Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis

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Joint work with Clint Scovel

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[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017. arXiv:1703.10761]

DARPA EQUiPS / AFOSR award no FA9550-16-1-0054
(Computational Information Games)

Question

- **Can we design a scalable solver that could be applied to nearly all linear operators?**

"Of course no one method of approximation of a "linear operator" can be universal. " [Sard, 1967]

Answer

Yes under two minor conditions: (1) The operator must be bounded and invertible (2) Its image space must have a regular multiresolution decomposition

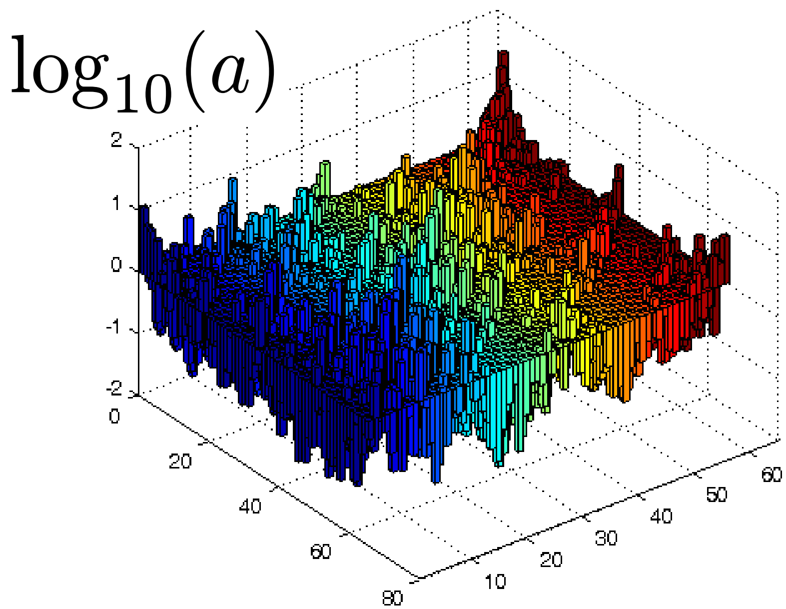
Problem: Solve (1) as fast as possible to a given accuracy

(1)
$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^d$ $\quad$ $\partial\Omega$ is piec. Lip.

a unif. ell.

$a_{i,j} \in L^\infty(\Omega)$



Multigrid Methods

Multigrid: [Fedorenko, 1961, Brandt, 1973, Hackbusch, 1978]

Multiresolution/Wavelet based methods

[Brewster and Beylkin, 1995, Beylkin and Coult, 1998, Averbuch et al., 1998]

[Beylkin, Coifman, Rokhlin, 1992] [Engquist, Osher, Zhong, 1992]

[Alpert, Beylkin, Coifman, Rokhlin, 1993]

[Cohen, Daubechies, Feauveau. 1992]

[Bacry, Mallat, Papanicolaou. 1993]

- **Linear complexity with smooth coefficients**

Problem **Severely affected by lack of smoothness**

Robust/Algebraic multigrid

[Mandel et al., 1999,Wan-Chan-Smith, 1999,
Xu and Zikatanov, 2004, Xu and Zhu, 2008], [Ruge-Stüben, 1987]
[Panayot - 2010]

Stabilized Hierarchical bases, Multilevel preconditioners

[Vassilevski - Wang, 1997, 1998]

[Panayot - Vassilevski, 1997]

[Chow - Vassilevski, 2003]

[Aksoylu- Holst, 2010]

- Some degree of robustness but problem remains open with rough coefficients

Why? **Interpolation operators are unknown**

Don't know how to bridge scales with rough coefficients!

Low Rank Matrix Decomposition methods

Fast Multipole Method: [Greengard and Rokhlin, 1987]

Hierarchical Matrix Method: [Hackbusch et al., 2002]

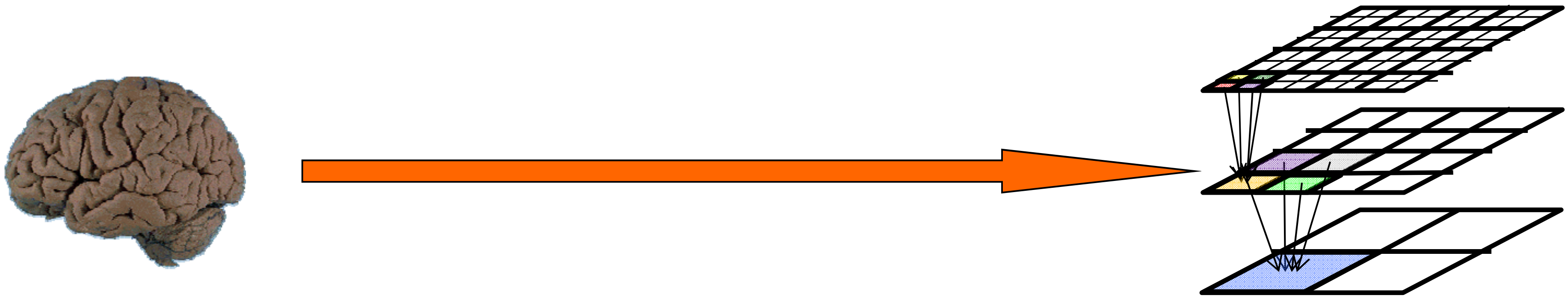
[Bebendorf, 2008]:

