Enclosing Ellipsoids and Elliptic Cylinders of Semialgebraic Sets and Their Application to Error Bounds in Polynomial Optimization

Masakazu Kojima and Makoto Yamashita
Tokyo Institute of Technology

October 2010

Modern Trends in Optimization and Its Application
Workshop II: Numerical Methods for Continuous Optimization

Outline

1. Problem and Some Formulations
2. Theory: Lifting and SDP Relaxation
3. Numerical Results
4. Applications to the Sensor Network Localization Problem
5. Concluding Remarks

Outline

1. Problem and Some Formulations
2. Theory: Lifting and SDP Relaxation
3. Numerical Results
4. Applications to the Sensor Network Localization Problem
5. Concluding Remarks

Problem

Given a nonempty compact semialgebraic subset

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0 \ (k = 1, 2, \dots, m)\}$$

of $\mathbb{R}^n$, find a "small" ellipsoid enclosing F. Here $f_k : \mathbb{R}^n \to \mathbb{R}$ denotes a polynomial $(k = 1, 2, \dots, m)$.

- "small" needs to be specified.

Formulation 1: Minimum volume ellipsoid

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

$\mathcal{E}(\boldsymbol{M}, \boldsymbol{c}) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{c}) \leq 1\}$.

$$\begin{array}{ll} \text{minimize} & \text{volume of } \mathcal{E}(\boldsymbol{M}, \boldsymbol{c}) \\ & \text{sub.to } F \subset \mathcal{E}(\boldsymbol{M}, \boldsymbol{c}),\ \boldsymbol{M} \succ \boldsymbol{O},\ \boldsymbol{c} \in \mathbb{R}^n. \end{array}$$

- The most popular in theory
- F consists of a finite number of points $\Rightarrow$ lots of studies $\supset$ (Khachiyan's method 1996)
- Ideal but too difficult in general

Formulation 2: Nie and Demmel 2005

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

$\mathcal{E}(\boldsymbol{P}^{-1}, \boldsymbol{c}) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x}-\boldsymbol{c})^T \boldsymbol{P}^{-1}(\boldsymbol{x}-\boldsymbol{c}) \leq 1\}$.

$$\begin{array}{ll} \text{minimize} & \text{Trace } \boldsymbol{P} \\ \text{sub.to} & F \subset \mathcal{E}(\boldsymbol{P}^{-1}, \boldsymbol{c}),\ \boldsymbol{P} \succ \boldsymbol{O},\ \boldsymbol{c} \in \mathbb{R}^n. \end{array}$$

$\Longleftarrow$ SOS (Sum Of Squares) relaxation

- A little more general to include parameters.
- Theoretical convergence.
- Still very expensive to apply it to large-scale problems.
  - The SOS relaxation problem becomes a dense problem.

$\Rightarrow$ Less expensive formulation: Fix the shape of the ellipsoid and minimize the size — Ours, next

Our Formulation:

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape). Define

$\varphi(\boldsymbol{x}, \boldsymbol{c}) \equiv (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{c}), \forall \boldsymbol{x}, \ \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),

Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$

A min. enclosing ellipsoidal set : $\gamma^* = \min_{\gamma, \ \boldsymbol{c}} \{\gamma : F \subset E(\boldsymbol{c}, \gamma)\}$.

Our Formulation:

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape). Define

$\varphi(\boldsymbol{x}, \boldsymbol{c}) \equiv (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{c}), \forall \boldsymbol{x}, \ \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),

Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).

Application to error bounds in Polynomial Optimization

POP : $f_0^* =$ min $f_0(\boldsymbol{x})$ subject to $f_k(\boldsymbol{x}) \geq 0 \ (k = 1, 2, \ldots, p)$.
Here $f_k : \mathbb{R}^n \to \mathbb{R}$: a polynomial $(k = 0, 1, \ldots, p)$.

Suppose that $\hat{f}_0 \geq f_0^*$ or $\hat{f}_0 = f_0(\hat{\boldsymbol{x}})$ for $\exists$ feasible $\hat{\boldsymbol{x}}$. Let

$$F = \{\boldsymbol{x} \in \mathbb{R}^n : f_k(\boldsymbol{x}) \geq 0, \ (k = 1, 2, \ldots, p), \ f_0(\boldsymbol{x}) \leq \hat{f}_0\}$$

$F \subset E(\boldsymbol{c}, \gamma) \Longrightarrow E(\boldsymbol{c}, \gamma)$ contains all opt. solutions of POP.

$\boldsymbol{M} = \boldsymbol{I} \ \Rightarrow \ \|\boldsymbol{x} - \boldsymbol{c}\| \leq \sqrt{\gamma}$ for $\forall$ opt. sol. $\boldsymbol{x}$

$\boldsymbol{M} = \mathrm{diag}(1, 0, \ldots, 0) \ \Rightarrow \ |x_1 - c_1| \leq \sqrt{\gamma}$ for $\forall$ opt. sol. $\boldsymbol{x}$

- This method can be combined with the SDP relaxation (Lasserre ’01) and its sparse variant (Waki et al. ’06).

Outline

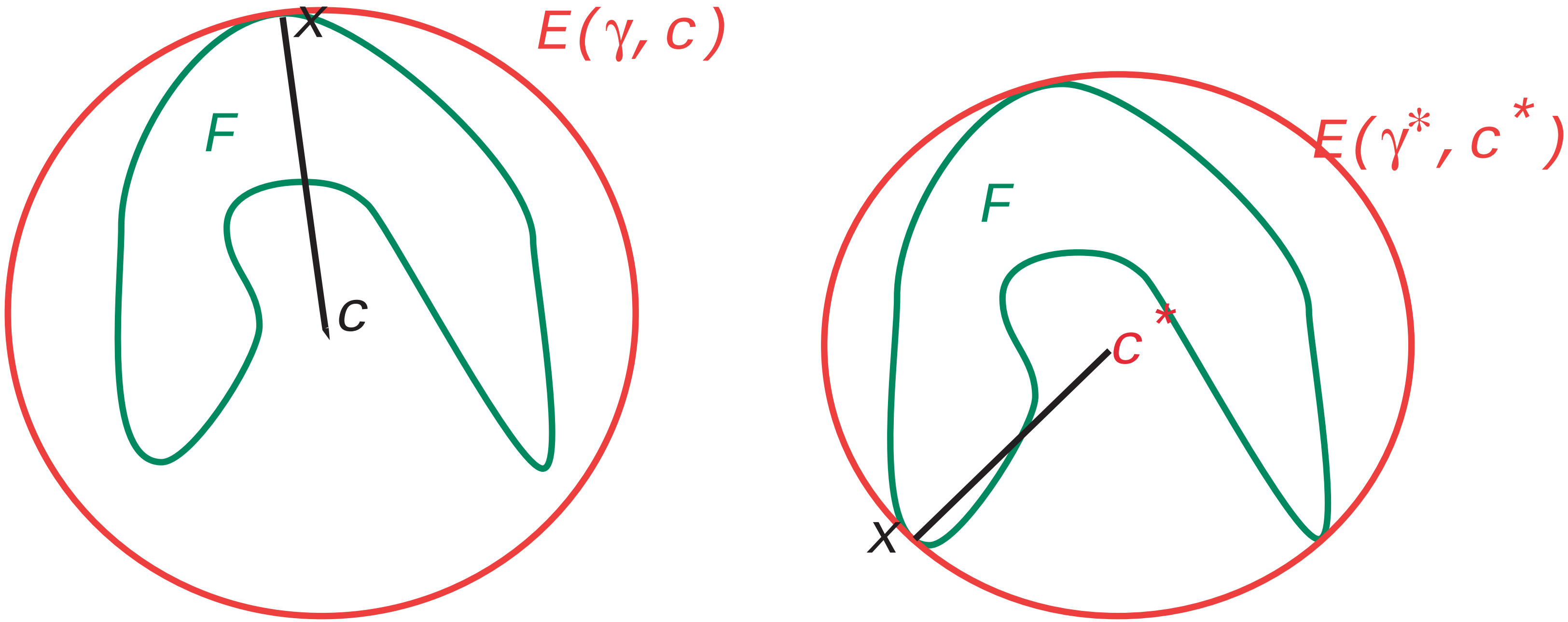
$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(\boldsymbol{x}, \boldsymbol{c}) \equiv (\boldsymbol{x} - \boldsymbol{c})^T \boldsymbol{M} (\boldsymbol{x} - \boldsymbol{c}), \forall \boldsymbol{x}, \ \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),
Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \le \gamma\}, \ \forall \gamma > 0$ (size).
F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

