Theory and Applications of Degeneracy in Cone Optimization

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IPAM, Oct 15, 2010
Outline

Part I: Degeneracy in Cone Optimization

- minimal representations and strong duality
- (strict) complementarity and duality gaps

(With: Y-L Cheung, L. Tuncel, S. Schurr, H. Wei)

Part II: Sensor Network Localization, SNL

- exploiting implicit degeneracy
- solving huge problems
- high accuracy (low rank) solutions

(With: N. Krislock, F. Rendl)
Cone Optimization, (e.g. $K = S^n_+, SDP, K = \mathbb{R}_+^n, LP$)

Primal-Dual Pair of Optimization Problems in Conic Form

(assumed finite) \[ v_P = \sup_y \{ \langle b, y \rangle : A^* y \preceq_K c \}, \] (P)

(v_P \leq) \[ v_D = \inf_x \{ \langle c, x \rangle : A x = b, x \succeq_{K^*} 0 \}. \] (D)

where

- $A$ - an onto linear transformation; adjoint is $A^*$
- $K$ - a proper convex cone with dual/polar cone $K^* = \{ x : \langle s, x \rangle \geq 0, \forall s \in K \}$.
- $s' \preceq_K s'' (s' \prec_K s'')$ - partial order, $s'' - s' \in K(\in \text{int}K)$
Primal-Dual Pair

\[ v_P = \sup_y \{ b^T y : c - \sum_{i=1}^{m} y_i A_i \succeq 0 \}, \quad (P) \]

\[ v_D = \inf_x \{ \text{trace } c x : (\text{trace } A_i x) = b \in \mathbb{R}^m, \ x \succeq 0 \}. \quad (D) \]

\[ c, A_i \in S^n, \forall i \]
Optimality Conditions

Strong Duality if a Constraint Qualification, CQ, holds

\[ v_P = v_D = \langle c, x \rangle, \quad x \text{ dual optimal} \]

Zero duality gap and dual attainment.

Strict Complementarity

\( x, z \) optimal pair;
\[ \langle x, z \rangle = 0 \text{ complementarity} \]
\[ x + z > 0 \text{ strict complementarity} \]
Part I: Motivation/Outline

In case of nonpolyhedral cones, Strong Duality and/or Strict Complementarity can Fail

- Many Instances: SDP relax. for hard comb. probs. (e.g. QAP, GP, strengthened MC, POP, SNL)
- Fresh look at known Characterizations of Optimality without a CQ using Subspace Formulation
- theme: use MINIMAL REPRESENTATIONS for regularization, efficient solutions
- Connections Complementarity of Homog. Probl. and duality/Numerical implications
### Face

A convex cone $F$ is a **face** of $K$, denoted $F \triangleleft K$, if

$$x, y \in K \text{ and } x + y \in F \implies x, y \in F.$$  

If $F \triangleleft K$ and $F \neq K$, write $F \triangleleft K$.

### Conjugate Face

If $F \triangleleft K$, the **conjugate face** (or complementary face) of $F$ is

$$F^c := F^\perp \cap K^* \triangleleft K^*.$$  

If $x \in \text{ri}(F)$, then $F^c = \{x\}^\perp \cap K^*$. 
Minimal Face (Minimal Cone)

Feasible sets

\[
\begin{align*}
\mathcal{F}^y_P & := \{ y : c - A^*y \succeq_K 0 \} & \text{primal} \\
\mathcal{F}^s_P & := \{ s : s = c - A^*y \succeq_K 0, \text{ for some } y \} & \text{primal slacks} \\
\mathcal{F}^x_D & := \{ x : Ax = b, x \succeq_K^* 0 \} & \text{dual}
\end{align*}
\]

Minimal Faces (Intersection of Faces is a Face)

\[
\begin{align*}
f_P := \text{face } \mathcal{F}^s_P \triangleleft K \\
f_D := \text{face } \mathcal{F}^x_D \triangleleft K^*
\end{align*}
\]
(Modified) SDP Example from Ramana, 1995

**Primal SDP,** $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$

$$0 = \nu_P = \sup_{y} \left\{ y_2 : \begin{pmatrix} 0 & 0 & y_2 \\ 0 & y_2 & 0 \\ y_2 & 0 & y_1 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$2 \times 2$ principal submatrix $\preceq 0 \implies y_2 = 0$

$y^* = \begin{pmatrix} y_1^* & 0 \end{pmatrix}^T, \quad y_1^* \leq 0, \quad s^* = c - \mathcal{A}^*y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -y_1^* \end{pmatrix}$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Slater’s CQ fails for primal and dual

in fact, positive duality gap: $\nu_D = 1 > \nu_P = 0$
Dual of SDP Example

**Dual Program**

\[ 1 = v_D = \inf_x \{ x_{22} : x_{33} = 0, \ x_{22} + 2x_{13} = 1, \ x \succeq 0 \} \]

\[ x_{33} = 0, \ x \succeq 0 \implies x_{13} = 0 \implies x_{22} = 1 \]

\[ x^* = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_{11} \geq (x_{12}^2) \]