$$\boxed{N \ln^{2d+8} N \text{ complexity}}$$

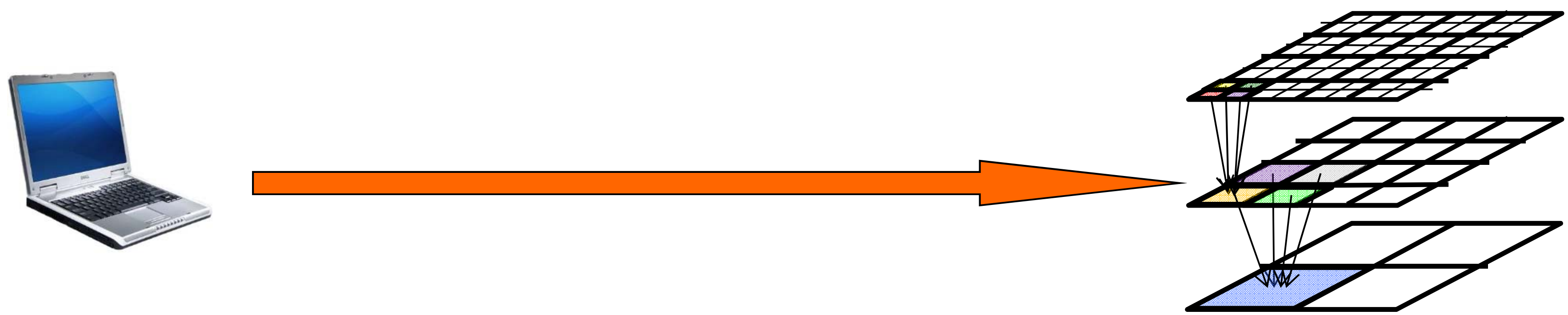
To achieve grid-size accuracy in L^2-norm

Common theme between these methods

Their process of discovery is based on intuition, brilliant insight, and guesswork



Can we turn this process of discovery into an algorithm?



[H. Owhadi, Multigrid with rough coefficients and Multiresolution operator decomposition from Hierarchical Information Games. SIAM Review, 2017]

Answer: YES

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

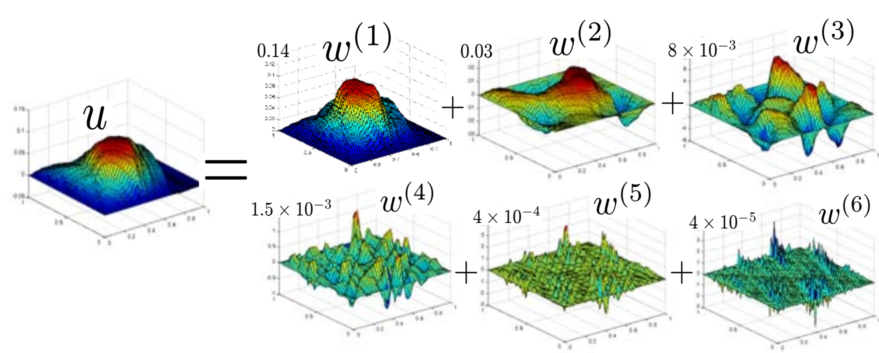
$$H^1_0(\Omega) = \mathfrak{W}^{(1)} \oplus_a \mathfrak{W}^{(2)} \oplus_a \cdots \oplus_a \mathfrak{W}^{(k)} \oplus_a \cdots$$

