A Robust Robust Optimization Result

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1. Problem and Goal

We consider the problem

$$\max\{v^T x : x \in C\},$$

$C \subset I\!R^n$: nonempty, convex, compact,
$v \in I\!R^n$.

- Traditional viewpoint: C uncertain, model so that result computationally tractable.
- Our viewpoint: v uncertain, how much is lost?

2. A Model Case

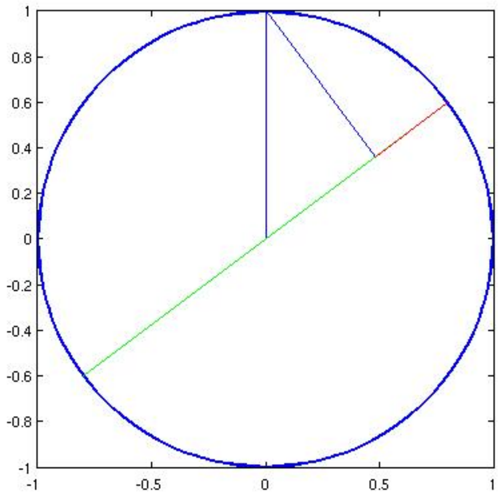
Suppose first that C is the unit ball, v has unit norm.
The solution to our problem with the nominal objective vector v is $x = v$, with objective value 1.

If the true objective vector is $w := w(\alpha)$, a unit vector making an angle α, $0 \leq \alpha \leq \pi$, with v, then v attains a true objective value of $\cos \alpha$, with a loss of $1 - \cos \alpha$. Since the range of $w^T x$ over C is 2 (from -1 to $+1$),

$$\text{scaled\_loss} = \frac{\text{loss}}{\text{range}} = \frac{1 - \cos \alpha}{2}.$$

We show that this scaled loss formula holds "on average" for arbitrary C.

Model Case, II



The loss is the length of the red line segment; the range is the combined lengths of the red and green line segments.

3. Definitions

$\max(v) := \max\{v^T x : x \in C\}$;

$\min(v) := \min\{v^T x : x \in C\}$;

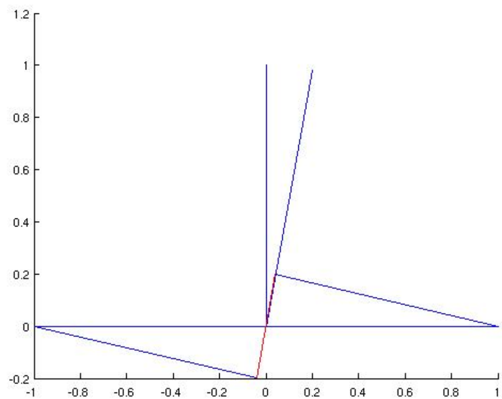
$\text{range}(v) := \max(v) - \min(v)$;

$\text{loss}(v, w) := \max(w) - \min\{w^T x : x \in C,\ v^T x = \max(v)\}$.
(The loss in the true objective $w^T x$ possible when implementing a best solution for the nominal objective $v^T x$.)

$\text{scaled\_loss}(v, w) := \frac{\text{loss}(v,w)}{\text{range}(w)}$.

4. A Very Bad Case

On the other hand, the scaled loss is terrible in the case that C is the line segment joining $[-1; 0]$ and $[+1; 0]$, v is $[0; 1]$, and $w := w(\alpha)$ is $[\sin\alpha; \cos\alpha]$.
Then $[-1; 0]$ is optimal for v but attains the worst objective value for w, so that scaled_loss(v, w) is 1.

