On the complexity of computing the handicap of a sufficient matrix

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Sufficient and $\mathcal{P}$-matrices

We look at several matrix classes related to positive semidefinite (PSD) matrices.

Definition (Sufficient matrices)

A matrix $M \in \mathbb{R}^{n\times n}$ is a column sufficient matrix if for all $x \in \mathbb{R}^n$

$$x_i(Mx)_i \leq 0 \; \forall \; i = 1, \ldots, n \quad \text{implies} \quad x_i(Mx)_i = 0 \; \forall \; i = 1, \ldots, n,$$

and row sufficient if M^T is column sufficient. Matrix M is sufficient if it is both row and column sufficient.

Definition ($\mathcal{P}$-matrices)

A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}$-matrix (resp. $\mathcal{P}_0$-matrix) if all its principal minors are positive (resp. nonnegative).

$\mathcal{P}_*(\kappa)$-matrices

Definition ($\mathcal{P}_*(\kappa)$-matrix)

Let $\kappa \geq 0$ be a nonnegative number. A matrix $M \in \mathbb{R}^{n \times n}$ is a $\mathcal{P}_*(\kappa)$-matrix if for all $x \in \mathbb{R}^n$

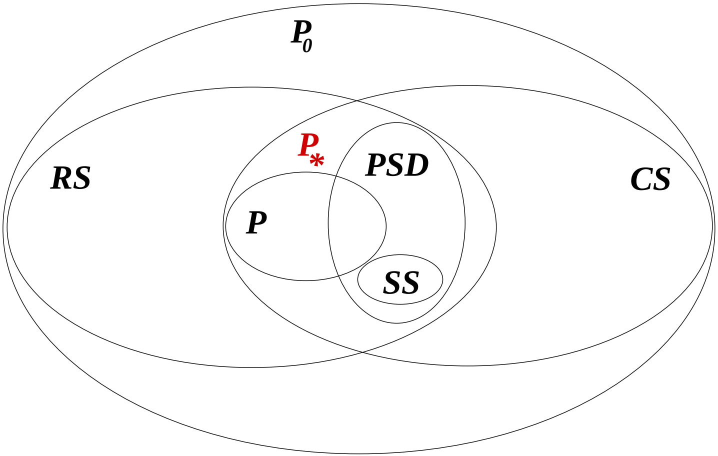
$$x^T Mx + 4\kappa \sum_{i \in \mathcal{I}_+(x)} x_i(Mx)_i \geq 0, \tag{1}$$

where

$$\mathcal{I}_+(x) := \{1 \leq i \leq n : x_i(Mx)_i > 0\}.$$

- Note that $\mathcal{P}_*(0)$ are the positive semidefinite (PSD) matrices.
- Define $\mathcal{P}_* := \bigcup_{\kappa \geq 0} \mathcal{P}_*(\kappa)$.
- The $\mathcal{P}_*$ and sufficient matrices are the same [Kojima et al. (1991), Guu and Cottle (1995), Väliaho (1996)].

Matrix classes: Venn diagram



CS = column sufficient, RS = row sufficient, SS = skew-symmetric, PSD = positive semidefinite.

Matrix classes: membership problem

Theorem (Tseng (2000))

The membership decision problem is co-NP complete in the Turing model for:

- $\mathcal{P}$ *and* $\mathcal{P}_0$ *matrices;*
- *Column sufficient matrices;*
- *Row sufficient matrices.*

P. Tseng. Co-NP-completeness of some matrix classification problems. *Mathematical Programming*, 88:183–192, 2000.

The linear complementarity problem (LCP)

LCP

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ such that

$$-Mx + q = s, \ \ x_i \geq 0, \ \ s_i \geq 0, \ \ x_i s_i = 0 \ \ (i = 1, \ldots, n).$$

Turing model complexity of LCP

Matrix class	**Complexity of LCP**	**Reference**
PSD	P	Kojima et al (1989)
$\mathcal{P}$	not NP-hard, unless NP=co-NP	Megiddo (1988)
$\mathcal{P}_*$	unknown	
$\mathcal{P}_0$	NP-complete	Kojima et al (1991)

Complexity of LCP with sufficient matrices

- LCP is NP-hard for general M, but ...
- ... may be solved by interior point methods if M is *sufficient*.
- The complexity is then polynomial in n, the bitsize of (M, q), and the handicap of M:

Definition (Handicap of a sufficient matrix)

Let $M \in \mathbb{R}^{n \times n}$. The handicap of M is:

$$\hat{\kappa}(M) := \inf\{\kappa \mid M \in \mathcal{P}_*(\kappa)\}.$$

F. A. Potra and X. Liu. Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path. *Optimization Methods & Software*, 20(1):145–168, 2005.