**Slater’s CQ for (primal) dual & complementarity fails**

positive duality gap: \[ v_D - v_P = 1 - 0 = 1, \]

\[ \text{trace } x^* s^* = \text{trace} \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -y_1^* \end{pmatrix} = 1 > 0 \]
Feasible Set/Minimal Face

\[ \mathcal{F}^y_P = \{ y \in \mathbb{R}^2 : y_1 \leq 0, \ y_2 = 0 \} \]

\[ f_P = \bigcap \{ F \leq K : \mathcal{F}^s_P = c - A^*(\mathcal{F}^y_P) \subset F \} \]
\[ = \begin{pmatrix} 0 & 0 \\ 0 & S_2^+ \end{pmatrix} < S_3^+ \]

Rotate/project to get Smaller Problem with Slater’s CQ

\[ y \in \mathcal{F}^y_P \text{ iff } \begin{bmatrix} 0 & I \end{bmatrix} (c - y_1A_1) \begin{bmatrix} 0 & I \end{bmatrix}^T \in S^2_+, \ A_2 \text{ disappears} \]

Slater CQ and Minimal Face

If \((\mathcal{P})\) is feasible, then

\[ c - A^*y \not\succ_K 0 \ \forall y \ ( \text{Slater’s CQ fails for } (\mathcal{P}) \ ) \iff f_P \prec K \]
Regularization of \((\mathcal{P})\) Using Minimal Face

Borwein-W (1981), \(f_P = \text{face } \mathcal{F}_P^s\)

\((\mathcal{P})\) is equivalent to regularized \((\mathcal{P})\)

\[
v_{RP} := \sup_y \{ \langle b, y \rangle : \mathcal{A}^* y \preceq_{f_P} c \}. \quad \text{(RP)}
\]

Lagrangian Dual DRP Satisfies Strong Duality:

\[
v_P = v_{RP} = v_{DRP} := \inf_x \{ \langle c, x \rangle : \mathcal{A} x = b, \ x \succeq_{f_P^*} 0 \} \quad \text{(DRP)}
\]

and \(v_{DRP}\) is attained

smaller cone in primal \(f_P \subset K\); larger cone in dual \(K^* \subset f_P^*\)
Assume Linear Feasibility for $\tilde{s}, \tilde{y}, \tilde{x}$; with data $A, b, c, K$

\[
A^*\tilde{y} + \tilde{s} = c \\
A\tilde{x} = b \\
L^\perp = R(A^*) \text{ (range)} \\
L = N(A) \text{ (nullspace)}
\]

Equivalent Primal-Dual Pair in Subspace Form, (e.g. N&N ’94)

Particular solution + solution of homogeneous equation

\[
v_P = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + L^\perp) \cap K \right\}. \quad (\mathbb{P})
\]

\[
v_D = \tilde{y}b + \inf_x \left\{ \tilde{s}x : x \in (\tilde{x} + L) \cap K^* \right\}. \quad (\mathbb{D})
\]
For $\mathbf{P}$ and $\mathbf{D}$

### Faces of Recession Directions (feasible case)

\[ f^0_P := \text{face } (\mathcal{L} \perp \cap K) \subset f_P, \quad f^0_D := \text{face } (\mathcal{L} \cap K^*) \subset f_D \]

### Recall

**minimal faces:** \[ f_P = \text{face } \mathcal{F}^s_P, \quad f_D = \text{face } \mathcal{F}^x_D \]

### Minimal Subspaces/Linear Transformations

**min. subsp.:** \[ \mathcal{L}^{\perp}_{PM} := \mathcal{L} \perp \cap (f_P - f_P), \quad \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D) \]

**min. Lin. Tr.:** \[ \mathcal{A}^{*}_{PM}, \quad \mathcal{A}_{DM} \]
Regularization of $\mathcal{P}$ Using Minimal Subspace

Assume $K$ Facialy Dual Complete, FDC (Pataki/07, 'nice')
i.e. $F \vartriangleleft K \implies K^* + F^\perp$ is closed. (e.g. $S^n_+, \mathbb{R}_+^n, \text{SOC}$).

\[ \mathcal{L}_{PM} = \mathcal{L}^\perp \cap (f_P - f_P) \]

\[ v_{RP} = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}_{MP}) \cap K \right\} \quad (\text{RP}) \]

Lagrangian Dual DRP Satisfies Strong Duality:

\[ v_P = v_{RP} = v_{DRP} = \tilde{y}b + \inf_x \left\{ \tilde{s}x : x \in (\tilde{x} + \mathcal{L}_{MP}) \cap K^* \right\} \quad (\text{DRP}) \]

and $v_{DRP}$ is **attained**
Lemma for SDP Case (Ramana, Tuncel, W./97)

Let $0 \neq F \subset S_+^n$. Then

$S_+^n + F^\perp$ is closed (nice)

