödinger operators on $L^2(\Omega)$ and $L^2(\Omega_s)$ with Robbin BCs.
Assume $\lambda_0 = \lambda(s_0) \in \mathrm{Sp}(L_{s_0})$ is an eigenvalue of L_{s_0} on $L^2(\Omega_s)$ of multiplicity m.
Consider eigenvalues $\lambda_j(s) \in \mathrm{Sp}(L_s)$, $j = 1, \dots, m$, of L_s for s near s_0.
Objective: Want asymptotic formulas for $\lambda_j(s)$ or for $\lambda_j^s = s^2\lambda_j(s) \in \mathrm{Sp}(\widehat{L_s})$ as above.

Perturbation theory for Schrödinger operators on star-shaped domains: the main lemma

- Let $\lambda_0 = \lambda(s_0)$, $s_0 \in (0,1]$, be a fixed eigenvalue of L_{s_0} of multiplicity m and let $\lambda_j(s)$, $j = 1, \dots, m$, be the eigenvalues of L_s for s near s_0.
- Denote: P^{s_0}, the Riesz projection in $L^2(\Omega)$ for $\widehat{L}_{s_0}$ onto $\ker(\widehat{L}_{s_0} - s_0^2\lambda_0 I_{L^2(\Omega)})$
- P^s, the Riesz projection in $L^2(\Omega)$ for $\widehat{L}_s$ that corresponds to the eigenvalues $\lambda_j^s = s^2\lambda_j(s)$ of $\widehat{L}_s$ that bifurcate from the eigenvalue $\lambda^{s_0} = s_0^2\lambda(s_0)$ of $\widehat{L}_{s_0}$.
- Introduce m-dimensional operators $T^{(1)}$ and $T^{(2)}$ acting in $\operatorname{ran} P^{s_0}$ by $T^{(1)} = P^{s_0}\big(\dot{V}^s\big|_{s=s_0} - \Theta_D\big)P^{s_0}$, where $\Theta_D = \gamma^*_{D,\partial\Omega}\Theta\gamma_{D,\partial\Omega} \in \mathcal{B}(H^1(\Omega), (H^1(\Omega))^*)$;
$T^{(2)} =$
$P^{s_0}\big(\frac{1}{2}\ddot{V}^s\big|_{s=s_0} - \dot{V}^s\big|_{s=s_0}S\dot{V}^s\big|_{s=s_0} - \Theta_D S\Theta_D + \dot{V}^s\big|_{s=s_0}S\Theta_D + \Theta_D S\dot{V}^s\big|_{s=s_0}\big)P^{s_0}$;
$S = (2\pi i)^{-1}\int_\Gamma(\zeta - s_0^2\lambda_0)^{-1}(\widehat{L}_{s_0} - \zeta)^{-1}\,d\zeta$ is the reduced resolvent of $\widehat{L}_{s_0}$.

Main Lemma:
For s near s_0 the restriction $\widehat{L}_s\big|_{\operatorname{ran} P^s}$ acting in $\operatorname{ran} P^s$ is similar to the operator $T(s) = \lambda^{s_0}P^{s_0} + (s - s_0)T^{(1)} + (s - s_0)^2T^{(2)} + o(s - s_0)^2$ as $s \to s_0$ on $\operatorname{ran} P^{s_0}$.
Hence, $\lambda_j^s = \lambda^{s_0} + (s - s_0)\lambda_j^{(1)} + o(s - s_0)$ as $s \to s_0$ where $\lambda_j^{(1)} \in \operatorname{Sp}(T^{(1)})$.

Maslov vs Hadamard: general Robbin eigenvalues

- Let $\lambda_0 = \lambda(s_0)$, $s_0 \in (0, 1]$, be a fixed eigenvalue of L_{s_0} of multiplicity m and let $\lambda_j(s)$, $j = 1, \dots, m$, be the eigenvalues of L_s for s near s_0.
- Consider the normalized basis $\{u_j^{s_0}\}_{j=1}^m$ of $\mathrm{ran}(P^{s_0})$ consisting of the eigenvectors of $T^{(1)}$ and corresponding to its eigenvalues $\lambda_j^{(1)}$ so that $T^{(1)} u_j^{s_0} = \lambda_j^{(1)} u_j^{s_0}$, $j = 1, \dots, m$, and denote by $q_j = (\gamma_{D,\partial\Omega} u_j^{s_0}, \frac{1}{s_0}\gamma_{N,\partial\Omega} u_j^{s_0})$ the respective rescaled traces located in $\Upsilon_{\lambda_0}(s_0) \cap \mathcal{G}$.