- We will show that the handicap of M can be exponential in its bit size ...
- ... proving that the best known complexity bounds for LCP with sufficient M are exponential in the input size.

Properties of the handicap

Definition

A *principal pivotal transformation* of a matrix $A = \begin{pmatrix} A_{\mathcal{J}\mathcal{J}} & A_{\mathcal{J}\mathcal{K}} \\ A_{\mathcal{K}\mathcal{J}} & A_{\mathcal{K}\mathcal{K}} \end{pmatrix}$ where $\mathcal{J} \cup \mathcal{K} = \{1, \dots, n\}$ and $A_{\mathcal{J}\mathcal{J}}$ is nonsingular, is the matrix

$$\begin{pmatrix} A_{\mathcal{J}\mathcal{J}}^{-1} & -A_{\mathcal{J}\mathcal{J}}^{-1} A_{\mathcal{J}\mathcal{K}} \\ A_{\mathcal{K}\mathcal{J}} A_{\mathcal{J}\mathcal{J}}^{-1} & A_{\mathcal{K}\mathcal{K}} - A_{\mathcal{K}\mathcal{J}} A_{\mathcal{J}\mathcal{J}}^{-1} A_{\mathcal{J}\mathcal{K}} \end{pmatrix}.$$

Theorem (Guu and Cottle (1995), Kojima et al. (1991), Väliaho (1997))

Let $M \in \mathbb{R}^{n \times n}$ be a sufficient matrix. Then:

1. *The handicaps of M and all its principal pivotal transforms are the same.*
2. *The handicap of M is at least as large as that of any of its proper principal submatrices.*
3. $\hat{\kappa} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \frac{1}{4} \left[\frac{m_{21}^2}{\left(\sqrt{m_{11}\, m_{22}} + \sqrt{m_{11}\, m_{22} - m_{12}\, m_{21}} \right)^2} - 1 \right].$

Size of the handicap

Theorem (De Klerk-Nagy)

There exists an $M \in \mathcal{P}$ with $\hat{\kappa}(M) > 2^{\sqrt{L(M)}}$, where $L(M)$ is the bitsize of M.

Proof sketch: Let

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & 0 & \dots & 0 & 0 \\ \\ \vdots & & & & \ddots & & \\ \\ -1 & -1 & -1 & -1 & \dots & -1 & 1 \end{pmatrix}$$

then $\hat{\kappa}(M) \geq 2^{2n-8} - \frac{1}{4}$ (via the theorem on the previous slide). □

Complexity of computing the handicap

Consider the following decision problem:

Decision problem

- **Input:** an integer $n > 0$, an integer $n \times n$ matrix M with bit size $L(M)$, and a positive integer t;
- **Question:** Is $\hat{\kappa}(M) > t$?

Conjecture

If M is sufficient, there is an upper bound on $\hat{\kappa}(M)$ with bit size polynomial in $L(M)$.

Theorem (De Klerk-Nagy)

The decision problem is in NP in the Turing model. If the conjecture holds, the decision problem is NP-complete.

Computing the handicap

There is an algorithm to compute the handicap of a sufficient M:

H. Väliaho. Determining the handicap of a sufficient matrix. *Linear Algebra and Its Applications*, 253:279–298, 1997.

Theorem (De Klerk-Nagy)

The complexity of the Väliaho algorithm is lower bounded by $\frac{1}{5}6^n$.

- In practice, the algorithm is prohibitively slow if $n \geq 7$ ...
- this motivates an alternative approach using sum-of-squares of polynomials and semidefinite programming.

Computing the handicap (ctd.)

Recall that, if $M \in \mathcal{P}_*(\kappa)$ then

$$x^T M x + 4\kappa \sum_{i \in \mathcal{I}_+(x)} x_i(Mx)_i \geq 0 \; \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{I}_+(x) := \{1 \leq i \leq n : x_i(Mx)_i > 0\}.$$

Lemma

Let $M \in \mathbb{R}^{n \times n}$ and

$$p_\kappa(x, \alpha) := x^T M x + 4\,\kappa \sum_{i=1}^{n} \alpha_i.$$

One has:

$$\hat{\kappa}(M) = \inf \left\{ \kappa \geq 0 : \; p_\kappa(x, \alpha) \geq 0, \; \forall (x, \alpha) \in \mathcal{K} \right\},$$

where $\mathcal{K} := \left\{ (x, \alpha) : \; \|x\| = 1, \;\; \alpha \geq x \circ Mx, \;\; \|\alpha\| \leq \|M\|_2, \;\; \alpha \geq 0 \right\}.$

Computing the handicap (ctd.)

