Positivity, Sums of Squares and Positivstellensätze for Noncommutative $*$-Algebras

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Contents

Artin's theorem and Reznick's theorem

Let us abbreviate: $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_d]$,
$\sum \mathbb{R}[x]^2$: set of all finite sums of squares p^2, where $p \in \mathbb{R}[x]$.

Artin's theorem (1927) on the solution of Hilbert 17th problem:

Let $p(x_1, \dots, p_d) \in \mathbb{R}[x]$. Suppose that $p(t) \geq 0$ for all $t \in \mathbb{R}^d$.

Then p is a sum of squares of rational functions, that is, there exists a $q \in \mathbb{R}[x]$, $q \neq 0$, such that $q^2 p \in \sum \mathbb{R}[x]^2$.

If one knows more about p, what can be said about the denominator q?
$\Longrightarrow$ Archimedean Positivstellensatz (1991)

Reznick's theorem (1995):

Let $p \in \mathbb{R}[x]$ be a homogeneous polynomial.
Suppose that $p(t) > 0$ for all $t \in \mathbb{R}^d$, $t \neq 0$.

Then there exists $n \in \mathbb{N}$ such that $(x_1^2 + \cdots + x_d^2)^n \; p \in \sum \mathbb{R}[x]^2$.

A Strict Positivstellensatz for Polynomials

Theorem:

$$p(x_1, x_2) = \sum_{j,k} \gamma_{jl} x_1^j x_2^l = \sum_{n=0}^{d_2} f_n(x_1) x_2^n = \sum_{k=0}^{d_1} g_k(x_2) x_1^k.$$

Suppose:

(i) $p(t_1, t_2) > 0$ for all $(t_1, t_2) \in \mathbb{R}^2$.

(ii) $\gamma_{d_1,d_2} > 0$, $f_{d_2}(t) > 0$ and $g_{d_1}(t) > 0$ for all $t \in \mathbb{R}$.

Let $\mathcal{M}$ be the set of all finite products of $x_1 \pm i$ and $x_2 \pm i$.

Then there exists $c \in \mathcal{M}$ such that $\overline{c} p c \in \sum \mathbb{R}[x_1, x_2]^2$.

Can be derived from the Archimedean Positivstellensatz applied to the "fraction algebra" generated by $(x_1 \pm i)^{-1}$ and $(x_2 \pm i)^{-1}$.

A Strict Positivstellensatz for the Weyl Algebra

Let $\mathcal{A}$ is the algebra of differential operators acting on $C_0^\infty(\mathbb{R})$:

$$a = \sum_{k=0}^{n} g_k(x)(\frac{d}{dx})^k, \quad g_k \in \mathbb{C}[x].$$

Set $p := i\frac{d}{dx}$ and $q = x$. Each element $a \in \mathcal{A}$, $a \neq 0$, can be written as

$$a = \sum_{j=0}^{d_1}\sum_{l=0}^{d_2} \gamma_{jl} p^j q^l = \sum_{n=0}^{d_2} f_n(p) q^n = \sum_{k=0}^{d_1} g_k(q) p^k,$$

where $\gamma_{jl} \in \mathbb{C}$, $f_n(p) \in \mathbb{C}[p]$, $g_k(q) \in \mathbb{C}[q]$ uniquely determined by a.
Set $d(a)=(d_1, d_2)$ if there are $j_0, l_0 \in \mathbb{N}_0$ such that $\gamma_{d_1,l_0} \neq 0$ and $\gamma_{j_0,d_2} \neq 0$.

A Strict Positivstellensatz for the Weyl Algebra

Theorem: K.S. Crelle (2010)

Let $a \in \mathcal{A}$, $a \neq 0$, and $d(a) = (d_1, d_2)$.

Let $\mathcal{S}$ be the set of all finite products of $p \pm i, q \pm i$.