$S_+^n + \text{span} F^c$ is not closed (devious)

$S_+^n + F^\perp = S_+^n + \text{span} F^c$

Infinite Duality Gap for Devious cones

Let $\mathcal{L} = \text{span} F^c$; choose $c = \tilde{s} = 0$ and

$\tilde{x} \in (S_+^n + F^\perp) \setminus (S_+^n + \text{span} F^c)$; (subspace repr. (P), (D): (13)).

then $0 = v_P < v_D = \infty$. 
### Strong Duality for (P) \((v_P = v_D \text{ and } v_D \text{ is attained})\)

<table>
<thead>
<tr>
<th>Minimal Face and Minimal Subspace CQs for (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (f_P = K) is a CQ (\text{(from BW: } f_P^* = K^* \text{)})</td>
</tr>
<tr>
<td>2. (\mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}^\perp_{PM} = \mathcal{L}^\perp) is a CQ (\text{(if } K \text{ is FDC (nice))})</td>
</tr>
<tr>
<td>(\tilde{s} \in f_P - f_P : x^* = x_K^* + x_f^* \in f_P^* = K^* + f_P^\perp \implies x^<em>(\tilde{s} + \mathcal{L}^\perp) = x_K^</em>(\tilde{s} + \mathcal{L}^\perp))</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Universal CQ, UCQ for (P) (\text{(i.e. independent of feasible data } c, b)}</th>
</tr>
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<tbody>
<tr>
<td>(\mathcal{L}^\perp \subset f_P^0 - f_P^0) is a UCQ (\text{(if } K \text{ is FDC)})</td>
</tr>
<tr>
<td>(\text{(wlog choose } \tilde{s} \in K, \tilde{x} \in K^*; \text{ shows that } f_P^0 \subset f_P, f_D^0 \subset f_D)</td>
</tr>
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</table>

\(17\)
Backward Stable Regularization (in progress)

Goals: Detect (near) Loss of Slater CQ/Regularize

- Solve a backward stable auxiliary problem
- **Alternate** projection onto smaller face/subspace to finally obtain a regularized problem

i.e. \( f_P = K \) and \( \mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}^\perp_{PM} = \mathcal{L}^\perp \)
Primal Slater condition implies strong duality, i.e. zero duality gap AND dual attainment.

(Near) loss of strict feasibility is used as a measure in complexity theory. (e.g. Renegar/95, Freund/01, Lara and Tuncel/02)

(Near) loss of strict feasibility correlates with number of iterations and loss of accuracy in interior-point methods (e.g. Freund/Ordonez/Toh 2006)
**Strict Complementary Optimal Primal-Dual Pair**

- There exists an optimal primal-dual pair $x, s$ such that
  \[ x + s \succeq 0 \quad (\in \text{int}(K + K^*)) \]

**Theoretical Difficulties/Convergence**

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

**Numerical Difficulties/Relation to Duality Gaps??**

increased number of iterations? loss of accuracy?
Maximal Complementary Solution Pair:

- A p-d pair of optimal solutions $(\bar{s}, \bar{x})$ is a **maximal complementary solution pair** if the pair maximizes the sum $\text{rank}(s) + \text{rank}(x)$ over all p-d optimal $(s, x)$.

Strict Complementarity Nullity, $g$:

- $g = n - \text{rank}(\bar{s}) - \text{rank}(\bar{x})$, where $(\bar{s}, \bar{x})$ is a maximal complementary solution pair

Hard SDP Instances:

- problems where nullity is nonzero
There is a strong correlation between the iteration number to achieve the desired stopping tolerance and the size of the complementarity nullity, when the accuracy requirement is high.

- Large nullity instances cause problems for SDP solvers.
- Local asymptotic convergence rate is slower when nullity is larger.
Theoretical Connections Complementarity/Duality?

Numerical Difficulties

(Both) loss of Slater CQ (strict feasibility) and loss of strict complementarity independently result in theoretical difficulties and numerical difficulties for interior-point methods.

Theoretical Connection?

Is there a theoretical connection between loss of duality (from loss of a CQ) and loss of strict complementarity?
Recall Faces of Recession Directions

\[ f_P^0 := \text{face} \left( \mathcal{L} \perp \cap K \right), \quad f_D^0 := \text{face} \left( \mathcal{L} \cap K^* \right) \]

The pair \( f_P^0, f_D^0 \) define a Complementarity Partition

- \( \text{face} (f_P^0) \subset \text{face} (f_D^0)^c \) and \( \text{face} (f_D^0) \subset \text{face} (f_P^0)^c \).

- It is a strict complementarity partition if both \( [\text{face} (f_P^0)]^c = \text{face} (f_D^0) \) and \( [\text{face} (f_D^0)]^c = \text{face} (f_P^0) \).