---

min-max formulation

$$\gamma^* = \min_{\boldsymbol{c} \in \mathbb{R}^n} \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}) = \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}^*).$$

Suppose that $\boldsymbol{M}$ = the 2×2 identity matrix



- $\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{c} + \boldsymbol{c}^T \boldsymbol{M} \boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$.

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),

$\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),

Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}$, $\forall \gamma > 0$ (size).

F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

---

min-max formulation

$$\gamma^* = \min_{\boldsymbol{c} \in \mathbb{R}^n} \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}) = \max_{\boldsymbol{x} \in F} \varphi(\boldsymbol{x}, \boldsymbol{c}^*).$$

Lifting $\Rightarrow$

Define $\psi(\boldsymbol{x}, \boldsymbol{W}, \boldsymbol{c}) \equiv \boldsymbol{M} \bullet \boldsymbol{W} - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}$,

$C^* \equiv$ the convex hull of $\{(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

convex-linear min-max formulation

$$\gamma^* = \min_{\boldsymbol{c} \in \mathbb{R}^n} \left(\max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \psi(\boldsymbol{x}, \boldsymbol{W}, \boldsymbol{c}) \right).$$

$\Updownarrow$

linear-convex max-min problem

$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \min_{\boldsymbol{c} \in \mathbb{R}^n} \psi(\boldsymbol{x}, \boldsymbol{W}, \boldsymbol{c}).$$

$\min_{\boldsymbol{c} \in \mathbb{R}^n} \boldsymbol{M} \bullet \boldsymbol{W} - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}$ $\Updownarrow$ $\boldsymbol{c}^* = \boldsymbol{x}$: a minimizer

concave maxization

$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$$

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),
Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}, \ \forall \gamma > 0$ (size).
F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

---

concave maxization $\quad \gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$

Here $C^* \equiv$ the convex hull of $\{(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

$\Downarrow$

- Relax the intractable C^* by a tractable convex $\widehat{C}$;

$$L \equiv \left\{ (\boldsymbol{x}, \boldsymbol{W}) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \right\} \supset \widehat{C} \supset C^*.$$

- Describe $\widehat{C}$ in terms of LMIs.

SDP-SOCP $\quad \hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \quad \Rightarrow \quad \gamma^* \leq \hat{\gamma}.$

- Under an assumption, $\{C^k$: described in terms of LMIs$\}$;

$$L \supset C^k \supset C^{k+1} \supset C^* \text{ and } \cap_k C^k = C^*$$

by using Lasserre's hierarchy of LMI relaxation '01.

$\boldsymbol{M} \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(\boldsymbol{x}, \boldsymbol{c}) = \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T - 2\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{c} + \boldsymbol{c}^T\boldsymbol{M}\boldsymbol{c}, \forall \boldsymbol{x}, \forall \boldsymbol{c} \in \mathbb{R}^n$ (center),
Ellipsoidal set $E(\boldsymbol{c}, \gamma) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \varphi(\boldsymbol{x}, \boldsymbol{c}) \leq \gamma\}$, $\forall \gamma > 0$ (size).
F : a nonempty compact semialgebraic subset of $\mathbb{R}^n$.

---

concave maxization
$$\gamma^* = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in C^*} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}.$$

Here $C^* \equiv$ the convex hull of $\{(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^T) \in \mathbb{R}^n \times \mathbb{S}^n : \boldsymbol{x} \in F\}$.

$\Downarrow$

- Relax the intractable C^* by a tractable convex $\widehat{C}$;

$$L \equiv \left\{ (\boldsymbol{x}, \boldsymbol{W}) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{W} \end{pmatrix} \succeq \boldsymbol{O} \right\} \supset \widehat{C} \supset C^*.$$

- Describe $\widehat{C}$ in terms of LMIs.

SDP-SOCP
$$\hat{\gamma} = \max_{(\boldsymbol{x}, \boldsymbol{W}) \in \widehat{C}} \boldsymbol{M} \bullet \boldsymbol{W} - \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} \quad \Rightarrow \quad \gamma^* \leq \hat{\gamma}.$$

- When $\widehat{C}$ is described in terms of sparse LMIs, take $\boldsymbol{M}$ which fits their sparsity.
  $\Rightarrow$ a sparse SDP-SOCP which we can solve efficiently.

$M \in \mathbb{S}^n_+$ ($n \times n$ positive semidefinite matrices, shape),
$\varphi(x, c) = M \bullet xx^T - 2x^T Mc + c^T Mc, \forall x, \forall c \in \mathbb{R}^n$ (center),
Ellipsoidal set $E(c, \gamma) \equiv \{x \in \mathbb{R}^n : \varphi(x, c) \leq \gamma\}, \ \forall \gamma > 0$ (size).

QOP case

$$
\begin{aligned}
F &= \left\{x \in \mathbb{R}^n : \alpha_k + 2b_k^T x + x^T Q_k x \geq 0 \ (1 \leq k \leq p)\right\} \\
&= \left\{x \in \mathbb{R}^n : \begin{pmatrix} \alpha_k & b_k^T \\ b_k & Q_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \geq 0 \ (1 \leq k \leq p)\right\},
\end{aligned}
$$

Let

$$
\widehat{C} = \left\{ (x, W) : \begin{array}{l} \begin{pmatrix} \alpha_k & b_k^T \\ b_k & Q_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \geq 0 \ (1 \leq k \leq p), \\ \begin{pmatrix} 1 & x^T \\ x & W \end{pmatrix} \succeq O \end{array} \right\},
$$

