Obstructions to determinantal representability

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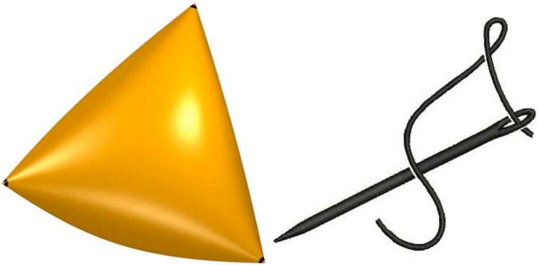


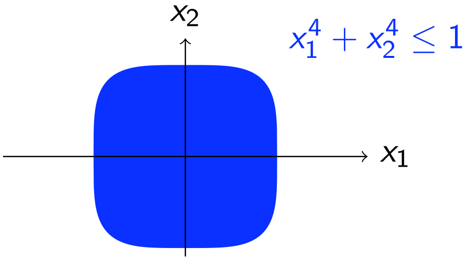
1. Spectrahedra and rigidly convex sets.
2. Spectrahedra = rigidly convex sets?
3. Counterexamples to RZ-polynomials = determinantal polynomials.
4. Counterexample to powers of RZ-polynomials = determinantal polynomials.
5. Further directions.

- A set $S \subseteq \mathbb{R}^n$ is a spectrahedron if there are symmetric $m \times m$ matrices, $A_0, A_1, \ldots, A_n$, such that

$$S = \{x \in \mathbb{R}^n : A_0 + x_1 A_1 + \cdots + x_n A_n \text{ is positive semidefinite}\}.$$

- Can we describe spectrahedra algebraically?
- S is convex.
- S is a closed semialgebraic set.
- All faces of S are exposed, that is, intersections of S with a supporting hyperplane.

Is the tv-screen a spectrahedron?



No it fails to be rigidly convex!

- Let us assume that $A_0 = I$, the identity matrix, and

$$S = \{x \in \mathbb{R}^n : I + x_1 A_1 + \cdots + x_n A_n \text{ is positive semidefinite}\}.$$

$$p(x) = \det(I + x_1 A_1 + \cdots + x_n A_n)$$

- Then S is the closure of the connected component of

$$\{x \in \mathbb{R}^n : p(x) \neq 0\}$$

that contains the origin. $S = \mathcal{C}_p$ (algebraic interior).

Real Zero polynomial (RZ polynomial)

A polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is a real zero polynomial if for all $\mu \in \mathbb{C}$ and $x \in \mathbb{R}^n$

$$p(\mu x) = 0 \quad \text{implies} \quad \mu \text{ is real}$$

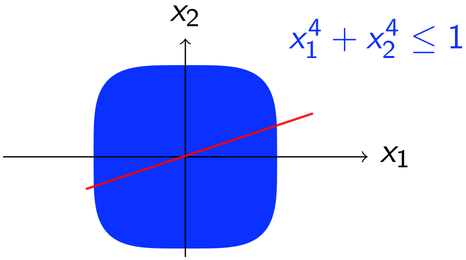
- If $p(x) = \det(I + x_1 A_1 + \cdots + x_n A_n)$, then $p(x)$ is a RZ polynomial:

$$p(\mu x) = \det(I + \mu(x_1 A_1 + \cdots + x_n A_n)) = \det(I + \mu A),$$

and $\det(I + tA)$ has only real zeros.

- S is rigidly convex if $S = \mathcal{C}_p$, where p is a RZ polynomial.
- Hence spectrahedra are rigidly convex.
- Rigidly convex sets are convex (Gårding).
- Their faces are exposed.

TV-screen is not rigidly convex



Conjecture (Helton and Vinnikov)

A set is a spectrahedron if and only if it is rigidly convex.

Solved for $n = 2$

Theorem (Helton and Vinnikov)

Let $p(x, y)$ be a RZ polynomial such that $p(0, 0) = 1$ and $\deg p = d$. Then there are symmetric $d \times d$ matrices A, B such that

$$p(x, y) = \det(I + xA + yB).$$

What about $n > 2$?

- The space of all RZ polynomials of degree d in n variables has nonempty interior (Nuij).
- Hence the dimension of the space of RZ polynomials with $p(0) = 1$ is $\binom{n+d}{n} - 1$.
- The space of all determinantal polynomials $\det(I + x_1 A_1 + \cdots + x_n A_n)$ has dimension at most $n\binom{d+1}{2}$.
- Hence if $d > 2$, an exact analog of the Helton–Vinnikov theorem does not hold. But we could allow matrices of a larger size.