Lemma

Let $M \in \mathbb{R}^{n\times n}$ a $\mathcal{P}$-matrix and

$$p_\kappa(x,\alpha) := x^T Mx + 4\,\kappa \sum_{i=1}^{n} \alpha_i.$$

One has:

$$\hat{\kappa}(M) = \inf\left\{\kappa \geq 0 : \; p_\kappa(x,\alpha) > 0, \; \forall\,(x,\alpha) \in \mathcal{K}\right\},$$

where $\mathcal{K} := \left\{(x,\alpha) : \; \|x\| = 1, \;\; \alpha \geq x \circ Mx, \;\; \|\alpha\| \leq \|M\|_2, \;\; \alpha \geq 0\right\}.$

- Now we can use Putinar's *positivstellensatz* for polynomials positive on compact semialgebraic sets ...
- ... and Lasserre's approach to obtain semidefinite programming approximations.

Putinar's positivstellensatz

Consider *semi-algebraic set* defined by polynomials g_i $(i = 1, \ldots, m)$:

$$\mathcal{K} = \left\{ x \in \mathbb{R}^k \ : \ g_i(x) \geq 0 \ (i = 1, \ldots, m) \right\}.$$

Quadratic module:

The *quadratic module* generated by functions $g_1, \ldots, g_m$ is defined as

$$\mathcal{M}(g_1, \ldots, g_m) = \left\{ s_0 + \sum_{j=1}^{m} s_j g_j \ : \ s_j \text{ sums of squares}, \ \ j = 0, \ldots, m \right\}.$$

Theorem (Putinar):

For a given polynomial p one has $p(x) > 0$ for all $x \in \mathcal{K}$ iff $p \in \mathcal{M}(g_1, \ldots, g_m)$, provided that $\mathcal{M}(g_1, \ldots, g_m)$ is Archimedean.

M. Putinar. Positive polynomials on compact semi-algebraic sets. *Ind. Univ. Math. J.* 42:969–984, 1993.

Lasserre's approach

Truncated quadratic module:

Given an integer $t > 0$, the *truncated quadratic module of degree $2t$* generated by functions $g_1, \ldots, g_m$ is defined as

$$\begin{aligned} \mathcal{M}_t(g_1, \ldots, g_m) &:= \left\{ s_0 + \sum_{j=1}^{m} s_j g_j : \; s_j \text{ sums of squares}, \;\; (j = 0, \ldots, m) \right\} \\ \text{degree}(g_j s_j) &\leq 2t \;\; (j = 0, \ldots, m), \; \text{degree}(s_0) \leq 2t. \end{aligned}$$

Approach of Lasserre:

For a given polynomial p the question: "Is $p \in \mathcal{M}_t(g_1, \ldots, g_m)$?", may be formulated as a semidefinite program (SDP).

J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIOPT*, 11:296–817, 2001.

SDP formulation

$$\kappa^{(t)} := \inf \quad \kappa$$

subject to

$$\begin{aligned}
x^T M x + 4\kappa \sum_{i=1}^{n} \alpha_i &= s_0(x,\alpha) + \sum_{j=1}^{n} \Big(\alpha_j - x_j (Mx)_j\Big)\, s_j(x,\alpha) \\
&+ \; + \sum_{j=1}^{n} \alpha_j \, s_{n+j}(x,\alpha) + \left(\|M\|_2^2 - \sum_{i=1}^{n} \alpha_i^2 \right) s_{2n+1}(x,\alpha) \\
&+ \; \left(1 - \sum_{i=1}^{n} x_i^2 \right) r(x,\alpha) \\
s_j(x,\alpha) & \quad \text{sums of squares}, \quad j = 0, \ldots, 2n+1 \\
\deg(s_0) &\le 2t, \\
\deg(s_j) &\le 2t - 2, \quad j = 1, \ldots, 2n+1 \\
r &\in \mathbb{R}[x,\alpha], \; \deg(r) \le 2t-2 \\
\kappa &\ge 0.
\end{aligned}$$

SDP approximation of the handicap: properties

For fixed t, $\kappa^{(t)}$ may be computed in polynomial time within any fixed accuracy.