Theorem (Hadamard-type formula for $L_s = -\Delta + V$ in $L^2(\Omega_s)$)

$$\frac{d\lambda_j(s)}{ds}\Big|_{s=s_0} = \frac{1}{s_0^2}\big(\lambda_j^{(1)} - 2s_0\lambda(s_0)\big) = \frac{1}{s_0}\mathfrak{m}_{s_0}(q_j, q_j),\; j = 1, \dots, m,$$

where the (Maslov) crossing form $\mathfrak{m}_{s_0}$ for the path $s \mapsto \Upsilon_{\lambda(s_0)}(s) = tr_s\big(\mathcal{K}_{\lambda(s_0),s}\big)$ at s_0 can be computed as follows:

$$\mathfrak{m}_{s_0}(q_j, q_j) = \frac{1}{s_0}\langle \dot{V}_{\lambda(s_0),s}\big|_{s=s_0} u_j^{s_0}, u_j^{s_0}\rangle_{L^2(\Omega)} - \frac{1}{s_0^2}\langle \gamma_{N,\partial\Omega} u_j^{s_0}, \gamma_{D,\partial\Omega} u_j^{s_0}\rangle_{1/2}.$$

Hadamard-type formula via the Maslov form

The "infinitesimal" formula relating the derivatives of the eigenvalues and the value of the crossing form implies the relations between the Maslov index, the spectral flow, and the Morse index.

Theorem (Hadamard-type formula for $L_s = -\Delta + V$ in $L^2(\Omega_s)$)

If $\mathrm{dom}(\widehat{L}_s) \subset H^2(\Omega)$ *and the strong trace* $\gamma^s_{N,\partial\Omega} u_j^{s_0}$ *is defined then*

$$\frac{d\lambda_j(s)}{ds}\Big|_{s=s_0} = \frac{1}{s_0}\mathfrak{m}_{s_0}(q_j, q_j)$$

$$= \frac{1}{(s_0)^3}\int_{\partial\Omega}\Big(\big(\nabla u_j^{s_0}\cdot\nabla u_j^{s_0}\big)(\nu\cdot x) - 2\langle\nabla u_j^{s_0}\cdot x, \gamma^{\mathrm{s}}_{N,\partial\Omega} u_j^{s_0}(x)\rangle_{\mathbb{R}^N}$$

$$+ (1-d)\langle\gamma^{\mathrm{s}}_{N,\partial\Omega} u_j^{s_0}(x), u_j^{s_0}(x)\rangle_{\mathbb{R}^N} + \langle(V(s_0x)-\lambda(s_0))u_j^{s_0}(x), u_j^{s_0}(x)\rangle_{\mathbb{R}^N}(\nu\cdot x)\Big)\,dx.$$

For the Dirichlet case:

$$\frac{d\lambda_j(s)}{ds}\Big|_{s=s_0} = \frac{1}{s_0}\mathfrak{m}_{s_0}(q_j, q_j) = -\frac{1}{(s_0)^3}\int_{\partial\Omega}\|\gamma^{\mathrm{s}}_{N,\partial\Omega} u_j^{s_0}\|^2_{\mathbb{R}^N}(\nu\cdot x)\,dx < 0.$$

Asymptotic formula up to quadratic terms

Renumber the eigenvalues of the operator $T^{(1)}$, and let $\{\lambda_i^{(1)}\}_{i=1}^{m'}$ denote the m' *distinct* eigenvalues of the operator $T^{(1)}$, so that $m' \le m$, let $m_i^{(1)}$ denote their multiplicities, and let $P_i^{(1)}$ denote the orthogonal Riesz spectral projections of $T^{(1)}$ corresponding to the eigenvalue $\lambda_i^{(1)}$. Let $\lambda_{ik}^{(2)}$, $i = 1, \ldots, m'$, $k = 1, \ldots, m_i^{(1)}$, denote the eigenvalues of the operator $P_i^{(1)} T^{(2)} P_i^{(1)}$ in $\mathrm{ran}(P_i^{(1)})$. Renumber the eigenvalues $\lambda_j(s)$ of L_s as $\lambda_{ik}(s)$.