- It is proper if \( f_P^0 \) and \( f_D^0 \) are both nonempty.
For SDP (after a rotation)

\[
\begin{bmatrix}
    f_0^0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & f_P^0
\end{bmatrix}
\]

Form Primal-Dual Pair

\[
\tilde{x} = \tilde{s} = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & v & 0 \\
    0 & 0 & 0
\end{bmatrix} \quad \Rightarrow \quad \langle s, x \rangle \geq \|v\|_F^2,
\]

for all feasible pairs \(s, x\). (gap is dimension of \(v\))
**Theorem:** $K$ is a proper cone

1. If $f^0_P, f^0_D$ define a proper complementarity partition with a gap of dimension 1, so, the partition is not a strict complementarity partition, then there exists $\bar{s}$ and $\bar{x}$ such that $(P)-(D)$ with data $(L, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap.

*(Partial Converse)*

2. If
   (a) $(P)-(D)$ with data $(L, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap with both optimal values attained, and
   (b) the objective functions are constant along all recession directions of $(P)$ and $(D)$,
   then $f^0_P, f^0_D$ has a proper complementarity partition but not a strict complementarity partition.
Minimal Representations of the data regularize (P)
min. face $f_P$ and/or the min. L.T. $\mathcal{A}_{PM}$ or $\mathcal{L}^*_{PM}$
goal: a stable algorithm to solve (feasible) conic problems for which Slater’s CQ fails
Failure of strict complementarity for the associated recession problems is closely related to the existence of instances having a finite nonzero duality gap; provides a means of generating instances for testing.

Numer. Difficulties

↑ →
Slater CQ

↕
pos. duality gap

↑ ←
str. compl. partit. homog. prob.

← ↑
str. compl.
Part II: Sensor Network Localization, SNL, Problem (Exploiting (Implicit) Degeneracy)

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grassmann 1886

- $n$ ad hoc wireless sensors (nodes) to locate in $\mathbb{R}^r$, ($r$ is embedding dimension; sensors $p_i \in \mathbb{R}^r$, $i \in V := 1, \ldots, n$)
- $m$ of the sensors are anchors, $p_i$, $i = n - m + 1, \ldots, n$ (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2$, $ij \in E$, are known within radio range $R > 0$

\[ P^T = [p_1 \ldots p_n] = [X^T \ A^T] \in \mathbb{R}^{r \times n} \]
Applications


Untethered micro sensors will go anywhere and measure anything - traffic flow, water level, number of people walking by, temperature. This is developing into something like a nervous system for the earth, a skin for the earth. The world will evolve this way.

Tracking Humans/Animals/Equipment/Weather (smart dust)

- geographic routing; data aggregation; topological control; soil humidity; earthquakes and volcanos; weather and ocean currents.
- military; tracking of goods; vehicle positions; surveillance; random deployment in inaccessible terrains.
- brain scans
Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- **node set** $\mathcal{V} = \{1, \ldots, n\}$
- **edge set** $(i, j) \in \mathcal{E}; \omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a **clique** (complete subgraph)
- **Realization of $\mathcal{G}$ in $\mathbb{R}^r$:** a mapping of node $v_i \rightarrow p_i \in \mathbb{R}^r$ with squared distances given by $\omega$.

**Corresponding Partial Euclidean Distance Matrix, EDM**

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors $p_i, p_j$; anchors correspond to a **clique**.
Sensor Localization Problem/Partial EDM

Sensors and Anchors

n = 100, m = 9, R = 2
Connections to Semidefinite Programming (SDP)

\( S^+_n \), Cone of (symmetric) SDP matrices in \( S^n \); \( x^T A x \geq 0 \)

inner product \( \langle A, B \rangle = \text{trace } AB \)

Löwner (psd) partial order \( A \succeq B, A \succ B \)

\( D = \mathcal{K}(B) \in \mathcal{E}^n \), \( B = \mathcal{K}^\dagger(D) \in S^n \cap S_C \) (centered \( Be = 0 \))

\( P^T = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n} \); \( B := PP^T \in S^+_n \);

\begin{align*}
\text{rank } B &= r; D \in \mathcal{E}^n \\
\text{(to } D \in \mathcal{E}^n) & \quad D = (\|p_i - p_j\|_2^2)_{i,j=1}^n \\
& = (p_i^T p_i + p_j^T p_j - 2p_i^T p_j)_{i,j=1}^n \\
& = \text{diag } (B) e^T + e \text{ diag } (B)^T - 2B \\
& =: \mathcal{D}_e(B) - 2B \\
& =: \mathcal{K}(B) \quad \text{(from } B \in S^+_n \).
\end{align*}
Current Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax rank $B$)

- $\min_{B \succeq 0, B \in \Omega} \| H \circ (\mathcal{K}(B) - D) \|; \text{rank } B = r$;
- typical weights: $H_{ij} = 1/\sqrt{D_{ij}}, \text{if } ij \in E$.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, **BUT**: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible $B$s)

Instead: (Shall) Take Advantage of Degeneracy!