SDP-SOCP

$$
\hat{\gamma} = \max_{(x, W) \in \widehat{C}} M \bullet W - x^T M x = M \bullet \widehat{W} - \hat{c}^T M \hat{c}
$$

$$
\Longrightarrow F \subset E(\hat{c}, \hat{\gamma}).
$$

Outline

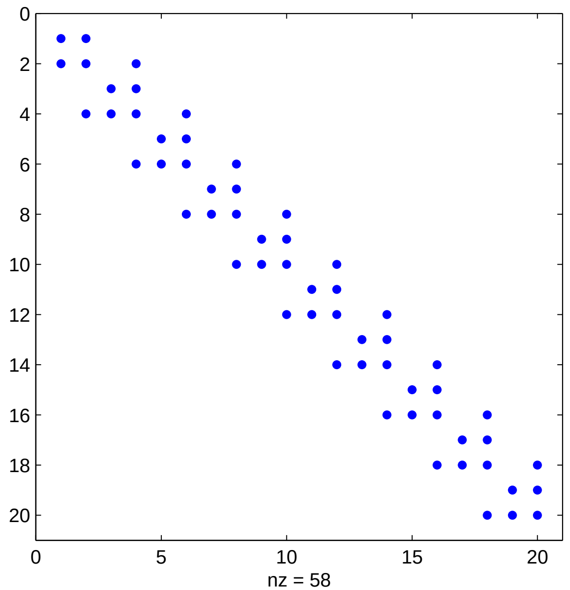
- SparsePOP (Waki et al. '08) for constructing sparse SDP relaxation problems of POPs.
- SeDuMi1.21 (Sturm, Polik '09) for solving SDP relaxation problems to compute an approx. opt. sol. of POPs and for solving SDP-SOCPs to compute error bounds.
- MATLAB Optimization Toolbox to refine the approx. opt. sol. obtained by SeDuMi for constrained optimization problems.
- 2.8GHz Intel Xeon with 4GB Memory.

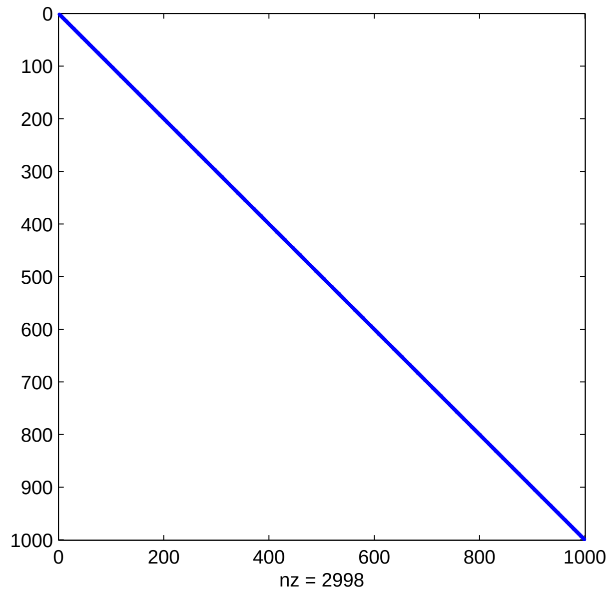
Unconstrained min. of ChainedWood function $f_C(\boldsymbol{x})$

$$f_C(\boldsymbol{x}) = 1 + \sum_{i \in J} \Big(100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2\Big)$$

Here $J = \{1, 3, 5, \dots, n-3\}$ and n is a multiple of 4.

Sparsity pattern of the Hessian matrix





$n = 20$, 20×20 matrix

no. of nonzeros = $\dfrac{58}{400}$

$n = 1000$, 1000×1000 matrix

no. of nonzeros = $\dfrac{2,988}{1,000,000}$

- Sparse enough to solve larger scale problems.

Unconstrained min. of ChainedWood function $f_C(\boldsymbol{x})$

$$f_C(\boldsymbol{x}) = 1 + \sum_{i \in J} \Big(100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2\Big)$$

Here $J = \{1, 3, 5, \ldots, n-3\}$ and n is a multiple of 4.

$\boldsymbol{M}$ = the $n \times n$ identity matrix.

	RelObjErr	E.Time	Error bound	
n	at $\hat{\boldsymbol{x}}$	for $\hat{\boldsymbol{x}}$	E.time	$\sqrt{\hat{\gamma}}/\Vert\hat{\boldsymbol{c}}\Vert$
1000	4.4e-4	2.4	4.7	4.9e-3
2000	8.8e-4	5.7	11.6	4.9e-3
4000	1.8e-3	14.6	30.3	1.5e-3

$\hat{\boldsymbol{x}}$ = an approx. optimal solution,

$$\text{RelObjErr} = \frac{|\text{lbd. for opt. val.} - f_C(\hat{\boldsymbol{x}})|}{|f_C(\hat{\boldsymbol{x}})|}$$

$\Vert\boldsymbol{x} - \hat{\boldsymbol{c}}\Vert / \Vert\hat{\boldsymbol{c}}\Vert \leq \sqrt{\hat{\gamma}}/\Vert\hat{\boldsymbol{c}}\Vert, \ \forall$ global minimizer $\boldsymbol{x}$

alkyl.gms from globallib

$$
\begin{array}{ll}
\min & -6.3x_5x_8 + 5.04x_2 + 0.35x_3 + x_4 + 3.36x_6 \\
\text{sub.to} & x_5x_{12} - x_2(1.12 + 0.132x_9 - 0.0067x_9^2) = 0,
\end{array}
$$

$$
x_8x_{13} - 0.01x_9(1.098 - 0.038x_9) - 0.325x_7 = 0.574,
$$

$$
0.98x_4 - x_7(0.01x_5x_{10} + x_4) = 0, \ -x_2x_9 + 10x_3 + x_6 = 0,
$$

$$
-0.820x_2 + x_5 - 0.820x_6 = 0, \ x_1x_{11} - 3x_8 = -1.33,
$$

$$
x_{10}x_{14} + 22.2x_{11} = 35.82, \ \text{lbd}_i \leq x_i \leq \text{ubd}_i \ (i = 1, 2, \ldots, 14).
$$

$$
\frac{|\text{lbd for opt.val.} - \text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|}{|\text{approx. opt.val } f_0(\hat{\boldsymbol{x}})|} = 6.7e\text{-}6
$$

max error in equalities at $\hat{\boldsymbol{x}} = 5.2e\text{-}9$

$$
F = \{\boldsymbol{x} \in \mathbb{R}^{14} : \ \text{feasible and } f_0(\boldsymbol{x}) \leq f_0(\hat{\boldsymbol{x}})\} \subset E(\hat{\boldsymbol{c}}, \hat{\gamma})
$$

$$
\boldsymbol{M} = \boldsymbol{I} \in \mathbb{S}^{14} \Rightarrow \hat{\boldsymbol{c}} = (1.7037030, 1.5847109, \ldots), \ \sqrt{\hat{\gamma}} = 1.6\text{e-}4.
$$

$\|\boldsymbol{x} - \hat{\boldsymbol{c}}\| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x} \in \mathbb{R}^{14}$.

$$
\boldsymbol{M} = \text{diag}(1, 0, \ldots, 0) \in \mathbb{S}^{14} \Rightarrow \hat{c}_1 = 1.7037017, \ \sqrt{\hat{\gamma}} = 1.0\text{e-}5.
$$

$|x_1 - \hat{c}_1| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x} \in \mathbb{R}^{14}$.