Gamblet Transform $\mathcal{O}(N \ln^{3d} N)$

$$u = w^{(1)} + w^{(2)} + \cdots + w^{(k)} + \cdots$$

Linear Solve $\mathcal{O}(N \ln^{d+1} N)$

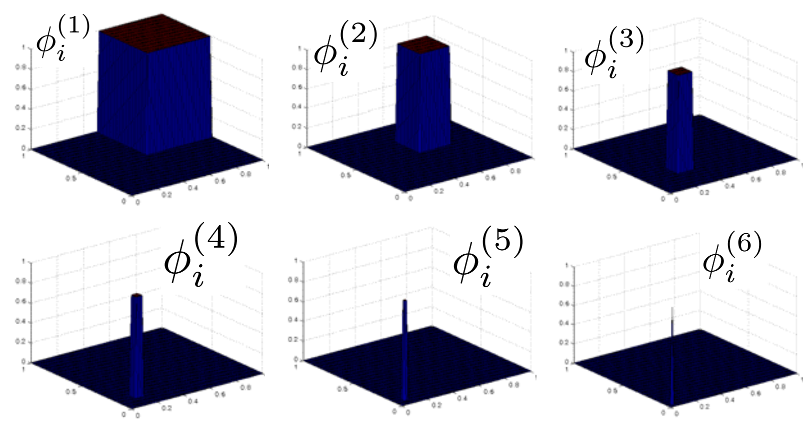
To achieve grid-size accuracy in H^1-norm



Hierarchy of nested Measurement functions

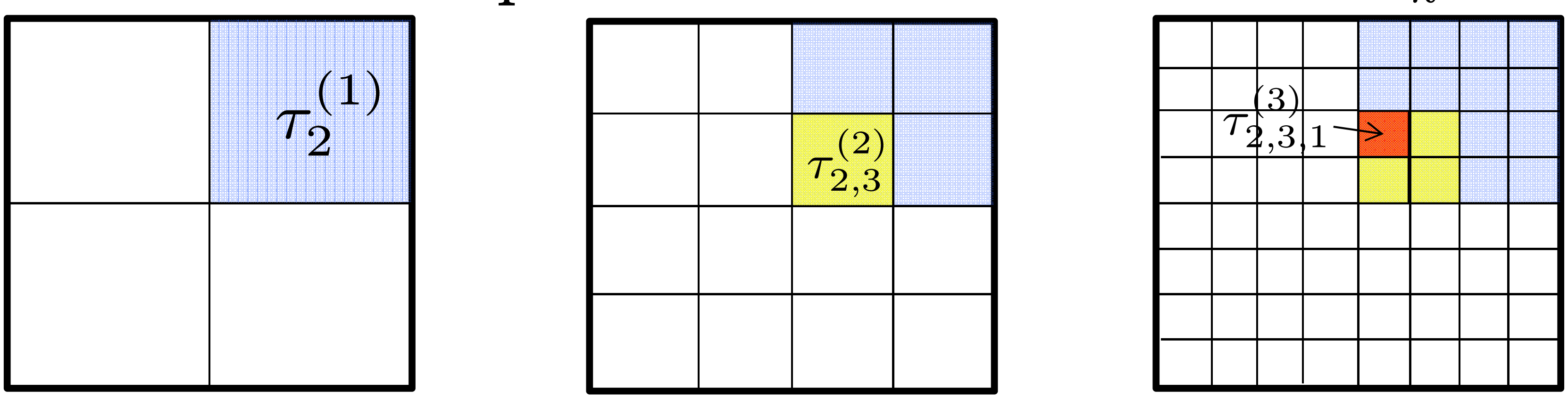
$$\phi_i^{(k)} \in L^2(\Omega) \text{ with } k \in \{1,\ldots,q\}$$

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$



Example

$\phi_i^{(k)}$: Indicator functions of a hierarchical nested partition of Ω of resolution $H_k = 2^{-k}$