(I) Suppose that there exists $\varepsilon > 0$ such that $a \geq \varepsilon$, that is,

$$\int_{-\infty}^{\infty} (af)(x)\overline{f(x)}dx \geq \varepsilon \int_{-\infty}^{\infty} |f(x)|^2 dx, \quad f \in C_0^\infty(\mathbb{R}).$$

(II) $\gamma_{d_1,d_2} > 0$, $f_{d_2}(t) > 0$ and $g_{d_1}(t) > 0$ for $t \in \mathbb{R}$.

Then there exists an element $s \in \mathcal{S}$ such that

$$s^* as \in \sum \mathcal{A}^2.$$

$s^* = (p - i)(q + i)$ if $s = (q - i)(p + i)$, that is, $p^* = p$, $q^* = q$.

$\sum \mathcal{A}^2$: finite sums of elements $b^* b$, where $b \in \mathcal{A}$.

A Strict Positivstellensatz for the Weyl Algebra

Idea of proof: "fraction algebra" generated by $(p \pm i)^{-1}$ and $(q \pm i)^{-1}$.

Possible Application: "Noncommutative Optimization"

Elements a of the algebra $\mathcal{A}$ act as differential operators

Idea:
Use Positivstellensatz to compute the **infimum of the spectrum** of a.

First attempt: J. Cimpric (2009)

What are positive polynomials?

When is $p \in \mathbb{R}[x_1, \ldots, x_d]$ positive (nonnegative)?

Answer 1:

p is *positive* if p is a sum of squares (of rational functions).

Answer 2:

p is *positive* if p is positive in all orderings of the field $\mathbb{R}(x_1, \cdots, x_d)$.

Answer 3:

p is *positive* if $p(t_1, \ldots, t_d) \geq 0$ for all $(t_1, \ldots, t_d) \in \mathbb{R}^d$.

Question:

How to generalize these concepts to noncommutative algebras?

Star Algebras

Let $\mathcal{A}$ be a complex or real unital algebra and let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.
Question:

How do to define "positive elements" of $\mathcal{A}$?

First Step:

An algebra involution on $\mathcal{A}$ is needed!

An **algebra involution** of $\mathcal{A}$ is a mapping $a \to a^*$ of $\mathcal{A}$ into $\mathcal{A}$ such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$, $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{K}$.

A $*$-**algebra** is an algebra equipped with an algebra involution.

In what follows **we suppose that $\mathcal{A}$ is a unital $*$-algebra**.

Star Algebras

Classical Real Algebraic Geometry:

$\mathcal{A} = \mathbb{R}[x_1, \ldots, x_d]$, $p^* := p$ or
$\mathcal{A} = \mathbb{C}[x_1, \ldots, x_d]$, $p^* = \overline{p}$, where $\overline{p}(x) = \sum \overline{a_\alpha} x^\alpha$ for $p(x) = \sum a_\alpha x^\alpha$.

Positivity of the Involution

All involutions occuring in this talk satisfy the following condition:

(P): If $x_1^* x_1 + \cdots + x_k^* x_k = 0$ for $x_1, \ldots, x_k \in \mathcal{A}$, then $x_1 = \cdots = x_k = 0$.

Matrix Algebra $M_n(\mathbb{K})$:

Let B be a diagonal matrix with non-zero real diagonal entries b_k.
Define $A^* := B\overline{A^t}B^{-1}$, where $\overline{A^t} = (\overline{a_{ji}})$ for $A = (a_{ij})$.
Then $A \to A^*$ defines an involution on $M_n(\mathbb{K})$.
Condition (P) is satisfied if and only if $b_k > 0$ for all k.

Quadratic Modules

Definition: Quadratic Modules

A **quadratic module** of $\mathcal{A}$ is a subset $\mathcal{C}$ of $\mathcal{A}_h := \{a{=}a^* : a \in \mathcal{A}\}$ s. t.

$1 \in \mathcal{C}$, $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$, $\mathbb{R}_+ \cdot \mathcal{C} \subseteq \mathcal{C}$ and $b^* \mathcal{C} b \in \mathcal{C}$ for all $b \in \mathcal{A}$.