Conjecture (Helton and Vinnikov)

Let $p(x_1, \ldots, x_n)$ be any RZ polynomial such that $p(0) = 1$. Then there are symmetric matrices $A_1, \ldots, A_n$ such that

$$p(x) = \det(I + x_1 A_1 + \cdots + x_n A_n).$$

A homogeneous polynomial $h(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to $e \in \mathbb{R}^n$ if

- $h(e) \neq 0$, and
- for each $x \in \mathbb{R}^n$ and $\mu \in \mathbb{C}$

$$h(x + \mu e) = 0 \quad \text{implies} \quad \mu \text{ is real.}$$

The hyperbolicity cone, Λ_{++}, of h at e is the connected component of

$$\{x \in \mathbb{R}^n : p(x) \neq 0\}$$

which contains e.

- $e' \in \Lambda_{++}$ if and only if all zeros of $t \mapsto h(e' + te)$ are negativ.
- Λ_{++} is convex (Gårding).
- The space of all degree d polynomials that are hyperbolic with respect to a fixed $e \in \mathbb{R}^n$ has nonempty interior (Nuij).
- A homogeneous polynomial $h(x, y, z)$, with $h(e_1, e_2, e_3) = 1$ and $\deg h = d$ is hyperbolic with respect e if and only if there are symmetric $d \times d$ matrices A, B, C such that $e_1 A + e_2 B + e_3 C = I$ and
$$p(x, y, z) = \det(xA + yB + zC).$$

Examples.

- Let $X = (x_{ij})_{i,j=1}^{n}$, where $x_{ij} = x_{ji}$ are variables. Then $\det(X)$ is hyperbolic with respect to I.

$$\det(X - \mu I) = \text{ characteristic polynomial.}$$

- The hyperbolicity cone is the cone of positive definite matrices.
- $h(x) = x_1^2 - x_2^2 - \cdots - x_n^2$, is hyperbolic with respect to $(1, 0, \ldots, 0)^T$.
- The hyperbolicity cone is the Lorentz cone

$$\{x \in \mathbb{R}^n : x_1 > 0 \text{ and } x_2^2 + \cdots + x_n^2 \leq x_1^2\}.$$

- If $h(x)$ is hyperbolic with respect to e, then $p(x) = h(x+e)$ is a RZ polynomial:

$$p(xt) = h(tx + e) = t^d h(x + t^{-1} e).$$

Theorem (B.)

Let $h(x)$ be a hyperbolic polynomial with respect to e, and let $p(x) = h(x+e)$. If p admits a representation

$$p(x) = \det(I + x_1 A_1 + \cdots + x_n A_n)$$

where A_j is symmetric (hermitian) and of size $N \times N$ for all j, then p admits a representation

$$p(x) = \det(I + x_1 B_1 + \cdots + x_n B_n)$$

where B_j is symmetric (hermitian) and of size $d \times d$ for all j, and d is the degree of h and p.

- Hence a count of parameters provides counterexamples to the Helton–Vinnikov conjecture.
- A relaxation of the conjecture is:

Conjecture

Let $p(x_1, \ldots, x_n)$ be any RZ polynomial such that $p(0) = 1$. Then there are symmetric matrices $A_1, \ldots, A_n$ and an integer $N > 0$ such that

$$p(x)^N = \det(I + x_1 A_1 + \cdots + x_n A_n).$$

- To disprove it we relate the problem to an old problem in matroid theory, namely representability of (poly-) matroids.

A polymatroid on a finite set E is a function $r : 2^E \to \mathbb{N}$ such that

- $r(\emptyset) = 0$;
- If $S \subseteq T \subseteq E$, then $r(S) \leq r(T)$;
- r is submodular, that is,

  $$r(S \cup T) + r(S \cap T) \leq r(S) + r(T),$$

  for all subsets S and T of E.
- Let $V_1, \ldots, V_n$ be subspaces of a vectorspace V over a field K. Then the function $r : 2^{\{1,\ldots,n\}} \to \mathbb{N}$ defined by

  $$r(S) = \dim \left(\sum_{j \in S} V_j \right)$$

  is a polymatroid, where $\sum_{j \in S} V_j$ is the smallest subspace containing $\cup_{j \in S} V_j$. These are called K-linear polymatroids.