Nonconvex QPs from globalib

$\boldsymbol{M}$ = the $n \times n$ identity matrix

Problem	n	RelObjErr	Error bound $\sqrt{\hat{\gamma}}$	Error bound $\sqrt{\hat{\gamma}}/\|\|\hat{\boldsymbol{c}}\|\|$	E.time sdpa
ex2_1_3	13	1.1e-9	4.9e-4	4.9e-4	0.5
ex2_1_5	10	3.5e-10	4.7e-4	1.7e-4	0.8
ex2_1_8	24	3.5e-9	5.4e-2	1.3e-3	9.5
ex9_1_2†	10	1.8e-2	4.2	0.53	0.2
ex9_1_5†	13	6.2e-2	4.7	1.0	0.3
ex9_2_3	16	2.8e-7	1.4e-2	2.6e-4	0.2

$$\text{RelObjErr} = \frac{|\text{approx. otp. val.} - \text{l. bd. for otp. val.}|}{|\text{approx. otp. val.}|}$$

$$\|\boldsymbol{x} - \hat{\boldsymbol{c}}\| \leq \sqrt{\hat{\gamma}}, \ \forall \text{ global minimizer } \boldsymbol{x}$$

$\dagger$: multiple solutions

More details on ex9_1_2[†]

min.	$-x_1 - 3x_2$
sub. to	5 linear equations in x_j $(j = 1, 2, \ldots, 10)$,
	$x_3 x_7 = 0,\ x_4 x_8 = 0,\ x_5 x_9 = 0,\ x_6 x_{10} = 0,$
	$0 \leq x_j \leq 5$ $(j = 1, 2, \ldots, 10)$.

$\boldsymbol{M}$ = diag(the ith unit coordinate vector) $(i = 3, 4, 5, 6, 8, 9)$

$\Rightarrow$ $|x_i - \hat{c}_i| \leq \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x}$

i	$\hat{c}_i$	$\sqrt{\hat{\gamma}}$		at $\forall$ opt. sol.
3	2.9995	0.0089	$\Rightarrow$	$x_3 > 0$, $x_7 = 0$
4	0.0002	0.0279		
5	0.0009	0.0148		
6	4.0002	0.0123	$\Rightarrow$	$x_6 > 0$, $x_{10} = 0$
8	1.000	1.0001		
9	3.000	2.0004	$\Rightarrow$	$x_9 > 0$, $x_5 = 0$

Fixing $x_5 = x_7 = x_{10} = 0$, we obtain the reduced problem $\Rightarrow$

Reduced ex9_1_2[†] with fixing $x_5 = x_7 = x_{10} = 0$

min.	$-x_1 - 3x_2$
sub. to	5 linear equations in x_j $(j = 1, 2, 3, 4, 6, 8, 9)$,
	$x_4 x_8 = 0,\ 0 \le x_j \le 5\ (j = 1, 2, 3, 4, 6, 8, 9)$.

$\boldsymbol{M}$ = diag(the ith unit coordinate vector) $(i = 1, 2, 3, 4, 6, 8, 9)$

$\Rightarrow |x_i - \hat{c}_i| \le \sqrt{\hat{\gamma}}$ for $\forall$ opt. sol. $\boldsymbol{x}$

i	$\hat{c}_i$	$\sqrt{\hat{\gamma}}$	i	$\hat{c}_i$	$\sqrt{\rho^*}$
1	4.0000	0.0002	2	4.0000	0.0002
3	3.0000	0.0006	4	0.0000	0.0006
6	4.0000	0.0004			
8	1.0000	1.0000	$\Rightarrow 0.0000 \le x_8 \le 2.0000$		
9	3.0000	2.0000	$\Rightarrow 1.0000 \le x_9 \le 5.0000$		

We can verify that the optimal solutions are:

$$x_1 = x_2 = x_6 = 4,\ x_3 = 3,\ x_4 = 0,$$

$$0 \le x_8 = (x_9 - 1)/2 \le 2,\ 1 \le x_9 \le 5.$$

Outline

1 Problem and Some Formulations

2 Theory: Lifting and SDP Relaxation

3 Numerical Results

4 Applications to the Sensor Network Localization Problem

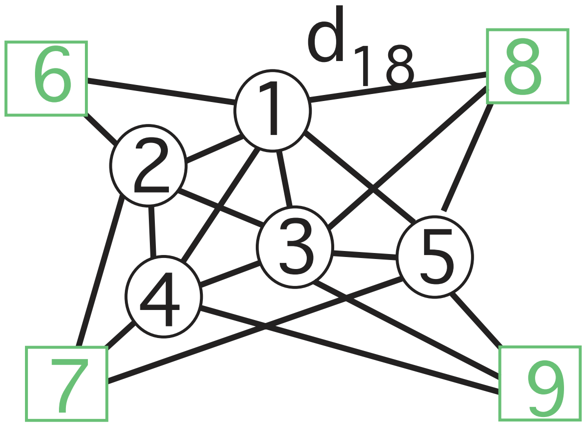
5 Concluding Remarks

A Sensor Network Localization Problem with Exact Distance

$m = 5,\ n = 9$.

$1, \ldots, 5$: sensors

$6, 7, 8, 9$: anchors



- Sensors' locations are unknown.
- Anchors' locatios are known.
- A distance is given for $\forall$ edge.

Compute locations of sensors.

$\Rightarrow$ Nonconvex QOPs

A Sensor Network Localization Problem with Exact Distance

$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

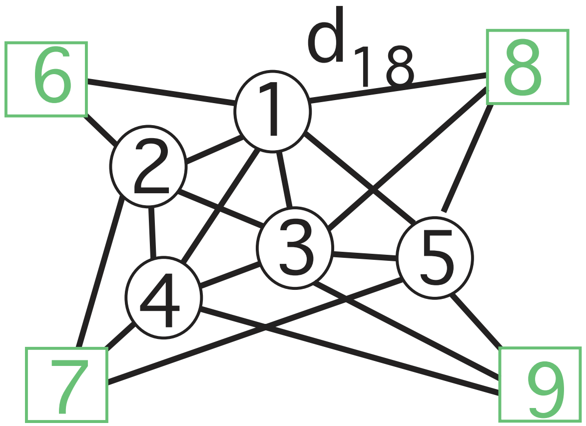
$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

$m = 5, \ n = 9$.

$1, \ldots, 5$: sensors

$6, 7, 8, 9$: anchors



- Sensors' locations are unknown.
- Anchors' locatios are known.
- A distance is given for ∀ edge.

Compute locations of sensors.
$\Rightarrow$ Nonconvex QOPs

A Sensor Network Localization Problem <u>with Exact Distance</u>

$$\begin{aligned} \boldsymbol{x}^p \in \mathbb{R}^2 \quad &: \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m), \\ \boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad &: \quad \text{known location of anchors } (r = m+1, \ldots, n), \\ d_{pq}^2 \quad &= \quad \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1) \\ \mathcal{E} \quad &= \quad \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

- FSDP by Biswas-Ye '06, SDP relaxation of (1)
  — Powerful in theory;
  FSDP computes exact locations $\boldsymbol{x}^p$ $(p = 1, 2, \ldots, m)$ if "(1) is uniquely localizable"
  = "the rigidity of $G(\{1, 2, \ldots, m\}, \mathcal{E})$ + a certain condition".
  But expensive in computation.
- SFSDP by Kim, Kojima, Waki '09 = a sparse version of FSDP which is as effective as FSDP in theory but is more efficient in computation.