Theorem (De Klerk-Nagy)

Let $M \in \mathbb{R}^{n\times n}$ with handicap $\hat{\kappa}(M)$ Then:

1. $\kappa^{(t)} = \infty$ *for all* $t \in \mathbb{N}$ *if* M *is not sufficient;*
2. $\kappa^{(t)} \geq \kappa^{(t+1)} \geq \hat{\kappa}(M)$ *if* $\kappa^{(t)}$ *is finite;*
3. $\hat{\kappa}(M) = \lim_{t\to\infty} \kappa^{(t)}$ *if* M *is a* $\mathcal{P}$*-matrix;*
4. $0 = \hat{\kappa}(M) = \kappa^{(1)}$ *if* M *is PSD;*
5. $\hat{\kappa}(M) = \kappa^{(1)}$ *if* $n = 2$*;*
6. $\kappa^{(1)} < \infty$ *iff* $\exists$ *a diagonal matrix* D *(positive diagonal entries) such that* DM *is PSD.*

Numerical examples

- We compared our approach numerically to the algorithm of Väliaho for small matrices ($n \leq 7$).
- The SDP problems with optimal values $\kappa^{(t)}$ ($t = 1, 2, \ldots$) were solved using SeDumi and Gloptipoly.

D. Henrion, J. B. Lasserre, J. Loefberg. GloptiPoly 3: moments, optimization and semidefinite programming. *Optimization Methods and Software*, **24**:4-5, 761–779, 2009.

- The test matrices were all $\mathcal{P}$-matrices (with finite handicap).
- $s = 1$ in the next table means Gloptipoly could verify global optimality, i.e. the handicap is obtained exactly.

Numerical examples

Matrix	Order of SOS relaxation (t) 1	2	Väliaho's algorithm
M_2 ($n=3$)	s=0 $\kappa^{(1)}=6$ 0.2s	s=1 $\kappa^{(2)}=6$ 0.6s	$\hat{\kappa}=6$ 0.3s
M_3 ($n=3$)	s=-1 $\kappa^{(1)}=\infty$ (infeasible) 0.2s	s=1 $\kappa^{(2)}=0.91886$ 0.5s	$\hat{\kappa}=0.91886$ 0.3s
M_4 ($n=3$)	s=0 $\kappa^{(1)}=0.08986$ 0.1s	s=1 $\kappa^{(2)}=0.08986$ 0.4s	$\hat{\kappa}=0.08986$ 0.6s
M_5 ($n=3$)	s=0 $\kappa^{(1)}=0.03987$ 0.2s	s=1 $\kappa^{(2)}=0.03987$ 0.4s	$\hat{\kappa}=0.03987$ 0.6s
M_6 ($n=6$)	s=0 $\kappa^{(1)}=15.75$ 0.3s	s=1 $\kappa^{(2)}=15.75$ 138.7s	$\hat{\kappa}=15.75$ 1737.7s
M_7 ($n=7$)	s=0 $\kappa^{(1)}=0.039866$ 0.3s	s=1 $\kappa^{(2)}=0.039866$ 413.1s	— >12h

Conclusions and summary

- We have shown that the handicap of a sufficient matrix M may be exponential in the bit size of M ...
- that implies the best known complexity bounds for LCP's with sufficient matrices are exponential in the input size.
- Lasserre's sum-of-squares approach may be used to compute the handicap ...
- and is a better choice in practice than Väliaho's algorithm.

Almost the End

Further reading:

Preprint at *Optimization Online*.

One more conjecture ...

A conjecture by Monique Laurent and myself

Identity:

$$x_1x_2 + \frac{1}{8} = \frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2 + \frac{1}{2}(x_1 - x_1^2) + \frac{1}{2}(x_2 - x_2^2).$$

Thus $x_1x_2 + \frac{1}{8}$ belongs to the truncated quadratic module of degree 2 generated by $x_1 - x_1^2,\ x_2 - x_2^2$.

Question:

What is the smallest constant $C_n > 0$ so that $\prod_{i=1}^n x_i + C_n$ belongs to the truncated quadratic module of degree n generated by $x_1 - x_1^2, \ldots, x_n - x_n^2$?

Conjecture: $C_n = \frac{1}{n(n+2)}$. (We know that $C_n \leq 1$.)

Conjecture from:

E. de Klerk, M. Laurent. Error bounds for some semidefinite programming approaches to polynomial minimization on the hypercube. *SIOPT*, to appear.