Theorem (Quadratic asymptotic formula)

$$\begin{aligned}\lambda_{ik}(s) &= \lambda(s_0) + \Big(\frac{1}{s_0^2}\lambda_i^{(1)} - \frac{2}{s_0}\lambda(s_0)\Big)(s - s_0) \\ &\quad + \Big(\frac{1}{s_0^2}\lambda_{ik}^{(2)} - \frac{2}{s_0^3}\lambda_i^{(1)} + \frac{3}{s_0^2}\lambda(s_0)\Big)(s - s_0)^2 \\ &\quad + \mathrm{o}(s - s_0)^2,\ i = 1, \ldots, m',\ k = 1, \ldots, m_i^{(1)} \ \textit{as } s \to s_0.\end{aligned}$$

Proofs: Pulses are unstable

The Schrödinger operators on the line and half-line

We study $Lu := -Du'' + V(x)u = \lambda u$, $D = \text{diag}\{d_i\} > 0$, $u \in \mathbb{R}^n$, $x \in \mathbb{R}$, where $\text{dom}(L) = H^2(\mathbb{R};\mathbb{R}^n)$, the Sobolev space, and $V(x) \in \mathbb{R}^{n\times n}$ satisfies the following hypotheses.

Hypothesis H

1. The potential $V \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ takes values in the set of symmetric matrices with real entries.
2. The limits $\lim_{x\to\pm\infty} V(x) = V_\pm$ exist, and are positive-definite matrices, i.e. $\text{Sp}(V_\pm) > 0$.
3. The functions $x \mapsto \big(V(x) - V_\pm\big)$ are in $L^1(\mathbb{R}_\pm;\mathbb{R}^{n\times n})$.

Hypothesis (H1): the spectrum of L is real. Hypothesis (H2): the essential spectrum of L is strictly positive, i.e., $k^2D + V_\pm > 0$ for every $k \in \mathbb{R}$ if and only if $V_\pm > 0$. Hypothesis (H3): applying the theory of exponential dichotomies. Our strategy is to first consider the eigenvalue problem on the half line

$$L_a u := -Du'' + V(x)u = \lambda u, \quad u \in \mathbb{R}^n, \quad x \in (-\infty, a],$$

$$\text{dom}(L_a) = \Big\{ u \in H^2((-\infty,a];\mathbb{R}^n) \,\Big|\, u(a) = 0 \Big\} \text{ and } a \in \mathbb{R} \text{ is fixed.}$$

The first order system

Setting $p_1 := u \in \mathbb{R}^n$, $p_2 := Du' \in \mathbb{R}^n$ and $\mathbf{p} := (p_1, p_2)^\top \in \mathbb{R}^{2n}$, we can write the eigenvalue problem $Lu = \lambda u$ as

$$\mathbf{p}' = A(x,\lambda)\mathbf{p}, \quad A(x,\lambda) = \begin{pmatrix} 0_n & D^{-1} \\ -\lambda I_n + V(x) & 0_n \end{pmatrix}.$$

This equation is asymptotically autonomous:

$$A_\pm(\lambda) = \lim_{x\to\pm\infty} A(x,\lambda) = \begin{pmatrix} 0_n & D^{-1} \\ -\lambda I_n + V_\pm & 0_n \end{pmatrix}.$$

By Hypothesis H, $\mathbf{p}' = A_\pm(\lambda)\mathbf{p}$ has exponential dichotomy on $\mathbb{R}_\pm$ and then $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ has exponential dichotomy on $\mathbb{R}_\pm$. We consider the dichotomy subspaces for $\mathbf{p}' = A(x,\lambda)\mathbf{p}$, in particular, $\mathbb{E}^u_-(x,\lambda)$, the unstable dichotomy subspace for $x \le 0$, and $\mathbb{E}^u_-(-\infty,\lambda)$, the spectral subspace of $A_-(\lambda)$, so that $\mathbb{E}^u_-(x,\lambda) \to \mathbb{E}^u_-(-\infty,\lambda)$ as $x \to -\infty$.
Let Y_λ denote the n-dimensional space of solutions $\mathbf{p}(\cdot)$ of the system $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ that decay at $-\infty$. Define the trace map $\Phi_x : Y_\lambda \to \mathbb{R}^{2n}$ by $\Phi_x : \mathbf{p}(\cdot) \mapsto \mathbf{p}(x) \in \mathbb{R}^{2n}$ for any $x \in \mathbb{R}$.
Then $\Phi_x(Y_\lambda) = \mathbb{E}^u_-(x,\lambda)$ for any $x \in [-\infty,+\infty)$.

Eigenvalues are the conjugate points

Writing the eigenvalue problem $Lu = \lambda u$ as

$$\mathbf{p}' = A(x,\lambda)\mathbf{p}, \quad A(x,\lambda) = \begin{pmatrix} 0_n & D^{-1} \\ -\lambda I_n + V(x) & 0_n \end{pmatrix}, x \in \mathbb{R},$$

we let Y_λ denote the n-dimensional space of solutions $\mathbf{p}(\cdot)$ of the system $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ that decay at $-\infty$.