A Sensor Network Localization Problem with Exact Distance

$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p,q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p,q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

Numerical Results: 20,000 sensors randomly distributed in $[0,1] \times [0,1]$, 4 anchors at the corner and $\rho = 0.1$

σ	RMSD	SDPA E.time
0.0	6.9e-6	182.9
0.1	7.6e-3	403.0
0.2	1.1e-2	402.6

$$\text{RMSD} = \left(\frac{1}{m} \sum_{p=1}^{m} \|\boldsymbol{x}^p - \boldsymbol{a}^p\| \right)^{1/2}.$$

$\boldsymbol{a}^p$: true location of p

$\sigma > 0 \Rightarrow d_{pq} = (1 + \xi) \times$ true distance, diferrent formulation:

$$\min \sum_{(p,q)\in\mathcal{E}} \left| \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 - d_{pq}^2 \right| \Leftarrow \text{ sparse SDP relaxation.}$$

Here ξ is chosen from $N(0, \sigma)$.

A Sensor Network Localization Problem with Exact Distance

$$
\begin{aligned}
\boldsymbol{x}^p \in \mathbb{R}^2 &: \text{ unknown location of sensors } (p = 1, 2, \ldots, m), \\
\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 &: \text{ known location of anchors } (r = m+1, \ldots, n), \\
d_{pq}^2 &= \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1) \\
\mathcal{E} &= \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}
\end{aligned}
$$

- Some numerical results of SFSD combined with our method for an ellipoidal set enclosing $F = \{(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m) : d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ for } (p, q) \in \mathcal{E}\}$.

A Sensor Network Localization Problem with Exact Distance

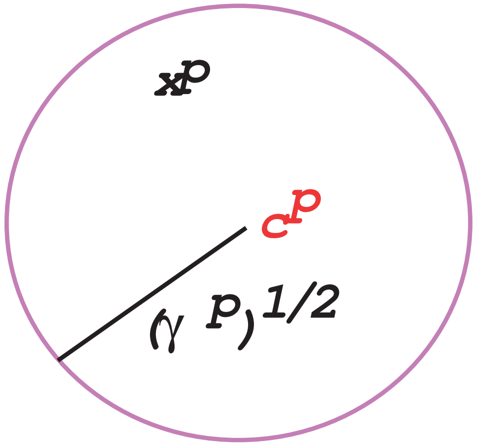
$$\boldsymbol{x}^p \in \mathbb{R}^2 \quad : \quad \text{unknown location of sensors } (p = 1, 2, \ldots, m),$$

$$\boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 \quad : \quad \text{known location of anchors } (r = m+1, \ldots, n),$$

$$d_{pq}^2 = \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1)$$

$$\mathcal{E} = \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\}$$

Problem: For each sensor $p = 1, 2, \ldots, m$, compute $\boldsymbol{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ such that the distance from $\boldsymbol{c}^p$ to its unknown location is bounded by $(\gamma^p)^{1/2}$.



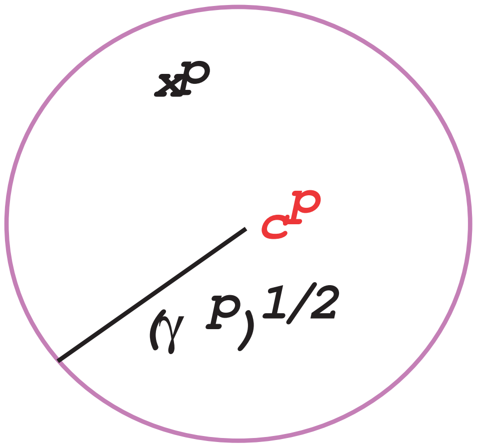
A Sensor Network Localization Problem with Exact Distance

$$\begin{aligned} \boldsymbol{x}^p \in \mathbb{R}^2 &: \text{ unknown location of sensors } (p = 1, 2, \ldots, m), \\ \boldsymbol{x}^r = \boldsymbol{a}^r \in \mathbb{R}^2 &: \text{ known location of anchors } (r = m+1, \ldots, n), \\ d_{pq}^2 &= \|\boldsymbol{x}^p - \boldsymbol{x}^q\|^2 \text{ — given for } (p, q) \in \mathcal{E} \qquad (1) \\ \mathcal{E} &= \{(p, q) : \|\boldsymbol{x}^p - \boldsymbol{x}^q\| \leq \rho = \text{a given radio range}\} \end{aligned}$$

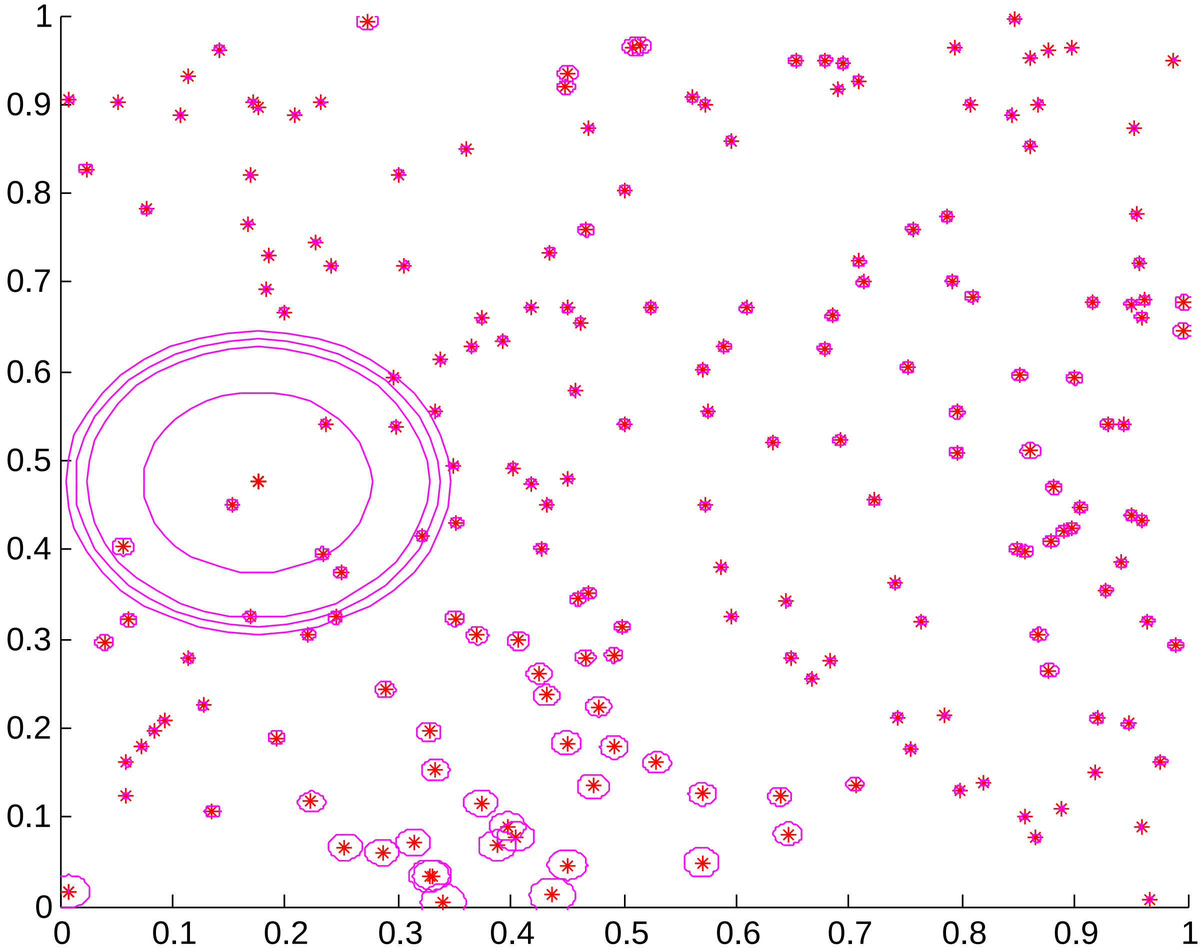
- When ρ is not large enough or $\mathcal{E}$ does not contain enough number of edges, (1) is underdetermined and/or its SDP relaxation is too weak to locate all sensors uniquely.
- Our method + SFSDP computes $\boldsymbol{c}^p \in \mathbb{R}^2$ and $\gamma^p > 0$ for each sensor p such that the distance from $\boldsymbol{c}^p$ to its unknown location $\boldsymbol{x}^p$ is bounded by $(\gamma^p)^{1/2}$.