- clique $\alpha, |\alpha| = k$ (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$
  $\implies \text{rank } \mathcal{K}^\dagger(D[\alpha]) = t \leq r \implies \text{rank } B[\alpha] \leq \text{rank } \mathcal{K}^\dagger(D[\alpha]) + 1$
  $\implies \text{rank } B = \text{rank } \mathcal{K}^\dagger(D) \leq n - (k - t - 1)$
- Slater’s CQ (strict feasibility) **fails**
Linear Transformations: $D_v(B), K(B), T(D)$

- allow: $D_v(B) := \text{diag}(B) v^T + v \text{diag}(B)^T$; $D_v(y) := yv^T + vy^T$
- adjoint $K^*(D) = 2(\text{Diag}(De) - D)$.
- $K$ is 1-1, onto between centered & hollow subspaces:
  
  $S_C := \{ B \in S^n : Be = 0 \}$;
  $S_H := \{ D \in S^n : \text{diag}(D) = 0 \} = \mathcal{R}(\text{offDiag})$
  
  $J := I - \frac{1}{n} ee^T$ (orthogonal projection onto $M := \{ e \}^\perp$);
  
  $T(D) := -\frac{1}{2} J \text{offDiag}(D) J$ ( = $K^\dagger(D)$)
F \subseteq K \text{ is a face of } K, \text{ denoted } F \triangleleft K, \text{ if } 
(x, y \in K, \frac{1}{2}(x + y) \in F) \implies (\text{cone } \{x, y\} \subseteq F).

\text{If } F \triangleleft K, \text{ if } F \triangleleft K, F \neq K; F \text{ is proper face if } \{0\} \neq F \triangleleft K.

F \triangleleft K \text{ is exposed if: intersection of } K \text{ with a hyperplane.}

\text{face}(S) \text{ denotes smallest face of } K \text{ that contains set } S.

\mathcal{S}^n_+ \text{ is a Facially Exposed Cone}

All faces are exposed.
Facial Structure of SDP Cone; Equivalent SUBSPACES

<table>
<thead>
<tr>
<th>Face $F \trianglelefteq S^n_+$ Equivalence to $\mathcal{R}(U)$ Subspace of $\mathbb{R}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F \trianglelefteq S^n_+$ determined by range of any $S \in \text{relint } F$, i.e. let $S = U\Gamma U^T$ be compact spectral decomposition; $\Gamma \in S^t_{++}$ is diagonal matrix of pos. eigenvalues; $F = US^t_+ U^T$</td>
</tr>
<tr>
<td>$(F$ associated with $\mathcal{R}(U))$</td>
</tr>
<tr>
<td>$\dim F = \frac{t(t + 1)}{2}$.</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>face $F$ representation by subspace $\mathcal{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(subspace) $\mathcal{L} = \mathcal{R}(T)$, $T$ is $n \times t$ full column, then:</td>
</tr>
<tr>
<td>$F := TS^t_+ T^T \trianglelefteq S^n_+$</td>
</tr>
</tbody>
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Further Notation

Matrix with Fixed Principal Submatrix

For $Y \in S^n$, $\alpha \subseteq \{1, \ldots, n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices $\alpha$.

Sets with Fixed Principal Submatrices

If $|\alpha| = k$ and $\tilde{Y} \in S^k$, then:

- $S^n(\alpha, \tilde{Y}) := \{ Y \in S^n : Y[\alpha] = \tilde{Y} \}$,
- $S^+_n(\alpha, \tilde{Y}) := \{ Y \in S^+_n : Y[\alpha] = \tilde{Y} \}$

i.e. the subset of matrices $Y \in S^n$ ($Y \in S^+_n$) with principal submatrix $Y[\alpha]$ fixed to $\tilde{Y}$.
\[ \bar{D} \in \mathcal{E}^k, \, \alpha \subset 1:n, \, |\alpha| = k \]

Define \( \mathcal{E}^n(\alpha, \bar{D}) := \{ D \in \mathcal{E}^n : D[\alpha] = \bar{D} \} \).

Given \( \bar{D} \); find a corresponding \( B \succeq 0 \); find the corresponding face; find the corresponding subspace.

\[ \text{if } \alpha = 1:k; \text{ embed. dim of } \bar{D} \text{ is } t \leq r \]

\[ D = \begin{bmatrix} \bar{D} \\ \cdot \end{bmatrix}, \]
**THEOREM 1: Single Clique/Facial Reduction**

Let: \( \tilde{D} := D[1:k] \in \mathcal{E}^k, \ k < n \), with embedding dimension \( t \leq r \);

\[ B := \mathcal{K}^\dagger(\tilde{D}) = \tilde{U}_B S \tilde{U}_B^T, \quad \tilde{U}_B \in \mathcal{M}^{k \times t}, \quad \tilde{U}_B^T \tilde{U}_B = I_t, \quad S \in S_t^{++}. \]

Furthermore, let

\[ U_B := \begin{bmatrix} \tilde{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k \times (t+1)}, \]

\[ U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}, \quad \text{and let} \quad \begin{bmatrix} V & U^T e \frac{1}{\| U^T e \|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1} \text{ be orthogonal.} \]