Examples

1. If $\mathcal{X}$ is a subset of $\mathcal{A}_h$ such that $1 \in \mathcal{X}$, then

$$\mathcal{C}_{\mathcal{X}} := \text{finite sums of elements} \;\; a^* x a, \;\; \text{where} \;\; a \in \mathcal{A}, \;\; x \in \mathcal{X},$$

is the **quadratic module** of $\mathcal{A}$ generated by the set $\mathcal{X}$.

2. The **smallest quadratic module** of $\mathcal{A}$ is the set

$\sum \mathcal{A}^2$ of all finite sums of **squares** $a^* a$, where $a \in \mathcal{A}$.

Quadratic Modules Defined by Representations

Let $\mathcal{D}$ be a vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$.

Definition: $*$-Representation

A $*$- **representation** of $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism π of $\mathcal{A}$ into the algebra $L(\mathcal{D})$ such that $\pi(1)\varphi = \varphi$ and $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$.

We write $\pi(a) \geq 0$ when $\langle \pi(a)\varphi, \varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}$.

Definition: Quadratic Module $\mathcal{A}(\mathcal{S})_+$

For a family $\mathcal{S}$ of $*$-representations of $\mathcal{A}$, we define a **quadratic module**

$$\mathcal{A}(\mathcal{S})_+ := \{a = a^* \in \mathcal{A} : \pi(a) \geq 0 \ \text{ for all } \ \pi \in \mathcal{S}\}.$$

Quadratic Modules Defined by $*$–Orderings

Let $\mathbb{K}$ be a formally real field, $\mathcal{A}$ a centrally simple $\mathbb{K}$-algebra and $a \to a^*$ an involution on $\mathcal{A}$. Further, let $\mathrm{tr} : \mathcal{A} \to \mathbb{K}$ be the reduced trace.

Definition (Procesi, Schacher, 1976)

A $*$-*ordering* is a preordering T on $\mathbb{K}$ s.t. $\mathrm{tr}\,(b^* b) \in T$ for $b \in \mathcal{A}$.

$$\mathcal{P}_{\mathcal{A}} = \{a{=}a^* \in \mathcal{A} : \mathrm{tr}\,(b^* b \cdot a) \in T \text{ for all } * \text{–orderings } \; T, \text{ all } b \in \mathcal{A}\}$$

is a **quadratic module** on $\mathcal{A}$.

This definition applies f. i. for the matrices over rational functions.

What are Noncommutative Positivstellensätze ?

Positivstellensätze

There is an interplay between quadratic modules
which are defined in algebraic terms (such as $\sum \mathcal{A}^2$ or $\mathcal{C}_{\mathcal{X}}$)
and those
which are defined by means of $*$-representations or $*$-orderings
(such as $\mathcal{A}(\mathcal{S})_+$ or $\mathcal{P}_{\mathcal{A}}$).

This is one of the most interesting challenges for the theory!

Positivstellensätze show how elements of $\mathcal{A}(\mathcal{S})_+$ or $\mathcal{P}_{\mathcal{A}}$ can be representated by means of $\sum \mathcal{A}^2$ or $\mathcal{C}_{\mathcal{X}}$.

Artin's theorem: $\mathcal{A} = \mathbb{R}[x]$, $\mathcal{S} = \{\pi_t(p) = p(t); t \in \mathbb{R}^d\}$

$p \in \mathcal{A}(\mathcal{S})_+$, that is $p \geq 0$ on $\mathbb{R}^d$, iff $q^2 p \in \sum \mathcal{A}^2$ for some $q \in \mathcal{A}$, $q \neq 0$.

Role of the Family $\mathcal{S}$ of Representations

Theorem: K.S. 1979

If $\mathcal{A}$ is the **commutative polynomial algebra** $\mathbb{C}[x_1, \ldots, x_d]$, the **Weyl algebra** $\mathcal{W}(d)$, the **enveloping algebra** $\mathcal{E}(g)$ or the **free polynomial algebra** $\mathbb{C} < x_1, \ldots, x_d >$, then $\sum \mathcal{A}^2$ is **closed** in the finest locally convex topology on $\mathcal{A}$.