Formulation of the hierarchy of games

Player I

$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

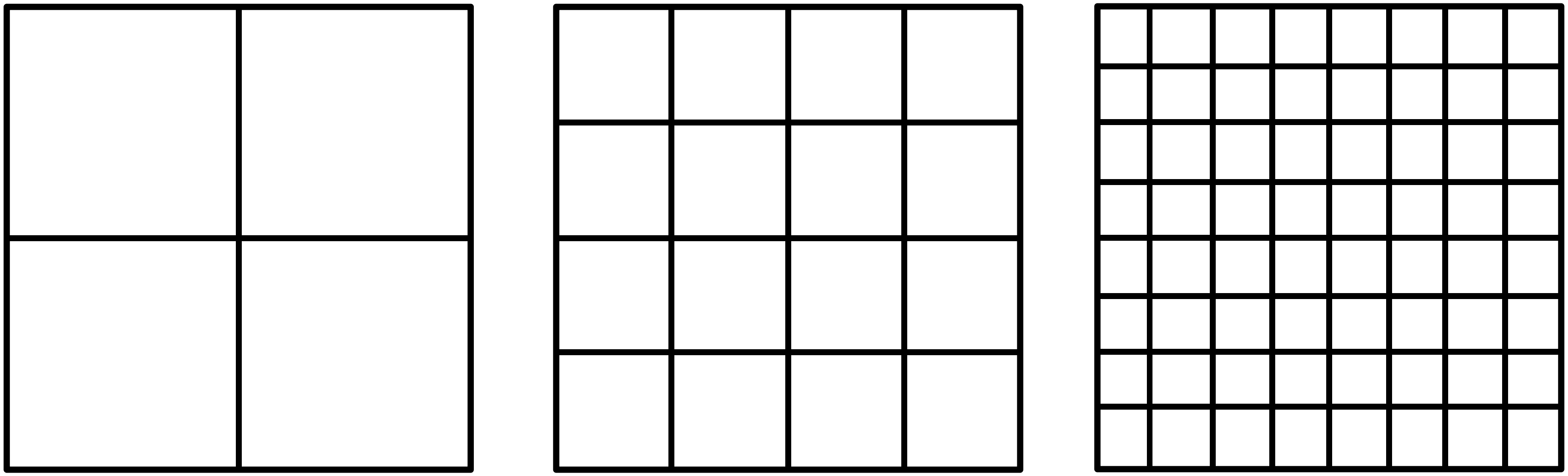
Player II

Chooses
$u \in H_0^1(\Omega)$

Sees $\{\int_\Omega u\phi_i^{(k)},\ i \in \mathcal{I}_k\}$

Must predict

u and $\{\int_\Omega u\phi_j^{(k+1)},\ j \in \mathcal{I}_{k+1}\}$



$$\begin{cases} -\operatorname{div}(a\nabla u) = g \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

Player II's best strategy

$$\xi \sim \mathcal{N}(0, G)$$

$$\text{Loss } \frac{\|u - u^{(k)}\|}{\|u\|}$$

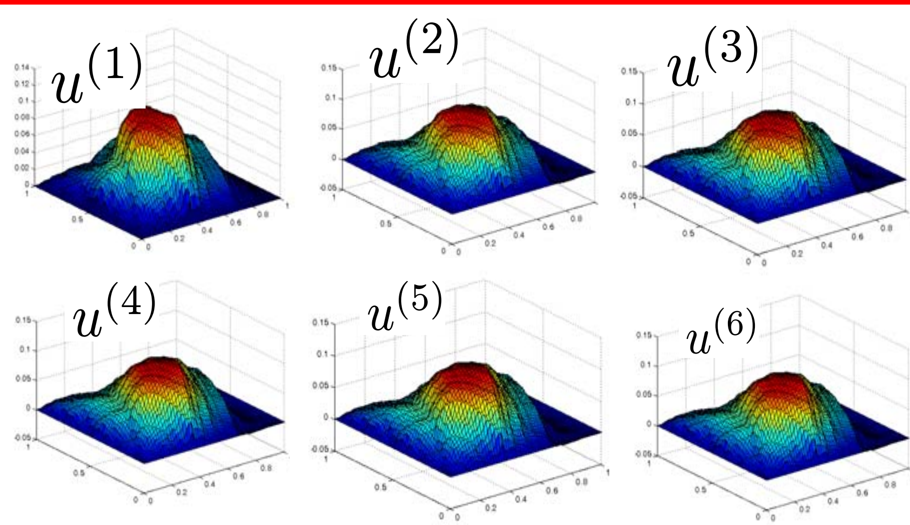
$$\|u\|^2 := \int_\Omega (\nabla u)^T a \nabla u$$

$$\int_\Omega \xi(x) f(x)\, dx \sim \mathcal{N}(0, \|f\|_*^2)$$

$$\|f\|_*^2 := \int_{\Omega^2} f(x) G(x,y) f(y)\, dx\, dy$$

Player II's bets

$$u^{(k)}(x) := \mathbb{E}\Big[\xi(x) \Big| \int_\Omega \xi(y)\phi_i^{(k)}(y)\, dy = \int_\Omega u(y)\phi_i^{(k)}(y)\, dy,\ i \in \mathcal{I}_k\Big]$$



Gamblets **Elementary gambles form a hierarchy of deterministic basis functions for player II's hierarchy of bets**