The Dirichlet boundary condition at $x = a$ for the operator $L_a u = -Du'' + V(x)u$ in $L^2((-\infty, a])^n$ corresponds to $\mathbf{p}(a) \in \mathcal{D}$, where $\mathcal{D}$ is the *Dirichlet subspace* defined by $\mathcal{D} = \{(p_1, p_2)^\top \in \mathbb{R}^{2n} \,\big|\, p_1 = 0\}$.

Recall the trace map $\Phi_a : \mathbf{p}(\cdot) \mapsto \mathbf{p}(a) \in \mathbb{R}^{2n}$ for any $a \in \mathbb{R}$.

A critical observation: λ is an eigenvalue of L_a on $(-\infty, a]$ if and only if the subspace $\Phi_a(Y_\lambda)$ intersects $\mathcal{D}$ nontrivially.

Fix $a \in \mathbb{R}$. For a given $\lambda \in \mathbb{R}$, a point $s \in (-\infty, a]$ is called a *λ-conjugate point* of $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ if $\Phi_s(Y_\lambda) \cap \mathcal{D} \neq \{0\}$. In the special case $\lambda = 0$, s is simply called a *conjugate point*. For any $\lambda \in \mathbb{R}$ and $s \in (-\infty, a]$ the following assertions are equivalent: (i) λ is an eigenvalue of L_s; (ii) s is a λ-conjugate point.

Moreover, the multiplicity of the eigenvalue λ is equal to the dimension of the subspace $\Phi_s(Y_\lambda) \cap \mathcal{D}$.

Lagrangian framework

Recall that Y_λ denotes the n-dimensional space of solutions $\mathbf{p}(\cdot)$ of the system $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ that decay at $-\infty$ and the trace map $\Phi_s : \mathbf{p}(\cdot) \mapsto \mathbf{p}(s) \in \mathbb{R}^{2n}$ for any $s \in \mathbb{R}$. A direct calculation gives:

Theorem

For all $s \in [-\infty, +\infty)$ and $\lambda \in (-\infty, 0]$ the plane $\Phi_s(Y_\lambda) = \mathbb{E}^u_-(s,\lambda)$ belongs to the space $\Lambda(n)$ of Lagrangian n-planes in $\mathbb{R}^{2n}$, with the Lagrangian structure $\omega(v_1, v_2) = \langle v_1, \Omega v_2\rangle$, where

$$\Omega = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Thus we can define the Maslov index $A_{i,a}$, $i = 1,2,3,4$, for the flow $\Gamma_{i,a} \ni (s,\lambda) \mapsto \Phi_s(Y_\lambda) \in \Lambda(n)$ parametrized by the sides $\Gamma_{i,a}$ of the magic square pictured below. Here, we fix any a (potentially, large) and any $\lambda_\infty > 0$ so large that L_s has no eigenvalues $\lambda \le -\lambda_\infty$ for all $s \in (-\infty, a]$. A homotopy property of the Maslov index yields $A_{1,a} + A_{2,a} + A_{3,a} + A_{4,a} = 0$.

The magic square

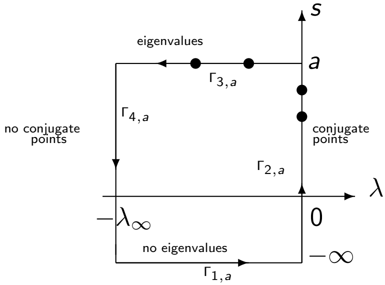


Figure: Illustrating the proof of Morse=Maslov Theorem: When λ_∞ is large enough, there are no crossings on $\Gamma_{1,a}$ and $\Gamma_{4,a}$, and the Morse index equals the number of crossings on $\Gamma_{3,a}$. By homotopy invariance, the Morse index is equal to the number of crossings on $\Gamma_{2,a}$; these are precisely the conjugate points in $(-\infty, a)$.

Counting eigenvalues for L_a via the Maslov index

Morse=Maslov Theorem on half line

Consider the operator $L_a = -D\partial_x^2 + V(x)$ defined in $L^2((-\infty, a])^n$. Fix large $a < +\infty$ and $\lambda_\infty > 0$, and let $A_{i,a} = \mathrm{Mas}(\Gamma_{i,a}, \mathcal{D})$. Then the following assertions hold.