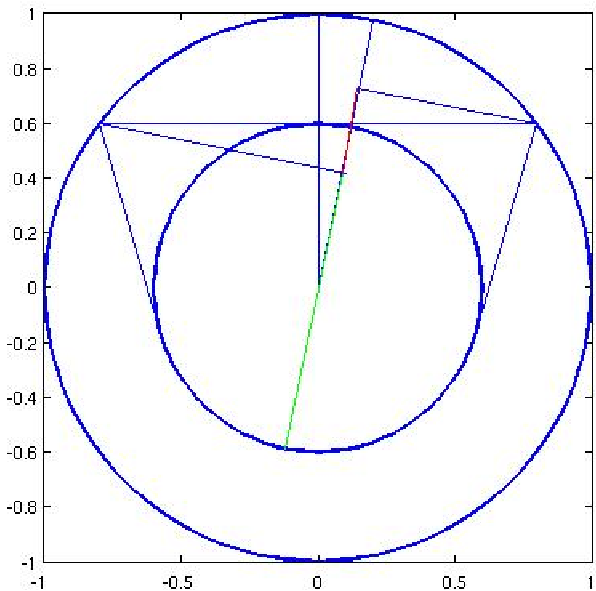


5. The Well-Rounded Case

Now let us assume that C is contained in a ball of radius 1, and contains a ball of radius r, both centered at the origin, and let $\rho := \sqrt{1-r^2}$. Without loss of generality, C is 2-dimensional. We suppose $v = [0;1]$ and $w := w(\alpha) = [\sin\alpha; \cos\alpha]$, and consider the feasible region that is the convex hull of the points $[-\rho; r]$, $[\rho; r]$, and the ball of radius r.
Then, as long as $\sin\alpha \le r \le \cos\alpha$, we have

$$\text{scaled\_loss}(v, w(\alpha)) = \frac{2\rho\sin\alpha}{r(1+\cos\alpha)+\rho\sin\alpha} \le \frac{\sin\alpha}{r}.$$

The Well-Rounded Case, II



6. Two Probabilistic Models

(i) Suppose v and u are independently drawn from the standard Gaussian distribution $N(0, I)$, and let $w := w(\alpha) := \cos\alpha\, v + \sin\alpha\, u$.

The angle between v and w is with high probability very close to α as n approaches infinity. Also, (w, v) has the same distribution as (v, w). We denote expectations with respect to this distribution by E_1.

(ii) Suppose $\bar{v}$ and $\bar{u}$ are independently drawn from $N(0, I)$. Let $\hat{u} := (I - \bar{v}\bar{v}^T/\bar{v}^T\bar{v})\bar{u}$, $v := \bar{v}/\|\bar{v}\|$, $u := \hat{u}/\|\hat{u}\|$, and $w := w(\alpha) := \cos\alpha\, v + \sin\alpha\, u$.

It is not hard to see that the angle between v and w is now exactly α, and again, (w, v) has the same distribution as (v, w). We denote expectations with respect to this distribution by E_2.

7. Results

Note: $-\min(v) = \max\{-v^T x : x \in C\}$. So

Proposition:

$$E_1[\max(w)] = E_1[\max(v)],$$

$$E_1[\mathsf{range}(w)] = E_1[\mathsf{range}(v)] = 2E_1[\max(v)].$$

Let $x_v \in C$ maximize $v^T x$ over C. Then

$$w(\alpha)^T x_v = \cos\alpha\, v^T x_v + \sin\alpha\, u^T x_v.$$

Proposition:

$$E_1[\mathsf{loss}(v, w(\alpha))] = (1 - \cos\alpha) E_1[\max(v)].$$

Results, II

Theorem:

$$\frac{E_1[\mathsf{loss}(v, w(\alpha))]}{E_1[\mathsf{range}(w(\alpha))]} = \frac{1 - \cos\alpha}{2}.$$