Theorem

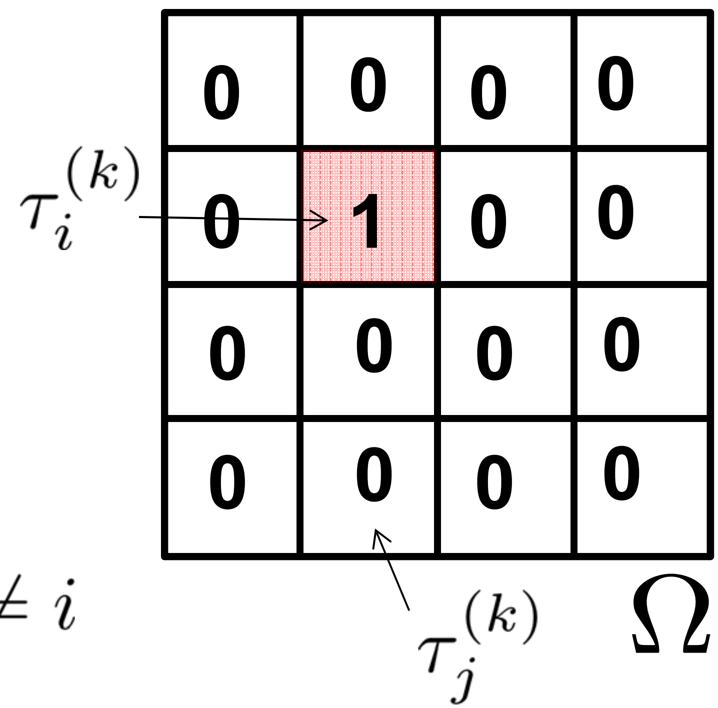
$$u^{(k)}(x) = \sum_i \psi_i^{(k)}(x) \int_\Omega u(y)\phi_i^{(k)}(y)\,dy$$

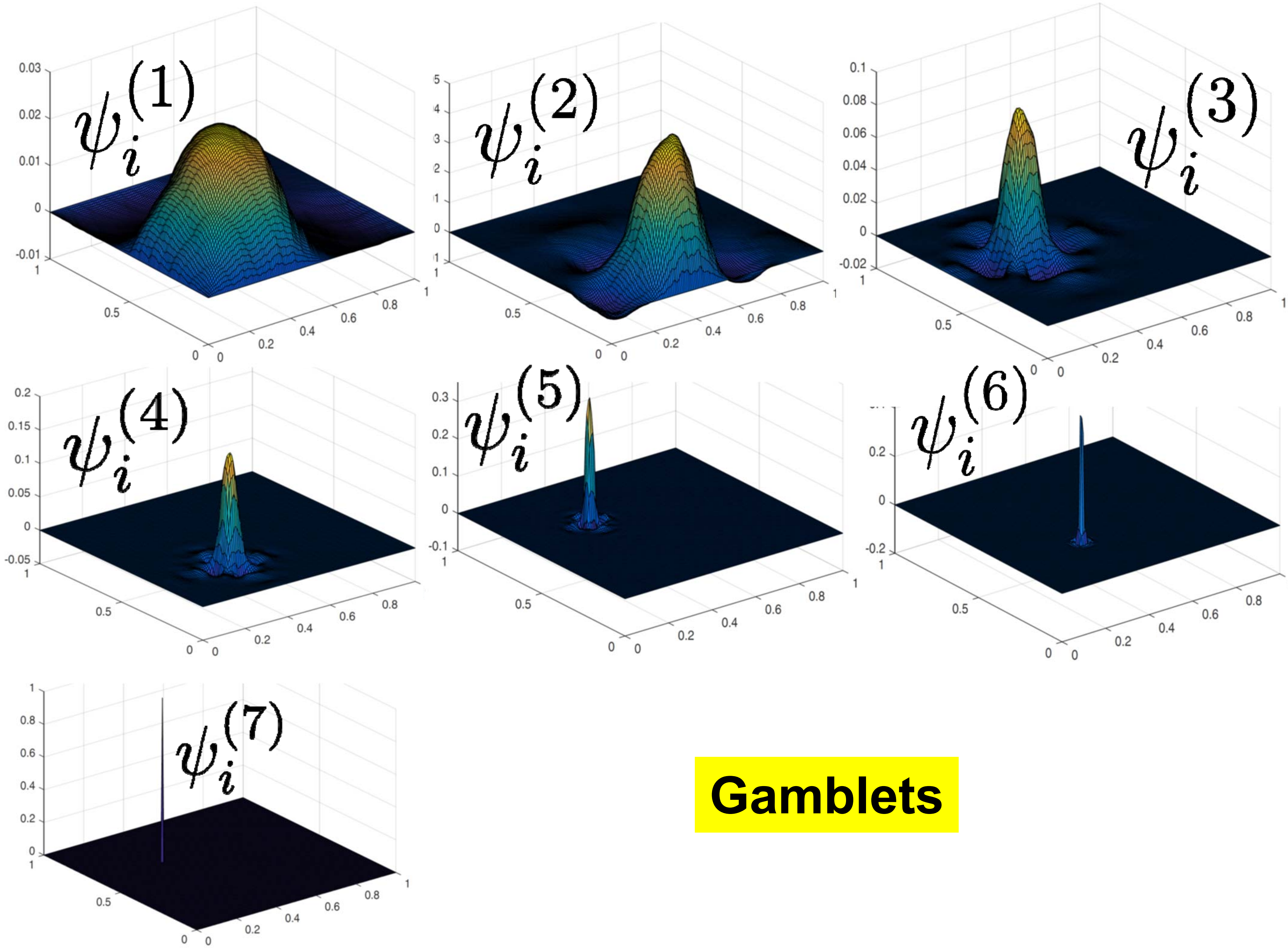
$\psi_i^{(k)}$: Elementary gambles/bets at resolution $H_k = 2^{-k}$