1. The Maslov index of the curve Γ_a is zero.
2. $A_{3,a} = -A_{2,a}$.
3. $A_{3,a} \leq 0$ and $|A_{3,a}|$ is equal to the number of nonpositive eigenvalues for L_a, counting multiplicities:
$$|A_{3,a}| = \mathrm{Mor}\left(L_a\right) + \dim\ker\left(L_a\right).$$
4. $A_{2,a} \geq 0$ and $A_{2,a}$ is equal to the number of the conjugate points in $(-\infty, a]$, counting multiplicities.
5. The Morse index $\mathrm{Mor}(L_a)$ is equal to the number of conjugate points in $(-\infty, a)$, counting multiplicities.

Counting eigenvalues for L via the Maslov index

The Morse=Maslov Theorem on half line can be extended to count eigenvalues of the Schrödinger operator $L = -D\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R};\mathbb{R}^n)$, the whole real line.

Morse=Maslov Theorem on whole line

There exists $a_\infty \in \mathbb{R}$ such that for all $a > a_\infty$ the following hold:

(i) $\mathrm{Mor}(L) = \mathrm{Mor}(L_a)$;

(ii) a is not a conjugate point;

(iii) L_a is invertible.

In particular, the number of conjugate points is finite and independent of a, hence

$$\mathrm{Mor}(L) = \#\,\text{conjugate points in } (-\infty, +\infty). \tag{3}$$

Since $A_{2,a}$ converges as $a \to \infty$ to the number of conjugate points in $(-\infty, +\infty)$, we can interpret this number as the Maslov index for the whole-line problem.
Proof: $\mathrm{Mor}(L)$ is the number of negative zeros of $\mathbb{E}^s_+(0,\lambda) \wedge \mathbb{E}^u_-(0,\lambda)$ while $\mathrm{Mor}(L_a)$ is that of $\mathcal{D} \wedge \mathbb{E}^u_-(a,\lambda)$; they are close for large a.

Pulse solutions to gradient reaction diffusion systems

Consider a reaction–diffusion system

$$u_t = Du_{xx} + G(u), \quad u \in \mathbb{R}^n$$

where $G(u) = \nabla F(u)$ for some C^2 function $F\colon \mathbb{R}^n \to \mathbb{R}$, and D is a diagonal diffusion matrix. We assume that there exists a stationary, spatially homogeneous solution $u_*(x,t) = u_0$; without loss of generality we take $u_0 = 0$. We also assume that $k^2 D - \nabla^2 F(0) > 0$ for all $k \in \mathbb{R}$. This ensures that the spectrum of the linearization of the reaction-diffusion equation about u_0, which is given by

$$\{\lambda \in \mathbb{R} : \det(k^2 D + \lambda - \nabla^2 F(0)) = 0 \text{ for some } k \in \mathbb{R}\},$$

lies in the open left half plane.
We further suppose there is a stationary solution $\phi(x,t) = \phi(x)$. A typical example is Allen-Cahn system [Chardar,Dias,Bridges]:

$$u_t = u_{xx} - 4u + 6u^2 - c(u - v), \quad v_t = v_{xx} - 4v + 6v^2 + c(u - v),$$

that has a pulse $\phi(x) = \big(\operatorname{sech}^2 x, \operatorname{sech}^2 x\big)$.
Since $\nabla^2 F(0)$ is nondegenerate, the invariant manifold theorem implies that ϕ decays exponentially as $|x| \to \infty$. We thus call this a pulse, or pulse-type solution.

The linearization

We now show how the Maslov index can be used to establish the instability of a generic pulse-type solution. The eigenvalues for the linearization about $\varphi(x)$ solve

$$\lambda v = D\partial_x^2 v + \nabla^2 F(\phi(x))v,$$

and so it suffices to prove that the operator

$$L = -D\partial_x^2 - \nabla^2 F(\phi(x)) \tag{4}$$

has at least one negative eigenvalue. We first show that L satisfies Hypothesis H for the Schrödinger operator, where $V(x) = -\nabla^2 F(\phi(x))$.

(H1) Since F is C^2, the matrix $-\nabla^2 F(\phi(x))$ is symmetric and continuous in x.

(H2) Since $|\phi(x)| \to 0$ as $|x| \to \infty$, the limits $\lim_{x\to\pm\infty} V(x) = -\nabla^2 F(0)$ exist. Moreover, $V_\pm = -\nabla^2 F(0) > 0$ because it was assumed that $Dk^2 - \nabla^2 F(0) > 0$ for all $k \in \mathbb{R}$ (and in particular $k = 0$).

(H3) The functions $V(x) - V_\pm = -\nabla^2 F(\phi(x)) + \nabla^2 F(0)$ are in $L^1(\mathbb{R}_\pm; R^{n\times n})$ since $\phi(x)$ approaches 0 exponentially fast as $|x| \to \infty$.