- Suppose that $V_1, \ldots, V_n \subset \mathbb{C}^m$, and $A_1, \ldots, A_n$ are PSD hermitian matrices such that $\mathrm{Im}(A_i) = V_i$, then

$$r(S) = \dim\left(\sum_{j \in S} V_j\right) = \mathrm{rank}\left(\sum_{j \in S} A_j\right) = \deg\det\left(I + t\left(\sum_{j \in S} A_j\right)\right)$$

Theorem (Gurvits, B.)

Let h be a hyperbolic polynomial with respect to e, and let $e_1, \ldots, e_n \in \Lambda_+$. Then $r : 2^{\{1,\ldots,n\}} \to \mathbb{N}$ defined by

$$r(S) = \deg h\left(e + t\left(\sum_{j \in S} e_j\right)\right)$$

is a polymatroid. Call it a hyperbolic polymatroid.

Obstructions to K-linearity.

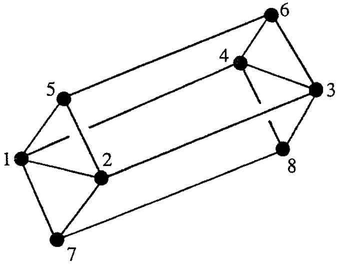
Ingleton's inequalities (1969)

Suppose that $r : 2^{\{1,\dots,n\}} \to \mathbb{N}$ is a K-linear polymatroid. Then

$$r(S_1 \cup S_2) + r(S_1 \cup S_3 \cup S_4) + r(S_3) + r(S_4) + r(S_2 \cup S_3 \cup S_4) \leq$$
$$r(S_1 \cup S_3) + r(S_1 \cup S_4) + r(S_2 \cup S_3) + r(S_2 \cup S_4) + r(S_3 \cup S_4)$$

for all $S_1, S_2, S_3, S_4 \in 2^{\{1,\dots,n\}}$.

Let $\mathcal{B}$ be the collection of all subsets of size 4 of $\{1, \ldots, 8\}$ such that the corresponding vertices do not lie in an affine plane in the following figure



The Vámos cube

Let further

$$h(x) = \sum_{B \in \mathcal{B}} \prod_{j \in B} x_j = x_1 x_2 x_3 x_5 + x_1 x_2 x_3 x_6 + \cdots .$$

Theorem (Wagner-Wei, 2009)

$h(x)$ is hyperbolic with hyperbolicity cone containing $\mathbb{R}^8_{++}$.

Theorem (Ingleton, 1969)

The polymatroid associated to $h(x)$ (with $e_1, \ldots, e_n$ the standard basis vectors) fails to satisfy Ingleton's inequalities.

- Hence the rank function of $h(x)^N$ does not satisfy Ingleton's inequalities.
- Thus we cannot have

$$h(x+e)^N = \det(I + x_1 A_1 + \cdots + \cdots x_8 A_8)$$

How is Wagner-Wei's theorem proved?

- Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
- A homogenous real polynomial $h(x)$ is hyperbolic with $\mathbb{R}^n_{++} \subseteq \Lambda_{++}$ if and only if it satisfies

$$z \in \mathbb{H}^n \quad \Longrightarrow \quad h(z) \neq 0. \qquad (*)$$

- Indeed, if $P(x + iy) = 0$ for some $x + iy \in \mathbb{H}^n$, then P cannot be hyperbolic with $\mathbb{R}^n_{++} \subseteq \Lambda_{++}$ since then all zeros of $t \mapsto P(x + ty)$ are real.
- Conversely, if P fails to be hyperbolic with respect to some $y \in \mathbb{R}^n_{+}$, then there is an $x \in \mathbb{R}^n$ for which $t \mapsto P(x + ty)$ has a non-real zero $a + bi$, where $b > 0$.
- But then $z = x + ay + iby \in \mathbb{H}$ and $P(z) = 0$.
- A polynomial $P(x) \in \mathbb{R}[x_1, \ldots, x_n]$ satisfying $(*)$ is called stable.

Theorem (B.)