Then:

\[
\text{face } \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \tilde{D})) = \left( US^{n-k+t+1}_+ U^T \right) \cap S_C = (UV)S^{n-k+t}_+ (UV)^T
\]

Note that we add \( \frac{1}{\sqrt{k}} e \) to represent \( \mathcal{N}(\mathcal{K}) \); then we use \( V \) to eliminate \( e \) to recover a centered face.
For each clique $|\alpha| = k$, we get a corresponding face/subspace ($k \times r$ matrix) representation. We now see how to handle two cliques, $\alpha_1, \alpha_2$, that intersect.
THEOREM 2: Clique/Facial Intersection Using Subspace Intersection

\{ \alpha_1, \alpha_2 \subseteq 1:n; \quad k := |\alpha_1 \cup \alpha_2| \}

For \( i = 1, 2 \): \( \bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i} \), embedding dimension \( t_i \);
\( B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^T, \quad \bar{U}_i \in \mathcal{M}^{k_i \times t_i}, \quad \bar{U}_i^T \bar{U}_i = I_{t_i}, \quad S_i \in S_{++}^{t_i}; \)
\( U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)} \); and \( \bar{U} \in \mathcal{M}^{k \times (t+1)} \) satisfies

\[ \mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{k_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^T \bar{U} = I_{t+1} \]

cont. . .
THEOREM 2 Nonsing. Clique/Facial Inters. cont. . .

cont. . . with
\[
\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{k_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^T\bar{U} = I_{t+1};
\]

let: \( U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}_{n \times (n-k+t+1)} \) and
\[
\begin{bmatrix} V & \frac{U^Te}{\|U^Te\|} \end{bmatrix} \in \mathcal{M}_{n-k+t+1} \text{ be orthogonal. Then}
\]
\[
\bigcap_{i=1}^{2} \text{ face } K^\dagger (\mathcal{E}^n(\alpha_i, \bar{D}_i)) = \left(US_{n-k+t+1}^+U^T\right) \cap S_C
\]
\[
= (UV)S_{n-k+t}^+ (UV)^T
\]
Suppose:

\[
U_1 = \begin{bmatrix}
U'_1 & 0 \\
U'' & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
U_2 = \begin{bmatrix}
I & 0 \\
0 & U''_2 \\
0 & U'_2
\end{bmatrix}
\]

Then:

\[
U := \begin{bmatrix}
U'_1 \\
U''_1 \\
U'_1(U''_1)^\dagger U''_1
\end{bmatrix}
\quad \text{or} \quad
U := \begin{bmatrix}
U'_1(U''_1)^\dagger U''_2 \\
U'_2 \\
U'_2
\end{bmatrix}
\]

(Efficiently) satisfies:

\[\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)\]
Completion: missing distances can be recovered if desired.
COR. Intersection with Embedding Dim. $r$/Completion

Hypotheses of Theorem 2 holds. Let $\bar{D}_i := D[\alpha_i] \in E^{k_i}$, for $i = 1, 2$, $\beta \subseteq \alpha_1 \cap \alpha_2$, $\gamma := \alpha_1 \cup \alpha_2$, $\bar{D} := D[\beta]$, $B := K^\dagger(\bar{D})$, $\bar{U}_\beta := \bar{U}(\beta, :)$, where $\bar{U} \in M^{k \times (t+1)}$ satisfies intersection equation of Theorem 2. Let $\left[\begin{array}{c} \bar{V} \\ \frac{\bar{U}_e^T}{\|\bar{U}_e^T\|} \end{array}\right] \in M^{t+1}$ be orthogonal. Let $Z := (J\bar{U}_\beta \bar{V})^\dagger B((J\bar{U}_\beta \bar{V})^\dagger)^T$. If the embedding dimension for $\bar{D}$ is $r$, THEN $t = r$ in Theorem 2, and $Z \in S^t_+$ is the unique solution of the equation $(J\bar{U}_\beta \bar{V})Z(J\bar{U}_\beta \bar{V})^T = B$, and the exact completion is $D[\gamma] = K(PP^T)$ where $P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{\mid\gamma\mid \times r}$.
Use $R$ as lower bound in singular/nonrigid case.
Hypotheses of previous COR holds. For $i = 1, 2$, let $\beta \subset \delta_i \subseteq \alpha_i$, $A_i := J\bar{U}_{\delta_i}\bar{V}$, where $\bar{U}_{\delta_i} := \bar{U}(\delta_i, :)$, and $B_i := \mathcal{K}^\dagger(D[\delta_i])$. Let $\bar{Z} \in S^t$ be a particular solution of the linear systems

$$A_1ZA_1^T = B_1$$
$$A_2ZA_2^T = B_2.$$ 

If the embedding dimension of $D[\delta_i]$ is $r$, for $i = 1, 2$, but the embedding dimension of $\bar{D} := D[\beta]$ is $r - 1$, then the following holds. cont...
The following holds:

1. $\dim \mathcal{N}(A_i) = 1$, for $i = 1, 2$.