$\Downarrow$

- If $\gamma^p = 0$ then $\boldsymbol{c}^p$ = the exact location of p (Biswas-Ye '06).



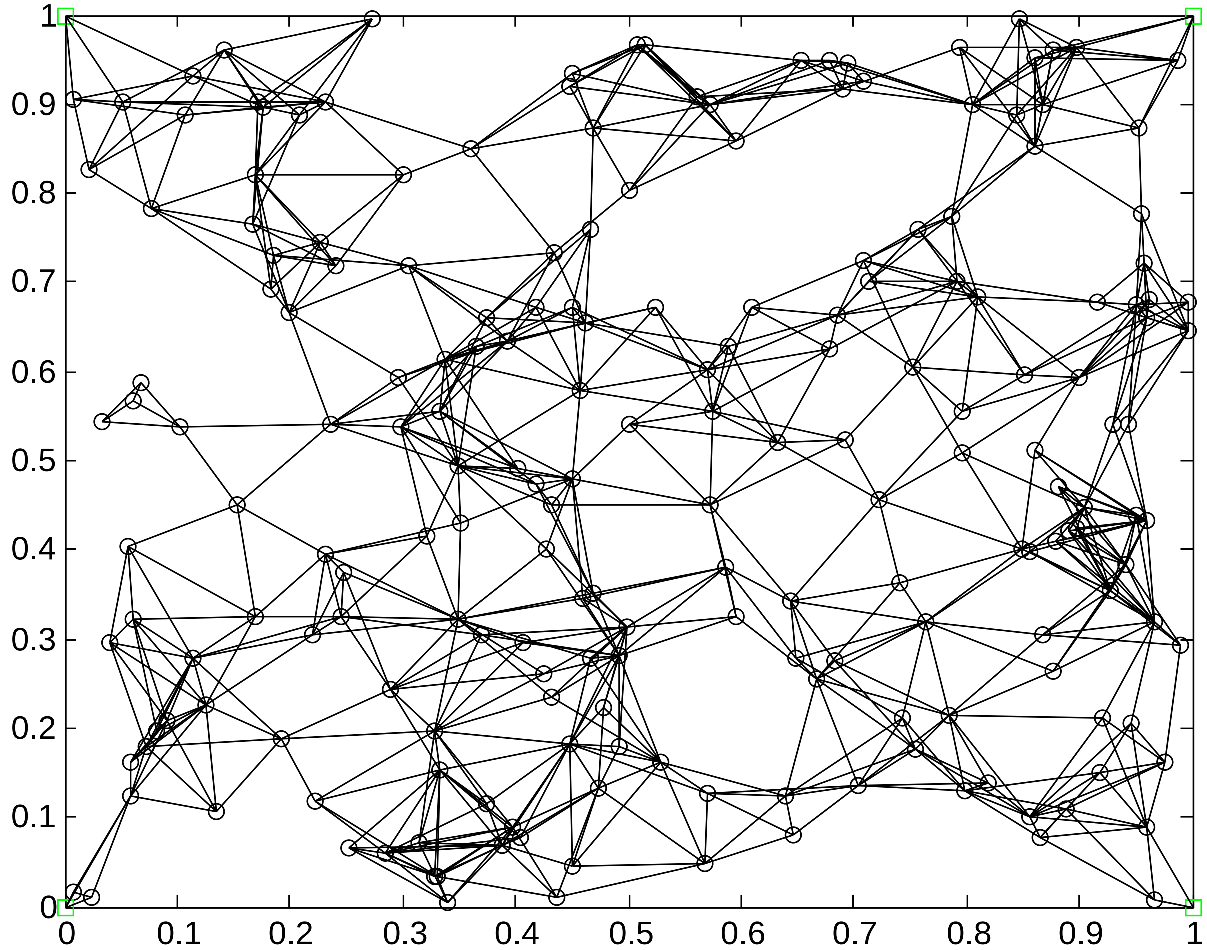
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.14$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.

the true location $\boldsymbol{x}^p$of sensor p is within $(\gamma^p)^{1/2} \leq 0.18$ from $\boldsymbol{c}^p$