Corollary:

Let $\mathcal{A}$ be one of the above four $*$-algebras and let $a \in \mathcal{A}$. Then: $a \in \sum \ \mathcal{A}^2$ **if and only if** $\pi(a) \geq 0$ **for all** $*$**-representations** π **of** $\mathcal{A}$**.**

$\mathcal{A} = \mathbb{C} < x_1, \ldots, x_d >$: This is Helton's theorem.

In order to get an interesting theory on has to select an appropriate **class of "good" representations**!

Some Interesting Examples

Example 1: Commutative Polynomial Algebra $\mathbb{R}[x]$

$\mathcal{S} := \{\pi_t : t \in \mathbb{R}\}$, where $\pi_t(p) = p(t)$, or

$\mathcal{S} = \{\pi_\mu\}$, where $\pi_\mu(p)q = p{\cdot}q$ for $p, q \in \mathbb{R}[x] \subseteq L^2(\mathbb{R}^d, \mu)$.

Example 2: Weyl Algebra
$\mathcal{W} = \mathbb{C} < a, a^* | aa^* - a^*a = 1 >= \mathbb{C} < p = p^*, q = q^* | pq - qp = -i >$

$\mathcal{S} = \{\pi_0\}$, where π_0 is the **Bargmann-Fock representation**

$\pi_0(a)e_n = n^{1/2}e_{n-1}$, $\pi_0(a^*)e_n = (n+1)^{1/2}e_{n+1}$ on $l^2(\mathbb{N}_0)$

or the **Schrödinger representation**

$\pi_0(q)f = xf(x)$, $\pi_0(p)f = -if'(x)$ on $L^2(\mathbb{R})$.

Some Interesting Examples

Example 3: Enveloping algebra $\mathcal{E}(g)$ of a real Lie algebra g with involution $x^* = -x$ for $x \in g$

$\mathcal{S} = \{dU; U \text{ unitary representation of } G\}$

Example 4: Free polynomial algebra $\mathcal{A} = \mathbb{C} < x_1, \ldots, x_d >$, $x_j^* = x_j$.

If $X_1, \ldots, X_d$ are **arbitrary** bounded self-adjoint operators, then there is a $*$-representation π such that $\pi(x_1) = X_1, \ldots, \pi(x_d) = X_d$.

Given $f = (f_1, \cdot, f_k), f_k \in \mathbb{C} < x_1, \ldots, x_d >$,
let $\mathcal{S}_f$ be the set of all bounded $*$-representations π such that

$$f_j(\pi(x_1), \cdots, \pi(x_d)) \geq 0, \quad j{=}1, \cdots, k.$$

$\Longrightarrow$ **"Free semialgebraic geometry"** of B. Helton and his coworkers

What about Artin's Theorem in the Noncommutative Case?

Artin's Theorem on the solution of Hilbert's 17th problem:

For each nonnegative polynomial p on $\mathbb{R}^d$ there exists a nonzero polynomial $q \in \mathbb{R}[x]$ such that $q^2 p \in \sum \mathbb{R}[x]^2$.

For a noncommutative $*$-algebra $\mathcal{A}$ it is natural to generalize the latter to

$$c^* a c \in \sum \mathcal{A}^2.$$

This will be our version 2 of Artin's theorem.

What about Artin's Theorem in the Noncommutative Case?

One might also think of

$$\sum_k c_k^* a c_k \in \sum \mathcal{A}^2,$$

but it can be shown that such a condition corresponds to a Nichtnegativstellensatz rather than a Positivstellensatz.

Positivstellensatz: $\langle \pi(a)\varphi, \varphi \rangle \neq 0$ for **all** vectors φ.

Nichtnegativstellensatz: $\langle \pi(a)\varphi, \varphi \rangle > 0$ for **at least one** vector φ.