$$\psi_i^{(k)}(x) := \mathbb{E}\Big[\xi(x)\Big| \int_\Omega \xi(y)\phi_j^{(k)}(y)\,dy = \delta_{i,j},\ j \in \mathcal{I}_k\Big]$$

$$\psi_i^{(k)}$$

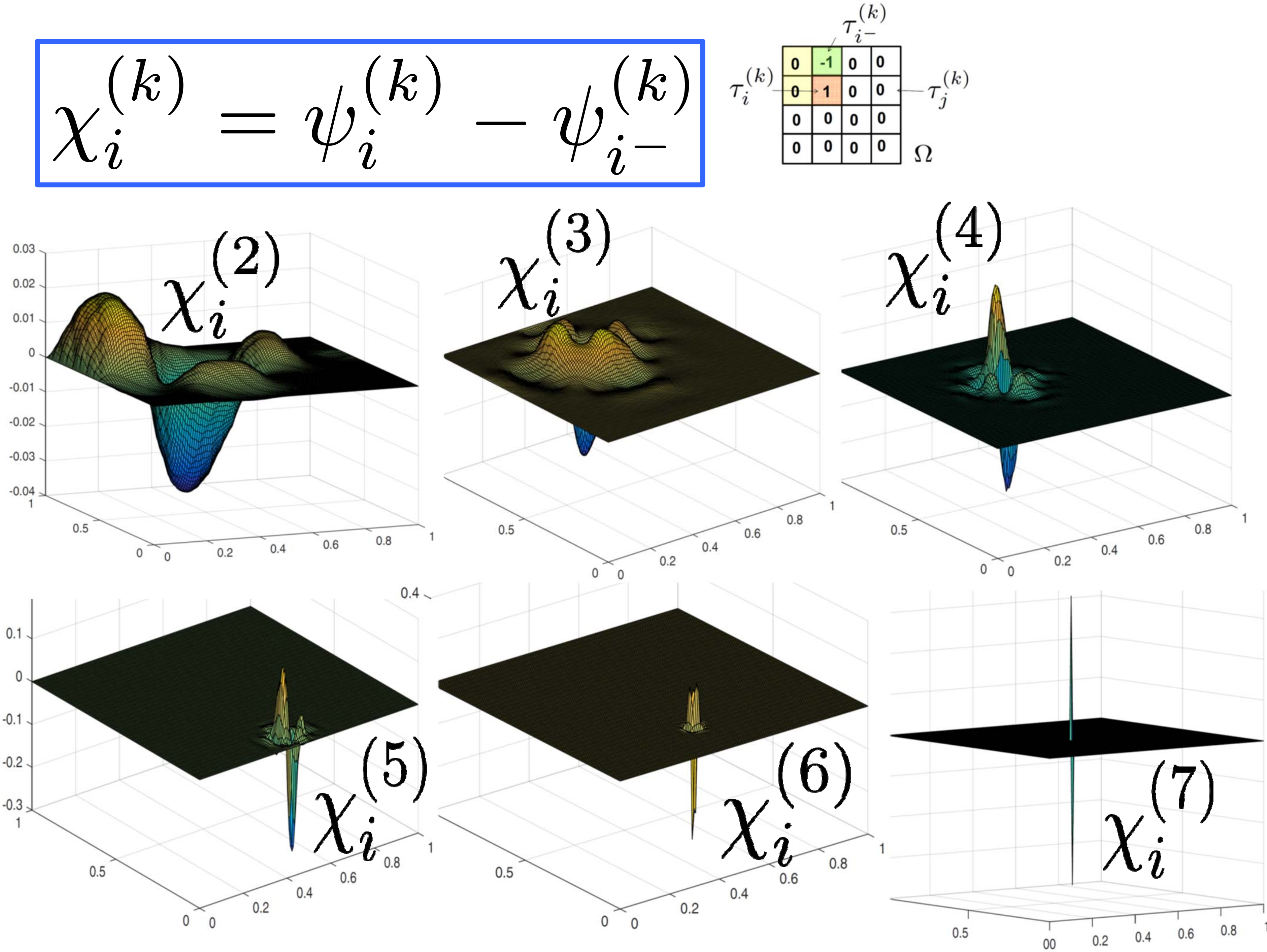
Your best bet on the value of u given the information that $\int_{\tau_i^{(k)}} u = 1$ and $\int_{\tau_j^{(k)}} u = 0$ for $j \neq i$