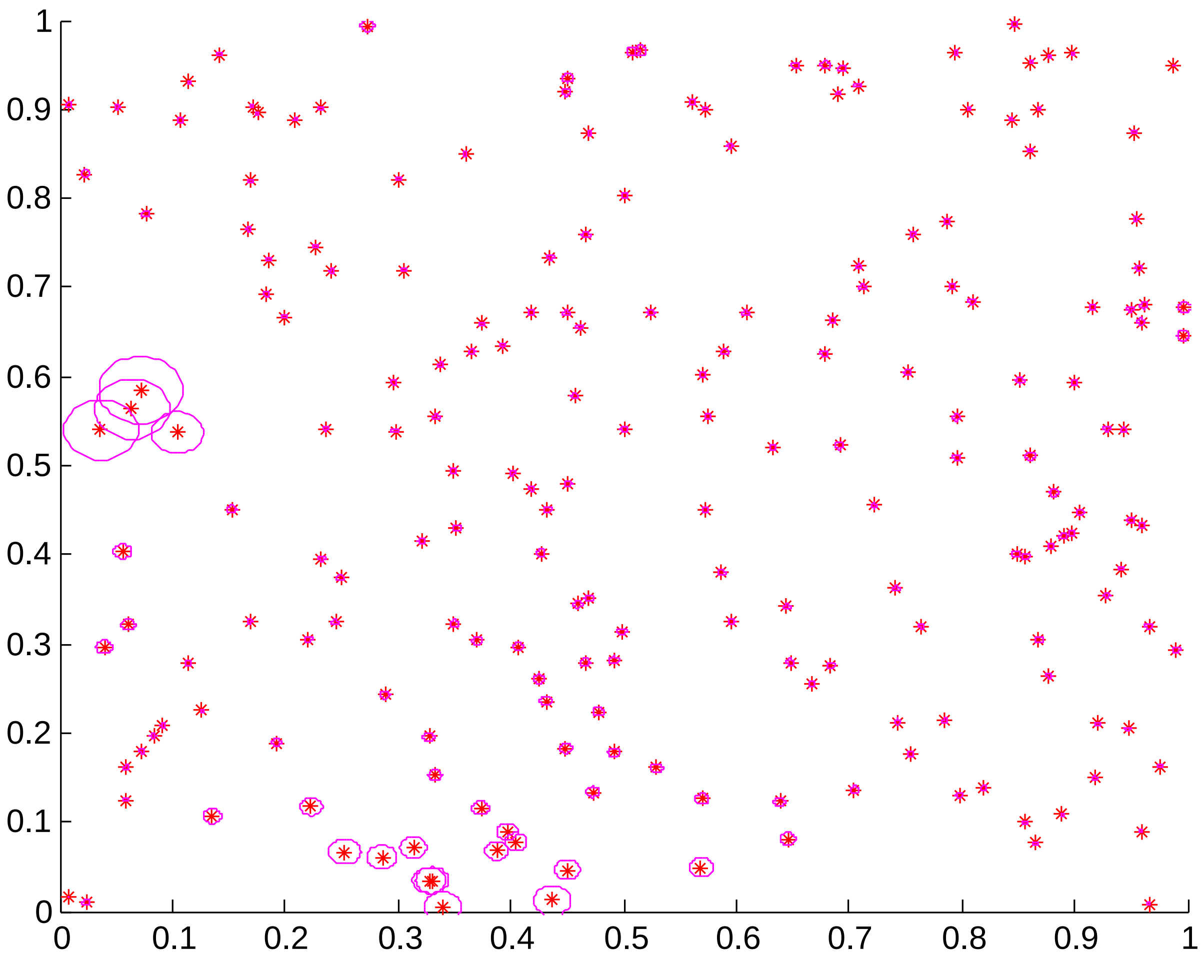
$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$ anchors at the corner of $[0, 1]^2$, $\rho = 0.14$.



the true location $\circ$ of sensor p

$\circ$——$\circ$: the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

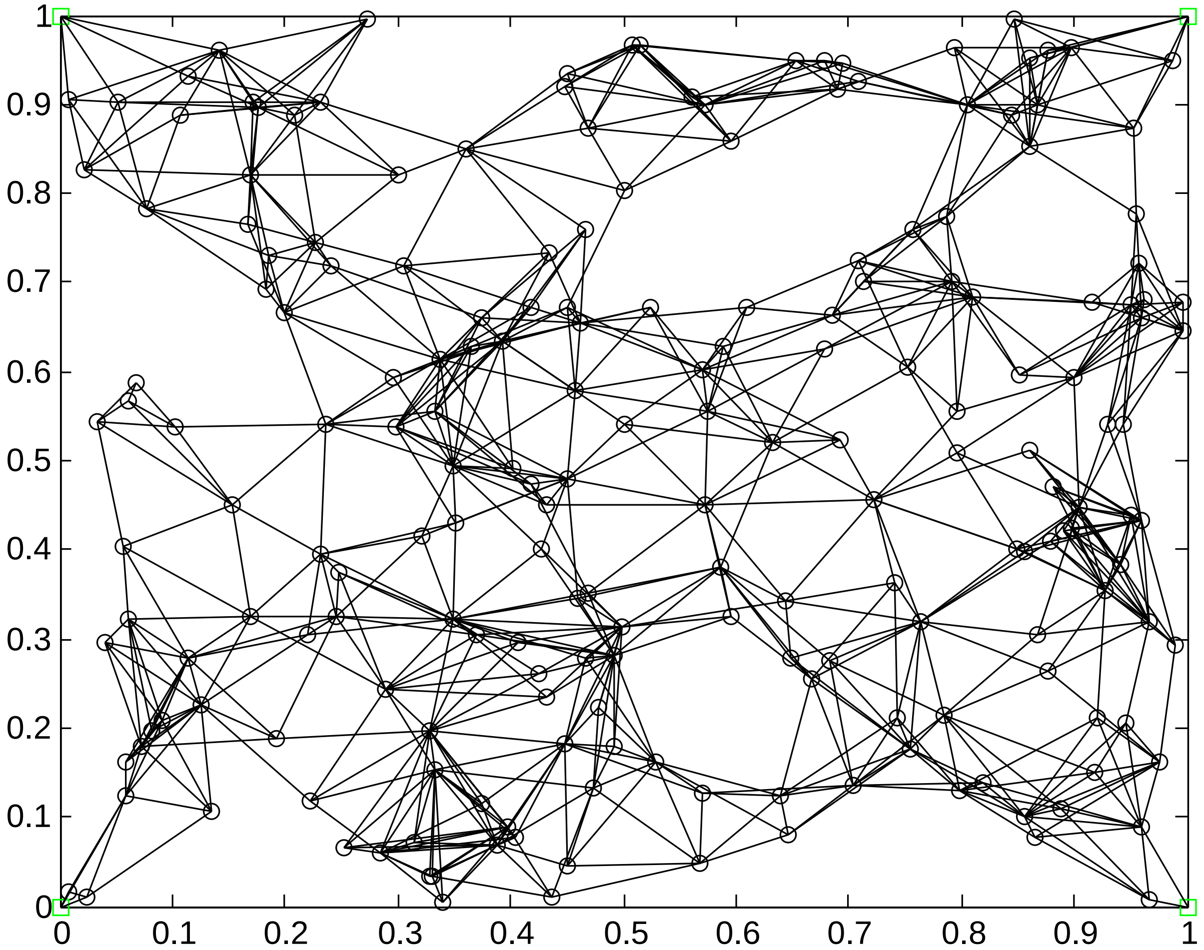
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.

the true location $\boldsymbol{x}^p$ of sensor p is within $(\gamma^p)^{1/2} \leq 0.04$ from $\boldsymbol{c}^p$