That is, $\pi(a) \leq 0$ does **not** hold.

An Essential Difference

In the commutative case $q^2 p \in \sum \mathbb{R}[t]^2$ implies that $p \geq 0$ on $\mathbb{R}^d$.

However, in the noncommutative case such a converse is not true.

Example: Weyl algebra

Let $\mathcal{A}$ be the Weyl algebra $\mathcal{W}$ and $\mathcal{S}=\{\pi_0\}$, see Example 2 above.

Set $N=a^*a$. Since $aa^*-a^*a=1$, we have

$$a(N-1)a^*=N^2+a^*a \in \sum \mathcal{A}^2.$$

But $\pi_0(N-1)$ has the eigenvalue -1, so it is not nonnegative,

since $\langle \pi_0(N-1)e_0, e_0 \rangle = -1$ for the vacuum vector e_0.

One needs additional conditions to ensure that then $c \in \mathcal{A}(\mathcal{S})_+$.

Version 1 of Artin's Theorem: Denominatorfree

Version 1: Denominator Free

For any $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$ we have

$$a \in \sum \mathcal{A}^2.$$

Examples of Version 1

Let $\mathcal{S}$ be the family of all $*$-representaions.

Each element $a \in \mathcal{A}(\mathcal{S})_+$ is a square $a = b^* b$, where $b \in \mathcal{A}$.

This assertion holds for the following algebras:

- $\mathbb{C} < z, z^{-1} | z^* z = zz^* = 1 >$ trigonometric polynomials (Riesz-Fejer theorem 1915)
- $\mathbb{C} < s, s^* | s^* s = 1 >$ - $*$-algebra generated by an isometry (Noncommutative Riesz-Fejer theorem: Y. Savchuk, K.S. 2010).
- $\mathbb{C} < x_1, x_1^*, \dots, x_d, x_d^* | \; x_k^* x_k = 1, \; x_1 x_1^* + \dots + x_d x_d^* = 1 >$ Algebraic Cuntz algebra (Zimmermann 2010)
- $M_n(\mathbb{C}[x_1])$ matrices of polynomials in one variable (Djokovic 1976)

In particular, version 1 holds for each of these algebras.

Examples of Version 1

Version 1 holds, that is, $\mathcal{A}(\mathcal{S})_+ \subseteq \sum \mathcal{A}^2$.

Version 1 holds for each of the following algebras:

- $\mathbb{R}[C]$ - **coordinate algebra** of an irreducible smooth affine **curve** C which has at least one **nonreal** point at infinity. (C. Scheiderer)

  Example: $x^3 + y^3 + 1 = 0 \Longrightarrow$ version 1 holds.

  Example: $y^3 = x^2$. Then $y \notin \sum \mathcal{A}^2$, so version 1 does not hold!

- $\mathbb{C} < x_1, x_1^*, \ldots, x_d, x_d^* | \; x_1^* x_1 + \cdots + x_d^* x_d = 1 >$

  Spherical Isometries (Helton/McCullough/Putinar)

- $\mathbb{C}[G]$ group algebra of a free group G with involution $g^* = g^{-1}$.

Version 2 of Artin's Theorem: With Denominators

Let $\mathcal{A}^\circ$ be the set of $a \in \mathcal{A}$ which are not zero divisors.

Version 2: With Denominators

For any $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$ there exists a $c \in \mathcal{A}^\circ$ such that

$$c^* ac \in \sum \mathcal{A}^2.$$

Example 1: Matrices of Poynomials
Gondard/Ribenboim (1974), Procesi/Schacher (1976)

$\mathcal{A} = M_n(\mathbb{R}[x_1, \ldots, x_d])$ and $\mathcal{S} = \{\pi_t((a_{ij})) = (a_{ij}(t)); t \in \mathbb{R}^d\}$.
Then version 2 holds.

There is a proof based on Schur complements of matrices.
This method can be extended to matrices over noncommutative algebras.