$\psi_i^{(1)}$
$\psi_i^{(2)}$
$\psi_i^{(3)}$
$\psi_i^{(4)}$
$\psi_i^{(5)}$
$\psi_i^{(6)}$
$\psi_i^{(7)}$
Gamblets

$$\chi_i^{(k)} = \psi_i^{(k)} - \psi_{i-}^{(k)}$$



Multiresolution decomposition of the solution space

$$\mathfrak{V}^{(k)} := \text{span}\{\psi_i^{(k)}, i \in \mathcal{I}_k\}$$

$$\mathfrak{W}^{(k)} := \text{span}\{\chi_i^{(k)}, i\}$$

$\mathfrak{W}^{(k+1)}$: Orthogonal complement of $\mathfrak{V}^{(k)}$ in $\mathfrak{V}^{(k+1)}$ with respect to $< \psi, \chi >_a := \int_\Omega (\nabla\psi)^T a \nabla\chi$

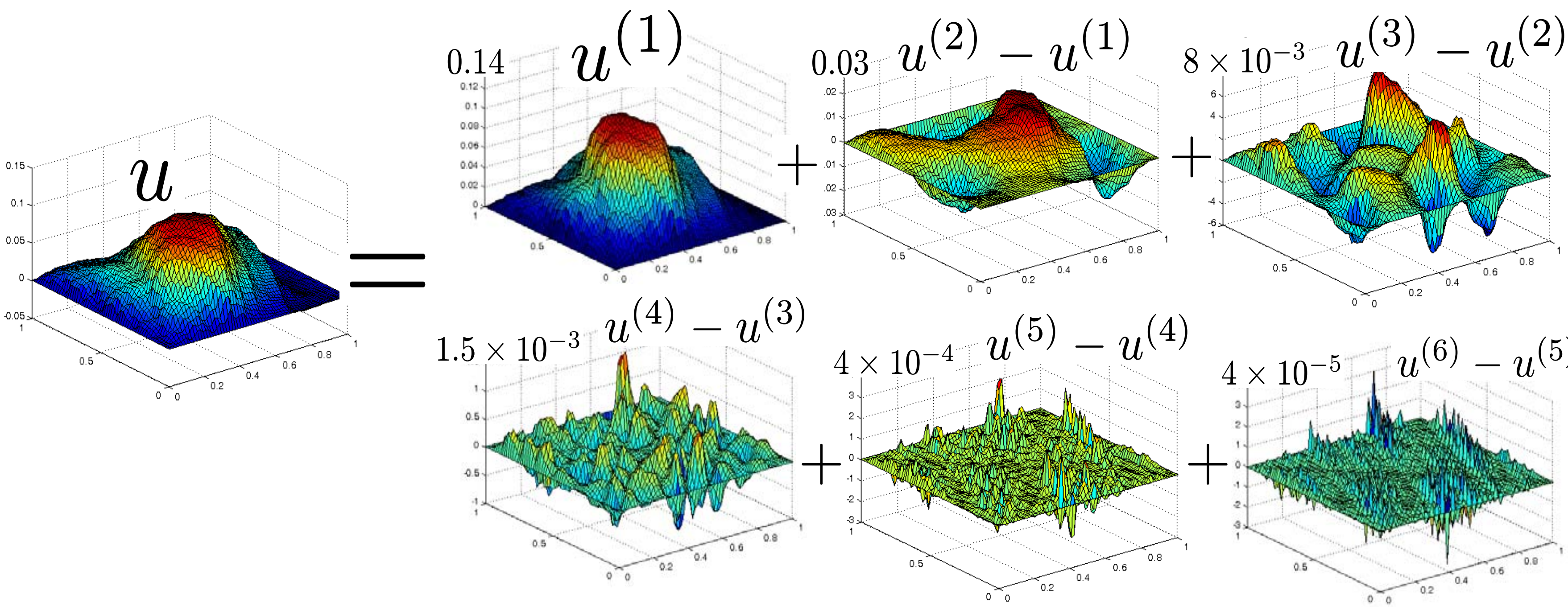
Theorem

$$H_0^1(\Omega) = \mathfrak{V}^{(1)} \oplus_a \mathfrak{W}^{(2)} \oplus_a \cdots \oplus_a \mathfrak{W}^{(k)} \oplus_a \cdots$$