Suppose $P(x) \in \mathbb{R}[x_1, \ldots, x_n]$ has degree at most one in each variable. Then P is stable if and only if

$$\frac{\partial P}{\partial x_i}\frac{\partial P}{\partial x_j} - \frac{\partial^2 P}{\partial x_i \partial x_j} \geq 0, \qquad \text{for all } x \in \mathbb{R}^n \text{ and } 1 \leq i < j \leq n.$$

- Wagner and Wei proved that the above polynomials are SOS for our choice of h.

$$
\begin{aligned}
&\frac{1}{4}(y_3y_4y_5+y_3y_4y_6+y_3y_4y_7+y_3y_4y_8+y_3y_5y_7+y_3y_5y_8\\
&\qquad+y_3y_6y_7+y_3y_6y_8+y_4y_5y_7+y_4y_5y_8+y_4y_6y_7\\
&\qquad+y_4y_6y_8+y_5y_6y_7+y_5y_6y_8+y_5y_7y_8+y_6y_7y_8)^2\\
&+\frac{1}{4}(y_3y_4y_5+y_3y_4y_6+y_3y_4y_7+y_3y_4y_8\\
&\qquad+y_3y_5y_7+y_3y_5y_8+y_4y_6y_7+y_4y_6y_8)^2\\
&+\frac{1}{4}(y_3y_4y_5+y_3y_4y_6+y_3y_4y_7+y_3y_4y_8\\
&\qquad+y_3y_6y_7+y_3y_6y_8+y_4y_5y_8+y_4y_5y_7)^2\\
&+\frac{1}{4}(y_3y_4y_5+y_3y_4y_6+y_3y_4y_7+y_3y_4y_8\\
&\qquad-y_6y_7y_8-y_5y_7y_8-y_5y_6y_7-y_5y_6y_8)^2\\
&+\frac{1}{8}(y_3y_6y_7-y_3y_5y_8+y_4y_6y_7-y_4y_5y_8+y_6y_7y_8-y_5y_6y_8)^2\\
&+\frac{1}{8}(y_3y_5y_7-y_3y_6y_8-y_4y_6y_8+y_4y_5y_7+y_5y_7y_8-y_5y_6y_8)^2\\
&+\frac{1}{8}(y_3y_5y_7+y_3y_6y_7+y_4y_6y_8+y_4y_5y_8+y_5y_6y_7+y_5y_7y_8)^2\\
&+\frac{1}{8}(y_3y_5y_8+y_3y_6y_8+y_4y_6y_7+y_4y_5y_7+y_5y_6y_7+y_6y_7y_8)^2\\
&+\frac{1}{8}(y_3y_6y_7-y_3y_5y_8+y_4y_6y_7-y_4y_5y_8+y_5y_6y_7-y_5y_7y_8)^2\\
&+\frac{1}{8}(y_3y_5y_7-y_3y_6y_8-y_4y_6y_8+y_4y_5y_7+y_5y_6y_7-y_6y_7y_8)^2\\
&+\frac{1}{8}(y_3y_5y_7+y_3y_6y_7+y_4y_6y_8+y_4y_5y_8+y_5y_6y_8+y_6y_7y_8)^2\\
&+\frac{1}{8}(y_3y_5y_8+y_3y_6y_8+y_4y_6y_7+y_4y_5y_7+y_5y_6y_8+y_5y_7y_8)^2
\end{aligned}
$$

- Netzer and Thom have very recently improved on some of our results:
- They find a degree two polynomial that does not have a determinantal representation.
- They prove that for each quadratic polynomial, some power of it has a determinantal representation.

- If h is a hyperbolic polynomial with respect to $e \in \Lambda_{++}$, define a rank function $\mathrm{rank}_h : \mathbb{R}^n \to \mathbb{N}$ by

$$\mathrm{rank}_h(x) = \deg p(e + tx).$$

- Does not depend on the choice of $e \in \Lambda_{++}$.
- Let Λ_+ be the closure of the hyperbolicity cone of h.
- the rank is constant on open line segments.
- The rank drops when going from the relative interior of a face to its boundary.

Theorem (Gurvits, B.)

Let $u, v, w \in \Lambda_+$, then

$$\mathrm{rank}(u + v + w) + \mathrm{rank}(w) \leq \mathrm{rank}(u + w) + \mathrm{rank}(v + w)$$

Problem 1. Given the closure of a hyperbolicity cone Λ_+, what are the possible rank functions on Λ_+?

Problem 2. Describe properties of the rank function on positive semidefinite matrices that are not valid for hyperbolicity cones in general.

Problem 3. Suppose that h is an irreducible hyperbolic polynomial and that

$$rh^N = \det(x_1 A_1 + \cdots + x_n A_n)$$

where r is is nonzero on Λ_{++}. How can the zero set of r intersect Λ_+?