Version 2 of Artin's Theorem: With Denominators

Let $\mathcal{A}$ be a unital $*$-algebra of operators on a pre-Hilbert space. Suppose $\mathcal{A}\backslash\{0\}$ satisfies a right Ore condition.

Theorem: Savchuk, K.S. (2010)

If $\mathcal{A}$ satisfies version 2, then also $M_n(\mathcal{A})$.

Let σ be a $*$-automorphim of order n of $\mathcal{A}$.
If $\mathcal{A}$ satisfies version 2, so does the cross product algebra $\mathcal{A} \times_\alpha \mathbb{Z}_n$.

If σ is an $*$-automorphism of order 3, then $\mathcal{A} \times_\alpha \mathbb{Z}_3$ is the set of matrices

$$\begin{pmatrix} a & b & c \\ \sigma(c) & \sigma(a) & \sigma(b) \\ \sigma^2(b) & \sigma^2(c) & \sigma^2(a) \end{pmatrix}, \quad a, b, c \in \mathcal{A}.$$

Some Open Problems

Problem 1:

Suppose version 1 holds for $\mathcal{A}$. Does it hold for the algebra $M_n(\mathcal{A})$?

Example: Let $\mathcal{A}$ the polynomial algebra on the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Then version 1 holds for $\mathcal{A}$ (C. Scheiderer).

Subproblem 1.1:

Does version 1 hold for $M_n(\mathcal{A})$?

An afffirmative answer would imply a number of other Positivstellensätze! For instance it would imply the following:

Let $A \in M_n(\mathbb{C}[x, y])$. If $A \geq 0$ on the unit circle $\{x^2 + y^2 \leq 1\}$, then

$$A \in \sum M_n(\mathbb{C}[x, y])^2 + (1 - x^2 - y^2) \sum M_n(\mathbb{C}[x, y])^2.$$

Some Open Problems

Let $\mathcal{A}$ be the $*$-algebra of operators

$$a = \sum_{k_1,\ldots,k_d} f_k(x) \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{k_d}, \quad f_k \in \mathbb{C}[x].$$

Problem 2:
Does Version 2 of Artin's Theorem hold for the Weyl Algebra?

Suppose $\langle a\varphi, \varphi\rangle \geq 0$ for $\varphi \in C_0^\infty(\mathbb{R}^d)$. Does there exist $c \in \mathcal{A}^\circ$ s. t.

$$c^* ac \in \sum \mathcal{A}^2?$$

A Version of a Noncommutative Stengle Theorem

Suppose $\langle a\varphi, \varphi\rangle \geq 0$ for $\varphi \in C_0^\infty(0, +\infty)$. Does there exist $c \in \mathcal{A}^\circ$ s. t.

$$c^* ac \in \sum \mathcal{A}^2 + x \sum \mathcal{A}^2?$$

Version 3: An Example

C.Procesi and M.Schacher (1976) asked if version 1 holds for a **centrally simple algebra** $\mathcal{A}$, that is,

$$\mathcal{P}_{\mathcal{A}} \subseteq \sum \mathcal{A}^2 \text{ ?}$$

No! Counterexample: Klep/Unger (2008).
We give another counterexample and propose a new type of Positivstellensatz.

Let $\mathfrak{A}$ the $*$-subalgebra of $M_3(\mathbb{C}[x, y, z])$ generated by the matrix

$$A = \begin{pmatrix} 0 & 0 & z \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}.$$

Let $\mathcal{A}$ be the localization of $\mathfrak{A}$ by $Z(\mathcal{A})$.

Version 3: An Example

Proposition:

- $A^*A \cdot AA^* \in \mathcal{P}_{\mathcal{A}}$.
- There is no $C \in \mathcal{A}^\circ$ such that $C^*(A^*A^2A^*)C \in \sum \mathcal{A}^2$.

That is, version 2 does not hold!

"Positivstellensatz":

For every $V \in \mathcal{P}_{\mathcal{A}}$ there exist $Y_i, Z_j \in \mathcal{A}$ such that

$$V = \sum_i Y_i^* Y_i + \sum_j Z_j^*(A^*A^2A^*)Z_j. \tag{1}$$

$A^*A \cdot AA^* = AA^* \cdot A^*A$ is a product of two commuting squares, but not a sum of squares.