Thus Morse=Maslov Theorem on whole line applies, so the existence of a conjugate point is enough to guarantee instability.

Dichotomy subspaces for the first order system

Writing the eigenvalue equation $Lv = \lambda v$ as a first order system, we obtain

$$\mathbf{p}' = A(x,\lambda)\mathbf{p}, \quad A(x,\lambda) = \begin{pmatrix} 0_n & D^{-1} \\ -\lambda I_n - \nabla^2 F(\phi(x)) & 0_n \end{pmatrix},$$

where $p_1 = v$, $p_2 = Dv'$ and $\mathbf{p} = (p_1, p_2)^\top \in \mathbb{R}^{2n}$. Differentiating the reaction-diffusion equation with respect to x, we find that $(\phi_x(x), \phi_{xx}(x))^\top$ is a solution to $\mathbf{p}' = A(x,\lambda)\mathbf{p}$ with $\lambda = 0$; thus $(\phi_x(x), \phi_{xx}(x))^\top \in \mathbb{E}_-^u(x,0)$. Let $(v_j(x), \partial_x v_j(x))^\top$, $j \in \{1, \ldots, n-1\}$, denote the remaining $n-1$ basis vectors for $\mathbb{E}_-^u(x,0)$, which are unknown. Denoting the ith component of $\phi(x)$ by $\phi_i(x)$ and the ith component of $v_j(x)$ by $v_{j,i}(x)$, we have

$$\mathbb{E}_-^u(x,0) = \operatorname{span}\left\{ \begin{pmatrix} \partial_x\phi_1(x) \\ \vdots \\ \partial_x\phi_n(x) \\ \partial_{xx}\phi_1(x) \\ \vdots \\ \partial_{xx}\phi_n(x) \end{pmatrix}, \begin{pmatrix} v_{1,1}(x) \\ \vdots \\ v_{1,n}(x) \\ \partial_x v_{1,1}(x) \\ \vdots \\ \partial_x v_{1,n}(x) \end{pmatrix}, \ldots, \begin{pmatrix} v_{n-1,1}(x) \\ \vdots \\ v_{n-1,n}(x) \\ \partial_x v_{n-1,1}(x) \\ \vdots \\ \partial_x v_{n-1,n}(x) \end{pmatrix} \right\}.$$

Existence of a conjugate point

As above, s is a conjugate point is and only if $\mathbb{E}^u_-(s,0) \cap \mathcal{D} \neq \{0\}$ if and only if

$$\det \begin{pmatrix} \partial_x \phi_1(s) & v_{1,1}(s) & \cdots & v_{n-1,1}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x \phi_n(s) & v_{1,n}(s) & \cdots & v_{n-1,n}(s) \end{pmatrix} = 0. \tag{5}$$

If we can find an x_0 so that all of the derivatives $\partial_x \phi_i(x_0)$ are *simultaneously* zero, then this is satisfied for $s = x_0$, regardless of the vectors $v_j(x_0)$.

We will show that such an x_0 exists by showing that the original pulse solution $\phi(x)$ is even-symmetric about some x_0 and therefore $\partial_x \phi(x)\big|_{x=x_0} = 0$.

Main assumption

Consider the first-order system of equations describing stationary solutions to the reaction-diffusion equation: $u_x = D^{-1}v$ and $v_x = -G(u)$, and let $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ denote the stable and unstable manifolds of $(u, v)^\top = 0$.

Hypothesis P

We assume that $(\phi(x), \phi_x(x))^\top$ is the unique, up to spatial translation, stationary solution of the reaction-diffusion equation contained in the intersection $\mathcal{W}^s(x) \cap \mathcal{W}^u(x)$.

This assumption is generic. Since $\phi(x)$ is a pulse solution to the reaction-diffusion equation, $\dim(\mathcal{W}^s(x) \cap \mathcal{W}^u(x)) \geq 1$. The assumption that this dimension is exactly equal to one is generic: Indeed, we append the x direction so that the manifolds $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ are $n+1$ dimensional manifolds in a $2n+1$ dimensional ambient space. Then it is a well known fact of differential topology that the dimension of a transverse intersection of two manifolds X and Z in the ambient space Y is given by $\dim(X \cap Z) = \dim(X) + \dim(Z) - \dim(Y)$ which in our case gives $\dim(\mathcal{W}^s(x) \cap \mathcal{W}^u(x)) = (n+1) + (n+1) - (2n+1) = 1$.

The even-symmetry of $\phi(x)$ and the main result

Claim

Assume Hypothesis P. Then there exists some $x_0 \in \mathbb{R}$ so that $\phi(x)$ is even-symmetric about $x = x_0$.