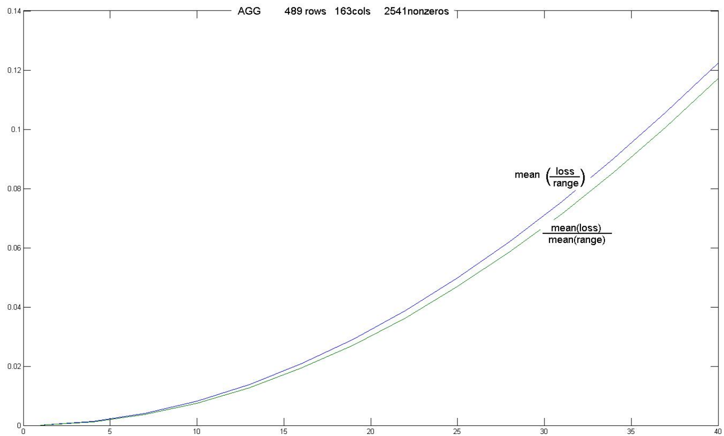
Similarly, we obtain

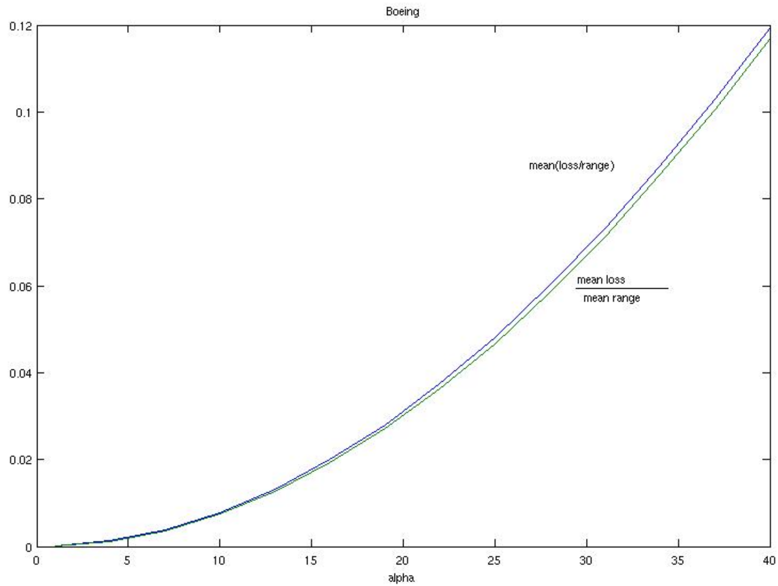
Theorem:

$$\frac{E_2[\mathsf{loss}(v, w(\alpha))]}{E_2[\mathsf{range}(w(\alpha))]} = \frac{1 - \cos\alpha}{2}.$$

Note that both results refer to the ratio of expectations, rather than the expectation of the ratio, the scaled loss.

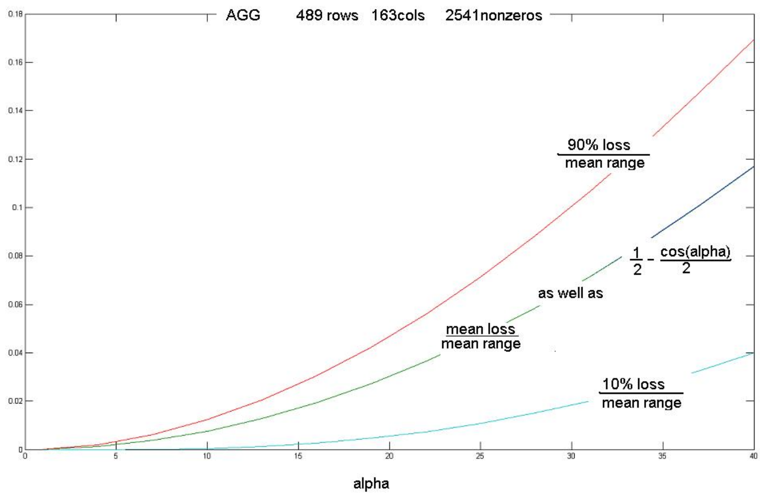
8. Comparison of Ratio of Expectations to Expectation of Ratio

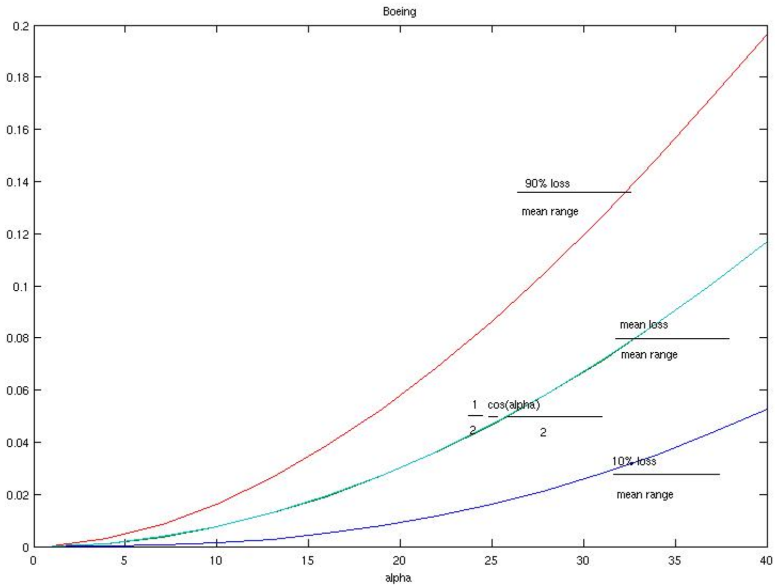




Boeing
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
40
alpha
mean(loss/range)
mean loss
mean range

9. Graphs of Percentiles





Boeing
90% loss / mean range
mean loss / mean range
1/2 − cos(alpha)/2
10% loss / mean range
alpha
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0
5
10
15
20
25
30
35
40

10. Conclusion

Under two probabilistic models, the loss in objective value from even a fairly large misspecification of a linear objective function is likely to be quite modest, for any compact convex feasible region.