2. For $i = 1, 2$, let $n_i \in \mathcal{N}(A_i)$, $\|n_i\|_2 = 1$, and
   $\Delta Z := n_1 n_2^T + n_2 n_1^T$. Then, $Z$ is a solution of the linear systems if and only if
   $Z = \tilde{Z} + \tau \Delta Z$, for some $\tau \in \mathcal{R}$

3. There are at most two nonzero solutions, $\tau_1$ and $\tau_2$, for the generalized eigenvalue problem $-\Delta Z v = \tau \tilde{Z} v$, $v \neq 0$. Set $Z_i := \tilde{Z} + \frac{1}{\tau_i} \Delta Z$, for $i = 1, 2$. Then the exact completion is
   one of $D[\gamma] \in \{ \mathcal{K}(\bar{U} \bar{V} Z_i \bar{V}^T \bar{U}^T) : i = 1, 2 \}$
Completing SNL (Delayed use of Anchor Locations)

Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^T)$

- Solve the orthogonal Procrustes problem:

$$\min \| A - P_2Q \| \quad \text{s.t.} \quad Q^TQ = I$$

- $P_2^TA = U\Sigma V^T$ SVD decomposition; set $Q = UV^T$; (Golub/Van Loan, Algorithm 12.4.1)

- Set $X := P_1Q$
## Algorithm: Four Cases

<table>
<thead>
<tr>
<th>Rigid</th>
<th>Clique Union</th>
<th>Node Absorption</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Rigid Clique Union" /></td>
<td><img src="image2" alt="Rigid Node Absorption" /></td>
<td></td>
</tr>
<tr>
<td>Non-rigid</td>
<td><img src="image3" alt="Non-rigid Clique Union" /></td>
<td><img src="image4" alt="Non-rigid Node Absorption" /></td>
</tr>
</tbody>
</table>
**ALGOR:** clique union; facial reduct.; delay compl.

### Initialize
Find initial set of cliques.

\[ C_i := \{ j : (D_p)_{ij} < (R/2)^2 \} , \quad \text{for } i = 1, \ldots, n \]

### Iterate
- For \(|C_i \cap C_j| \geq r + 1\), do Rigid Clique Union
- For \(|C_i \cap \mathcal{N}(j)| \geq r + 1\), do Rigid Node Absorption
- For \(|C_i \cap C_j| = r\), do Non-Rigid Clique Union (lower bnds)
- For \(|C_i \cap \mathcal{N}(j)| = r\), do Non-Rigid Node Absorp. (lower bnds)

### Finalize
When \(\exists\) a clique containing all anchors, use computed facial representation and positions of anchors to solve for \(X\).
2.16 GHz Intel Core 2 Duo, 2 GB of RAM
Dimension $r = 2$
Square region: $[0, 1] \times [0, 1]$
$m = 9$ anchors
Using only Rigid Clique Union and Rigid Node Absorption
Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left( \frac{1}{n} \sum_{i=1}^{n} \left\| p_i - p_i^{\text{true}} \right\|^2 \right)^{1/2}$$
### Results - Large $n$ (SDP size $O(n^2)$)

$n$ # of Sensors Located

<table>
<thead>
<tr>
<th>$n$ # sensors \ $R$</th>
<th>0.07</th>
<th>0.06</th>
<th>0.05</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>6000</td>
<td>6000</td>
<td>6000</td>
<td>6000</td>
<td>6000</td>
</tr>
<tr>
<td>10000</td>
<td>10000</td>
<td>10000</td>
<td>10000</td>
<td>10000</td>
</tr>
</tbody>
</table>

CPU Seconds

<table>
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<tr>
<th># sensors \ $R$</th>
<th>0.07</th>
<th>0.06</th>
<th>0.05</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6000</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>10000</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

RMSD (over located sensors)

<table>
<thead>
<tr>
<th>$n$ # sensors \ $R$</th>
<th>0.07</th>
<th>0.06</th>
<th>0.05</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>$4e-16$</td>
<td>$5e-16$</td>
<td>$6e-16$</td>
<td>$3e-16$</td>
</tr>
<tr>
<td>6000</td>
<td>$4e-16$</td>
<td>$4e-16$</td>
<td>$3e-16$</td>
<td>$3e-16$</td>
</tr>
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<td>10000</td>
<td>$3e-16$</td>
<td>$5e-16$</td>
<td>$4e-16$</td>
<td>$4e-16$</td>
</tr>
</tbody>
</table>
## Results - $N$ Huge SDPs Solved