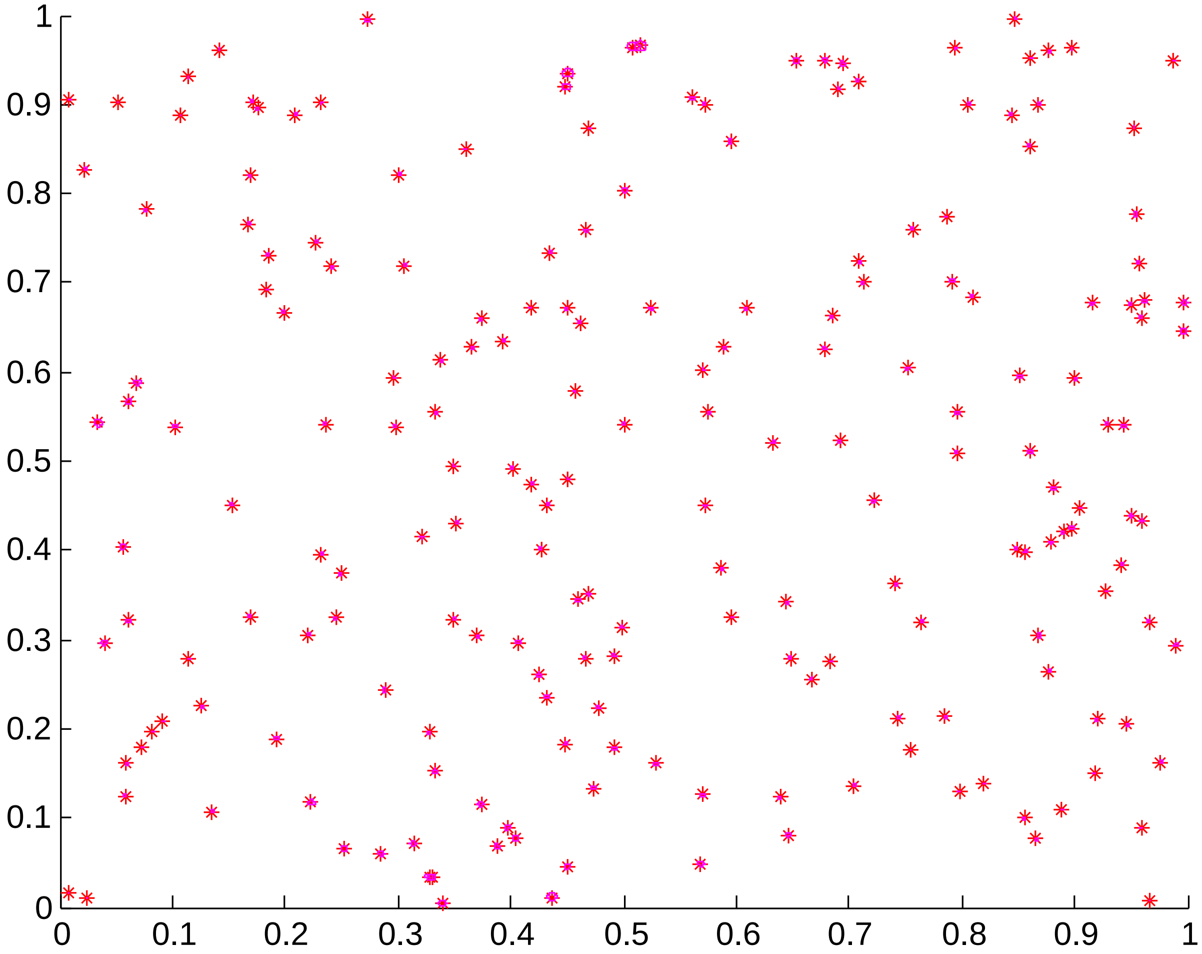
$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.15$.



the true location $\boldsymbol{x}^p$ of sensor p

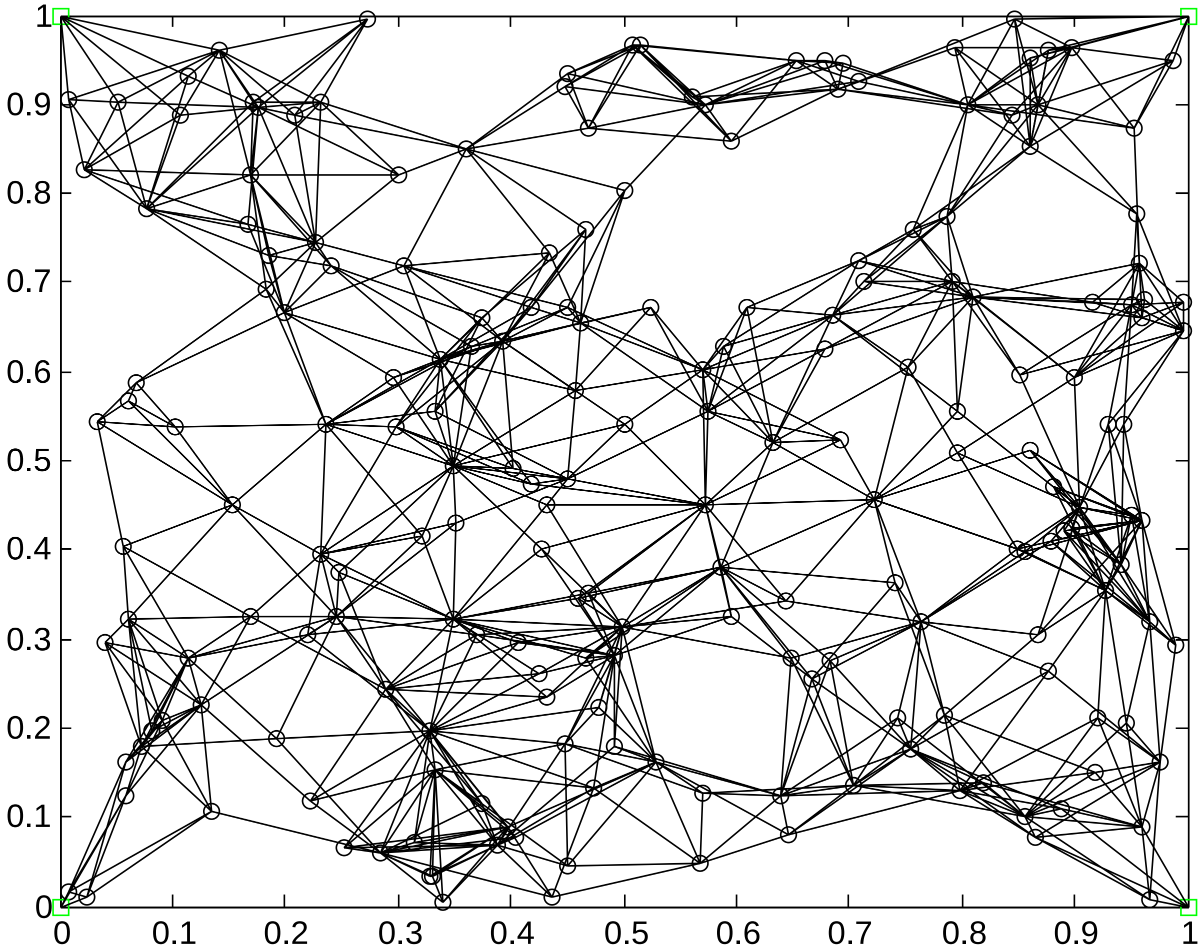
○——○ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

$m = 200$ sensors randomly distributed in $[0,1]^2$, $n - m = 4$ anchors at the corner of $[0,1]^2$, $\rho = 0.16$.



* : $\boldsymbol{c}^p$ = a computed location of censor p.
the true location $\boldsymbol{x}^p$ of sensor p is within $(\gamma^p)^{1/2} \leq 6.0$e-3 from $\boldsymbol{c}^p$

$m = 200$ sensors randomly distributed in $[0, 1]^2$, $n - m = 4$ anchors at the corner of $[0, 1]^2$, $\rho = 0.16$.



the true location $\boldsymbol{x}^p$ of sensor p

○——○ : the edge $(\boldsymbol{x}^p, \boldsymbol{x}^q)$ with a given exact distance

Outline

1. Problem and Some Formulations
2. Theory: Lifting and SDP Relaxation
3. Numerical Results
4. Applications to the Sensor Network Localization Problem
5. Concluding Remarks

Concluding Remarks

- We can apply the proposed method to sensor network localization problems with inexact distance involving measurement error, but the results are not sharp.
- Polynomial optimization problems with a 0-1 variable x to determine whether $x = 0$ or $x = 1$.