Version 3 of Artin's Theorem: NC Sums of Squares

Denominators Sets $\mathcal{D}_a$ and Right Hand Sides $\sum_{nc}$

Let $a \in \mathcal{A}_h$. We form a set $\mathcal{D}_a \subseteq \mathcal{A}_h$ such that $a \in \mathcal{D}_a$ and

(i) If $b \in \mathcal{D}_a$ and $x \in \mathcal{A}$, then $x^* bx \in \mathcal{D}_a$.

(ii) If $c = \sum_j c_j^* c_j$ commutes with $b \in \mathcal{D}_a$, then $cb \in \mathcal{D}_a$.

Let $\sum_{nc}$ be the set of finite sums of elements of $\mathcal{D}_1$, that is, $\sum_{nc}$ is the smallest set containing the unit 1 which is closed under sums and operations (i) and (ii).

If $\mathcal{A}$ is a $*$-algebra of *bounded* operators on a Hilbert space and $a \geq 0$, then all elements of $\mathcal{D}_a$ and $\sum_{nc}$ are positive operators.

$\sum_{nc}$ is a **"noncommutative preorder"**.

Version 3 of Artin's Theorem: NC Sums of Squares

Version 3: Most General Denominators and Right Hand Sides

Suppose that $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$.

Then there exist a $s_a \in \mathcal{D}_a$ such that $s_a \in \sum_{nc}$.

Example:

$$x^*(c_1^* c_1 + c_2^* c_2)ax = y_1^*(c_3^* c_3(c_4^* c_4 + c_5^* c_5))y_1 + \cdots .$$

An Example Concerning Versions 1, 2 and 3

Weyl Algebra $\mathcal{A} = \mathbb{C} < a, a^* | \; aa^* - a^* a = 1 >$

Fock representation $ae_n = n^{1/2} e_{n-1}$, $a^* e_n = (n+1)^{1/2} e_{n+1}$ on $l^2(\mathbb{N}_0)$.

Let $N := a^* a$ and $f(N) \in \mathbb{C}[N]$. Then $Ne_n = ne_n$ and we have:

$f \in \mathcal{A}(\mathcal{S})_+$ **iff** $f(n) \geq 0$ **for all** $n \in \mathbb{N}_0$.

$f \in \sum \mathcal{A}^2$ **iff** $f \in N \sum^2 + N(N-1) \sum^2 + \cdots + N(N-1) \cdots (N-k) \sum^2$.

From this it follows that $(N-1)(N-2) \in \mathcal{A}(\mathcal{S})_+ \setminus \sum \mathcal{A}^2$.

An Example Concerning Versions 1, 2 and 3

Weyl Algebra $\mathcal{A} = \mathbb{C} < a, a^* | \ aa^* - a^*a = 1 >$

We have $a^{*k} a^k = N(N-1) \cdots (N-(k-1))$.

If $f \in \mathcal{A}(\mathcal{S})_+$, then version 3 holds for f, there are $c_1, \cdots, c_k \in \sum \mathcal{A}^2$ such that $c_j f = f c_j$, $c_j c_k = c_k c_j$, and $c_1 \cdots c_k f \in \sum \mathcal{A}^2$.

For instance, for $f = (N-1)(N-2) \in \mathcal{A}(\mathcal{S})_+$, we have

$(a^*a)f = N(N-1)(N-2) \ = a^{*3}a^3$.

