Wave equation imaging by the Kaczmarz method

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Kaczmarz’ method for linear problems

\[ R_s f = g_s, \quad s = 1, \ldots, p \]

\( R_s \) linear bounded operators \( H \to H_s, \)

\( H, H_s \) Hilbert spaces.

Update:

\[
\begin{align*}
f &\leftarrow f - \alpha R_s^* C_s^{-1} (R_s f - g_s), \quad s = 1, \ldots, p \\
\end{align*}
\]

Convergence:

\[
0 < \alpha < 2, \quad C_s \geq R_s R_s^* > 0
\]
The Kaczmarz method and SOR

\[ (L/D/U) = RR^*, \]

\[ C_\omega = I - \omega(D + \omega L)^{-1}RR^*, \]

\[ c_\omega = \omega(D + \omega L)^{-1}g. \]

Theorem: Let \( u^k \) be the SOR iterates for \( RR^* f = g \), i.e.

\[ u^{k+1} = C_\omega u^k + c_\omega. \]

Then \( f^k = R^* u^k \) are the Kaczmarz iterates for \( Rf = g \).
The model problem

\[
\frac{\partial^2 u}{\partial t^2}(x, t) = c^2(x) (\Delta u(x, t) + q(t)p(x - s)), \quad 0 < t < T,
\]

\[
u = 0, \quad t < 0,
\]

\[
g_s(x', t) = u(x', 0, t) = (R_s(f))(x', t) \text{ seismogram for source } s,
\]

\[
c^2(x) = \frac{c_0^2}{(1 + f(x))}.
\]
Kaczmarz’ method for nonlinear problems (consecutive time time reversal)

Solve \( R_s(f) = g_s \) for all sources \( s \).

Update:

\[
f \leftarrow f - \alpha (R_s'(f))^\ast (R_s(f) - g_s)
\]

Compute the adjoint by time reversal:

\[
\frac{\partial^2 z}{\partial t^2} = c^2(x)\Delta z \quad \text{for} \quad x_2 > 0,
\]

\[
\frac{\partial z}{\partial x_2} = r \quad \text{on} \quad x_2 = 0,
\]

\[
z = 0 \quad \text{for} \quad t > T.
\]

\[
(R_s'(f))^\ast r(x) = \int_0^T z(x,t) \frac{\partial^2 u(x,t)}{\partial t^2} dt
\]
Kaczmarz' method for breast phantom, eight sources

1 sweep

3 sweeps
Rays

Kaczmarz’s method for breast phantom, eight sources

1 sweep  3 sweeps
Reconstruction in the presence of focal points

7.5  5.0  2.5  mm  Luneberg lens  200 kHz
wavelength 7.5 mm
Reconstruction in the presence of trapped rays

200 kHz
Source encoding

\[ g_s(x_1, t) = u_s(x_1, 0, t) \]

is the usual seismogram for source s. Let \( w \) be a random vector, and let

\[ g_w(x_1, t) = \sum_s w_s g_s(x_1, t). \]

\( g_\alpha \) is the value at \( x_2 = 0 \) of the solution for the source

\[ q_w(x, t) = \sum_s w_s p(x - s) q(t). \]
Plane wave stacking

\[ g_s(x_1, t) = u_s(x_1, 0, t) \]

is the usual seismogram for source \( s \). Let \( \alpha, |\alpha| \leq \pi/2 \) be an angle, and let

\[ g_\alpha(x_1, t) = \int_{R^1} g_s(x_1, t - \frac{s}{c} \sin \alpha) ds. \]

\( g_\alpha \) is the value at \( x_2 = 0 \) of the solution

\[ u_\alpha(x, t) = \int_{R^1} u_s(x, t - \frac{s}{c} \sin \alpha) ds \]

that exhibits a wave front making an angle \( \alpha \) with the surface \( x_2 = 0 \).

Coverage in Fourier domain

\[ \hat{f}(\sigma + \rho, \kappa(\sigma) - \kappa(\rho)) \quad \hat{f}(\sigma + \rho, \kappa(\sigma) + \kappa(\rho)) \]

\[ \kappa(\sigma) = \sqrt{k^2 - \sigma^2} \]

P. Mora, 1989
Easy case Nr. 1: Clutter

Original 12 cm

5 sweeps of Kaczmarz

Diameter of dots 5 mm

Frequency range 50 to 150 kHz
Easy case Nr. 2: Source wavelet $q$ is Gaussian peak.

Original

6 sweeps
Difficult case

10 kHz - 150 kHz
Suggestion of K. Richter, 1995
Mammography reflection imaging

3D scanner of U-Systems
What can we achieve in reflection mammography?

- Aperture $A = 15\text{ cm}$
- Frequencies $15-500\text{ kHz}$
- Wavelength $3\text{ mm}$
- Depth $5\text{ cm}$
- Tumor diameter $1.5\text{ mm}$
- Step size $0.5\text{ mm}$
Plane wave stacking in reflection mammography

frequencies 15-500 kHz, wavelength 3mm, tumor diameter 1.5 mm, stepsize 0.5 mm, 20 sources, depth 5 cm
Reconstruction with various methods

Hesse & Schmitz 2012

Kaczmarz
SA synthetic aperture focusing
DAS (delay and sum)
Layered medium

\[ f(x_1, x_2) = f(x_2). \]

Born approximation, one source at \( x_1 = 0, \ x_2 = 0: \)

\[ g_k(x) = (2\pi)^{-1/2} \int e^{-ix\xi} \hat{f}(-2\kappa(\xi))d\xi, \ \kappa = \sqrt{k^2 - \xi^2}. \]

Finite aperture: Data available for \( |x| \leq A \) only.

All we can determine: \[ \int \delta_A(\eta - \xi) \hat{f}(-2\kappa(\xi)d\xi, \ \delta_A(\xi) = \frac{A}{\pi} sinc(A\xi). \]
Determine $\hat{f}$ from

$$\int \delta_A(\eta - \xi) \hat{f}(-2\kappa(\xi)) d\xi, \quad \delta_A(\xi) = \frac{A}{\pi} \text{sinc}(A\xi), \quad \kappa = \sqrt{k^2 - \xi^2}.$$  

peaks in $\eta$, bandwidth $A$

$\hat{f}(-2\kappa(\xi))$ can be stably determined for $A > 2z|\xi|/\kappa(\xi)$

i.e. $\frac{2k}{\sqrt{1 + A^2/4z^2}} < 2\kappa < 2k$.

bandwidth $2z|\kappa'(\xi)| = 2z|\xi|/\kappa(\xi)$

for line object at depth $z$:

$f(x) = \delta(x-z), \quad \hat{f}(\xi) \sim e^{-iz\xi}$,

$\hat{f}(-2\kappa(\xi)) \sim e^{-2iz\kappa(\xi)}$ for $|\xi| < k$.

Sirgue & Pratt 2004:

$\hat{f}(\xi)$ can be determined for $|\xi| \geq \frac{2k}{\sqrt{1 + A^2/4z^2}}$
Kaczmarz' method, frequencies 5-25 Hz

- **true profile**
- Kaczmarz starting at f=0
- Kaczmarz with analytic continuation
20 cm Asphaltschicht
15 cm Schotter- schicht
Sand-Lehm
Falling weight deflectometer (FWD)