Proof: The reaction-diffusion equation is reversible: if $u(x)$ is a solution, so is $u(-x)$. By the definition of a pulse, both $\phi(x)$ and $\phi(-x)$ are contained in the intersection $\mathcal{W}^s(x) \cap \mathcal{W}^u(x)$. By Hypothesis P, $\phi(x)$ and $\phi(-x)$ are the same up to spatial translations. This can only be true if $\phi(x)$ is even-symmetric about some point x_0. More precisely, if $\phi(x) = \phi(-x + \delta)$ for all $x \in \mathbb{R}$ and some fixed δ, then $\phi(x_0 + x) = \phi(x_0 - x)$ for all $x \in \mathbb{R}$, where $x_0 = \delta/2$.

Pulse Instability Theorem

Assume Hypothesis P. Then the pulse $\phi(x)$ of the reaction-diffusion gradient system is unstable.

Proof: Since $\phi(x)$ is even-symmetric about some x_0 we have $\partial_x \phi(x)\big|_{x=x_0} = 0$ and so $\mathbb{E}_-^u(s, 0) \cap \mathcal{D} \neq \{0\}$ is satisfied for $s = x_0$. Thus $\mathrm{Mor}(H) = \mathit{Maslov} > 0$; hence, instability.

Maslov Index: the Definition

Maslov Index of a Path of Lagrangian Planes

We will now define the Maslov index of a path of Lagrangian planes in a complex Hilbert space relative to a reference plane. Let $\omega : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ be a sesquilinear, bounded, skew-Hermitian ($\omega(u, v) = -\overline{\omega(v, u)}$), non-degenerate form. We denote the annihilator of a subset $\mathcal{F} \subset \mathcal{X}$ by

$$\mathcal{F}^\circ := \{u \in \mathcal{X} : \omega(u, v) = 0 \text{ for all } v \in \mathcal{F}\}.$$

The subspace $\mathcal{F}$ is called *Lagrangian* if $\mathcal{F} = \mathcal{F}^\circ$. A pair of Lagrangian planes $\mathcal{F}, \mathcal{Z}$ is called Fredholm pair if

$$\dim(\mathcal{F} \cap \mathcal{Z}) < \infty,\ \mathcal{F} + \mathcal{Z} \text{ is closed in } \mathcal{X}, \text{ and } \operatorname{codim}(\mathcal{F} + \mathcal{Z}) < \infty.$$

The Fredholm-Lagrangian-Grassmannian is the space

$$F\Lambda(\mathcal{Z}) := \{\mathcal{F} \subset \mathcal{X} : \mathcal{F} \text{ is Lagrangian, and the pair } (\mathcal{F}, \mathcal{Z}) \text{ is Fredholm}\},$$

equipped with metric

$$d(\mathcal{F}_1, \mathcal{F}_2) := \|P_{\mathcal{F}_1} - P_{\mathcal{F}_2}\|_{\mathcal{B}(\mathcal{H})},\ \mathcal{F}_1, \mathcal{F}_2 \in F\Lambda(\mathcal{Z}),$$

where $P_{\mathcal{F}}$ denotes the orthogonal projection onto $\mathcal{F}$.

Maslov Index of a Path of Lagrangian Planes, Contd.

There exists a bounded operator $J : \mathcal{X} \to \mathcal{X}$, such that

$$\omega(u, v) = \langle Ju, v\rangle_{\mathcal{X}}, \quad u, v \in \mathcal{X},$$

and

$$J^2 = -I_{\mathcal{X}}, J^* = -J.$$

Moreover,

$$\mathcal{X} = \ker(J - \mathbf{i}I) \oplus \ker(J + \mathbf{i}I).$$

The Lagrangian plane $\mathcal{F}$ can be uniquely represented as a graph of a bounded operator $U \in \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}}))$, i.e., one has

$$\mathcal{F} = \text{graph}(U) := \{y + Uy : y \in \ker(J + \mathbf{i}I_{\mathcal{X}})\}.$$

Moreover, U is isometric, one-to-one, and onto. Here, Uy is defined as the unique (a lemma) vector in $\ker(J - \mathbf{i}I_{\mathcal{X}})$ such that $y + Uy \in \mathcal{F}$.

Maslov Index of a Path of Lagrangian Planes, Contd.