### Large-Scale Problems

<table>
<thead>
<tr>
<th># sensors</th>
<th># anchors</th>
<th>radio range</th>
<th>RMSD</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>20000</td>
<td>9</td>
<td>0.025</td>
<td>$5 \times 10^{-16}$</td>
<td>25s</td>
</tr>
<tr>
<td>40000</td>
<td>9</td>
<td>0.02</td>
<td>$8 \times 10^{-16}$</td>
<td>1m 23s</td>
</tr>
<tr>
<td>60000</td>
<td>9</td>
<td>0.015</td>
<td>$5 \times 10^{-16}$</td>
<td>3m 13s</td>
</tr>
<tr>
<td>100000</td>
<td>9</td>
<td>0.01</td>
<td>$6 \times 10^{-16}$</td>
<td>9m 8s</td>
</tr>
</tbody>
</table>

### Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

\[ \mathcal{E}_n(\text{density of } G) = \pi R^2; \quad M = \mathcal{E}_n(|E|) = \pi R^2 N \quad (# \text{ constraints}) \]

Size of SDP Problems:

\[ M = [3,078,915 \quad 12,315,351 \quad 27,709,309 \quad 76,969,790] \]
\[ N = 10^9 [0.2000 \quad 0.8000 \quad 1.8000 \quad 5.0000] \]
Locally Recover Exact EDMs

Nearest EDM

- Given clique \( \alpha \); corresp. EDM \( D_\epsilon = D + N_\epsilon \), \( N_\epsilon \) noise.
- We need to find the smallest face containing \( \mathcal{E}_n(\alpha, D) \).

\[
\begin{align*}
\min & \| \mathcal{K}(X) - D_\epsilon \| \\
\text{s.t.} & \text{rank}(X) = r, X e = 0, X \succeq 0 \\
& X \succeq 0.
\end{align*}
\]

- Eliminate the constraints: \( Ve = 0 \), \( V^T V = I \),

\[
\mathcal{K}_V(X) := \mathcal{K}(VX V^T):
\]

\[
U_r^* \in \arg\min \quad \frac{1}{2} \| \mathcal{K}_V(UU^T) - D_\epsilon \|_F^2 \\
\text{s.t.} \quad U \in M^{(n-1)r}.
\]

The nearest EDM is \( D^* = \mathcal{K}_V(U_r^*(U_r^*)^T) \).
Solve Overdetermined Nonlin. Least Squares Prob.

Newton (expensive) or Gauss-Newton (less accurate)

\[ F(U) := \text{us2vec} \left( \mathcal{K}_V(UU^T) - D_\epsilon \right), \quad \min_U f(U) := \frac{1}{2} \| F(U) \|^2 \]

Derivatives: gradient and Hessian

\[ \nabla f(U)(\Delta U) = \langle 2 \left( \mathcal{K}_V^* \left[ \mathcal{K}_V(UU^T) - D_\epsilon \right] \right) U, \Delta U \rangle \]

\[ \nabla^2 f(U) = 2 \text{vec} \left( \mathcal{L}_U^* \mathcal{K}_V^* \mathcal{K}_V S_\Sigma \mathcal{L}_U + \mathcal{K}_V^* \left( \mathcal{K}_V(UU^T) - D_\epsilon \right) \right) \text{Mat} \]

where \( \mathcal{L}_U(\cdot) = \cdot U^T; \quad S_\Sigma(U) = \frac{1}{2}(U + U^T) \)
random noisy probs; \( r = 2, m = 9, nf = 1 \times 10^{-6} \)

Using only Rigid Clique Union, preliminary results:

### remaining cliques

<table>
<thead>
<tr>
<th>( n / R )</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1.00</td>
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<td>11.00</td>
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<td>124.00</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>7.00</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### cpu seconds

<table>
<thead>
<tr>
<th>( n / R )</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>9.43</td>
<td>6.98</td>
<td>5.57</td>
<td>5.04</td>
<td>4.05</td>
</tr>
<tr>
<td>2000</td>
<td>12.46</td>
<td>12.18</td>
<td>12.43</td>
<td>11.18</td>
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<tr>
<td>3000</td>
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<td>18.50</td>
<td>19.07</td>
<td>18.33</td>
<td>16.33</td>
</tr>
<tr>
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<td>24.01</td>
<td>24.02</td>
<td>23.80</td>
<td>22.12</td>
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<tr>
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<td>38.13</td>
<td>31.66</td>
<td>30.26</td>
<td>30.32</td>
<td>29.88</td>
</tr>
</tbody>
</table>

### max-log-error

<table>
<thead>
<tr>
<th>( n / R )</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>–3.28</td>
<td>–4.19</td>
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<td>Inf</td>
<td>Inf</td>
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<tr>
<td>4000</td>
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<td>–4.05</td>
<td>–3.52</td>
<td>–3.04</td>
<td>–3.33</td>
</tr>
</tbody>
</table>
SDP relaxation of SNL is highly (implicitly) degenerate: The feasible set of this SDP is restricted to a low dim. face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail.

We take advantage of this degeneracy by finding explicit representations of intersections of faces of the SDP cone corresponding to unions of intersecting cliques.

*Without* using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation.
Thanks for your attention!

Theory and Applications of Degeneracy in Cone Optimization

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University of Waterloo

IPAM, Oct 15, 2010