Positivstellensätze for some CSA

Definition

An **centrally simple algebra** $\mathfrak{A}$ over $\mathbb{K}$ is called **cyclic algebra** if there exists a Galois extension $\mathbb{L}/\mathbb{K}$ with the group $\mathbb{Z}/n\mathbb{Z}$ and fixed elements $e \in \mathfrak{A},\ a \in \mathbb{K}^\circ$ such that

$$\mathfrak{A} = \mathbb{L} \cdot 1 \oplus \mathbb{L} \cdot e \oplus \ldots \mathbb{L} \cdot e^{n-1},\ e^n = a \cdot 1,\ \text{and}$$
$$\lambda \cdot e = e \cdot \sigma(\lambda) \text{ for } \lambda \in \mathbb{L},$$

where σ is a fixed automorphism of $\mathbb{L}$ which generates $\mathbb{L}/\mathbb{K}$.

Note that $\mathfrak{A}$ is a $\mathbb{Z}/n\mathbb{Z}$-graded algebra:

$$\mathfrak{A} = \bigoplus_{k=0}^{n-1} \mathfrak{A}_k,\ \text{where } \mathfrak{A}_k = \mathbb{L} \cdot e^k.$$

Positivstellensätze for some CSA

Let $\mathcal{A}$ be a complex $*$-algebra of operators on a pre-Hilbert space s. t.:

- center $Z(\mathcal{A})$ is an integral domain,
- $\mathfrak{A}$ obtained from localization of $\mathcal{A}$ by $Z(\mathcal{A})$ is a **cyclic algebra**;
- $\mathfrak{A}_k^* = \mathfrak{A}_{-k}$.

Let $\mathcal{M}$ be the quadratic module generated by the elements

$$e^*e, (e^*)^2e^2, \ldots, (e^*)^{n-1}e^{n-1} \in \sum \mathcal{A}^2$$

and their **products**. (These products are no longer squares!)

Theorem: Yu.Savchuk , K. S. (2010)

$\pi(X) \geq 0$ for all bounded $*$-representations π if and only if $X \in \mathcal{M}$.

A Strict Positivstellensatz for the Enveloping Algebra of the $ax+b$-Group

Let $\mathcal{A}$ is the complex universal enveloping algebra of the Lie algebra of the affine group of the real line. Then $\mathcal{A}$ is the unital $*$-algebra with two generators $a=a^*$ and $b=b^*$ and defining relation

$$ab-ba=ib.$$

Each nonzero element $c\in\mathcal{A}$ can be written as

$$c=\sum_{j=0}^{d_1}\sum_{l=0}^{d_2}\gamma_{jl}a^jb^l=\sum_{n=0}^{d_2}f_n(a)b^n=\sum_{k=0}^{d_1}g_k(b)a^n.$$

Here $\gamma_{jl}\in\mathbb{C}$ and $f_n(a)$, $g_k(b)$ are polynomials uniquely determined by c. Set $d(c)=(d_1,d_2)$ if there are $j_0,l_0\in\mathbb{N}_0$ such that $\gamma_{d_1,l_0}\neq 0$ and $\gamma_{j_0,d_2}\neq 0$.

A Strict Positivstellensatz for the Enveloping Algebra of the $ax+b$-Group

Let α and β be reals such that $\alpha < -1$, $\beta \neq 0$ and α is not an integer. Let $\mathcal{S}$ denote the unital monoid generated by $b \pm \beta i$; $a \pm (\alpha+n)i, n \in \mathbb{Z}$.

Theorem: K.S. Crelle (2010)

Let $c{=}c^* \in \mathcal{A}$, $c{\neq}0$, $d(c){=}(2n_1, 2n_2)$, where $n_1, n_2 \in \mathbb{N}_0$. Assume :

(I)There is a bounded selfadjoint operators $T_\pm > 0$ on $L^2(\mathbb{R})$ such that

$$\pi_\pm(c) = \sum_{k=0}^{2n_1} g_k(\pm e^x)\big(i\frac{d}{dx}\big)^k \geq T_\pm.$$

(II) $\gamma_{2n_1,2n_2} > 0$. The polynomials $f_{2n_2}(\cdot + n_2 i)$ and g_{2n_1} are positive on $\mathbb{R}$.

Then there exists an element $s \in \mathcal{S}$ such that

$$s^* cs \in \sum \mathcal{A}^2.$$