Let $\mathcal{I} = [\alpha, \beta] \subset \mathbb{R}$ be an interval of parameters. Let us fix a continuous path in $F\Lambda(\mathcal{Z})$

$$\Upsilon : \mathcal{I} \to F\Lambda(\mathcal{Z}), \quad \Upsilon(s) = \mathcal{F}_s, \ \Upsilon \in C(\mathcal{I}, F\Lambda(\mathcal{Z})),$$

and introduce the corresponding family of unitary operators U_s and V such that

$$\mathcal{F}_s = \text{graph}(U_s), \ s \in \mathcal{I}, \ \mathcal{Z} = \text{graph}(V),$$
$$\upsilon : \mathcal{I} \to \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}})), \ \upsilon(s) = U_s.$$

Then

- $\upsilon \in C(\mathcal{I}, \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}})))$
- $U_s V^{-1}$ is unitary in $\ker(J - \mathbf{i}I_{\mathcal{X}}), \ s \in \mathcal{I}$
- $U_s V^{-1} - I_{\mathcal{X}}$ is Fredholm in $\ker(J - \mathbf{i}I_{\mathcal{X}}), \ s \in \mathcal{I}$
- $\dim(\mathcal{F}_s \cap \mathcal{Z}) = \dim\ker(U_s V^{-1} - I_{\mathcal{X}}), \ s \in \mathcal{I}$

Maslov Index of a Path of Lagrangian Planes, Contd.

The Maslov index of $\Upsilon(s)$ is the spectral flow through the point $1 \in \mathbb{C}$ of the family $U_s V^{-1}, s \in \mathcal{I}$.

In other words, the Maslov index is the net number of eigenvalues of $U_s V^{-1}$ that crossed through 1. In formulas: Since $U_s V^{-1} - I_{\mathcal{X}}$ is Fredholm, there exists a partition $a = s_0 < s_1 < \cdots < s_N = b$ of $[a, b]$ and positive numbers $\varepsilon_j \in (0, \pi)$ such that $e^{\pm \mathbf{i} \varepsilon_j} \notin \mathrm{Sp}(U_s V^{-1})$ if $s \in [s_{j-1}, s_j]$, for each $1 \leq j \leq N$ (a lemma). For any $\varepsilon > 0$ and $s \in [a, b]$ we let

$$k(s, \varepsilon) := \sum\nolimits_{0 \leq \varkappa \leq \varepsilon} \dim \ker(U_s V^{-1} - e^{\mathbf{i}\varkappa}),$$

and define the Maslov index

$$\mathrm{Mas}(\Upsilon, \mathcal{Z}) := \sum_{j=1}^{N} \left(k(s_j, \varepsilon_j) - k(s_{j-1}, \varepsilon_j) \right). \tag{6}$$

The number $\mathrm{Mas}(\Upsilon, \mathcal{X})$ is well defined, i.e., it is independent on the choice of the partition s_j and ε_j.

Maslov index via the crossing forms

Assume $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{X}))$ and let $s_* \in \mathcal{I}$. There exists a neighbourhood $\mathcal{I}_0$ of s_* and a family $R_s \in C^1(\mathcal{I}_0, \mathcal{B}(\Upsilon(s_*), \Upsilon(s_*)^\perp))$, such that (a lemma)

$$\Upsilon(s) = \{u + R_s u \big| u \in \Upsilon(s_*)\}, \text{ for } s \in \mathcal{I}_0.$$

Let $\mathcal{Z}$ be a Lagrangian subspace and $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{Z}))$.
(i) We call $s_* \in \mathcal{I}$ a conjugate point or crossing if $\Upsilon(s_*) \cap \mathcal{Z} \neq \{0\}$.
(ii) The finite dimensional form is called the crossing form at the crossing s_*:

$$\mathfrak{m}_{s_*,\mathcal{Z}}(u, v) := \frac{d}{ds}\omega(u, R_s v)\Big|_{s=s_*} = \omega(u, \dot{R}_{s=s_*} v), \text{ for } u, v \in \Upsilon(s_*) \cap \mathcal{Z}.$$

(iii) The crossing s_* is called regular if the form $\mathfrak{m}_{s_*,\mathcal{Z}}$ is non-degenerate, positive if $\mathfrak{m}_{s_*,\mathcal{Z}}$ is positive definite, and negative if $\mathfrak{m}_{s_*,\mathcal{Z}}$ is negative definite.

Theorem. If all crossings are regular then they are isolated, and one has

$$\text{Mas}\,(\Upsilon, \mathcal{Z}) = -n_-(\mathfrak{m}_{a,\mathcal{Z}}) + \sum_{a<s<b} \text{sign}(\mathfrak{m}_{s,\mathcal{Z}}) + n_+(\mathfrak{m}_{b,\mathcal{Z}}),$$

where n_+, n_- are the numbers of positive and negative squares, $\text{sign} = n_+ - n_-$.