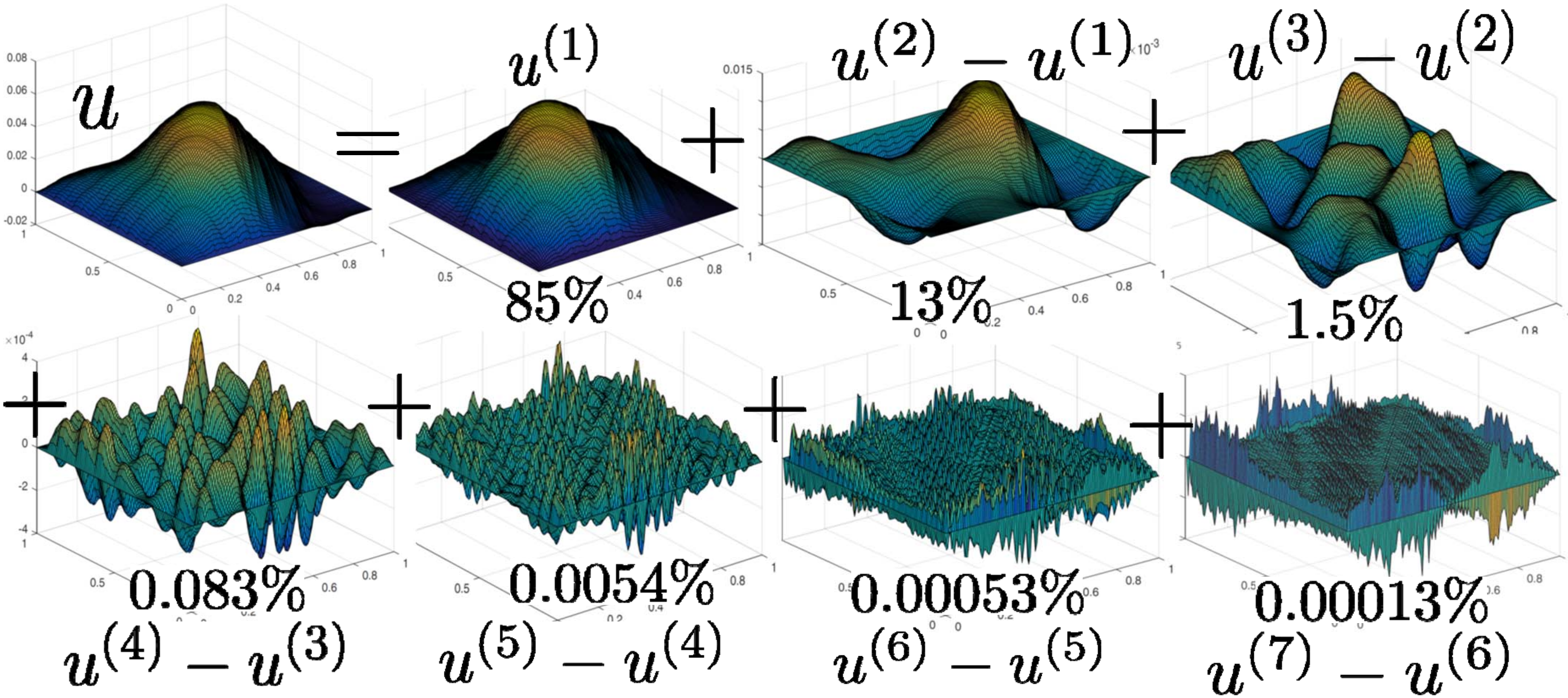
Multiresolution decomposition of the solution

Theorem

$$u^{(k+1)} - u^{(k)} = \text{F.E. sol. of PDE in } \mathfrak{W}^{(k+1)}$$



Subband solutions $u^{(k+1)} - u^{(k)}$
can be computed independently



Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$g \in C^\infty(\Omega)$$

If r.h.s. is regular we don’t need to compute all subbands

Numerical Homogenization

Harmonic Coordinates Babuska, Caloz, Osborn, 1994

Kozlov, 1979 Allaire Brizzi 2005; Owhadi, Zhang 2005

MsFEM [Hou, Wu: 1997]; [Efendiev, Hou, Wu: 1999]

[Fish - Wagiman, 1993] [Chung-Efendiev-Hou, JCP 2016]

Variational Multiscale Method, Orthogonal decomposition

[Hughes, Feijóo, Mazzei, Quincy. 1998]

[Malqvist-Peterseim 2012] Local Orthogonal Decomposition

Projection based method Nolen, Papanicolaou, Pironneau, 2008

HMM Engquist, E, Abdulle, Runborg, Schwab, et Al. 2003-...

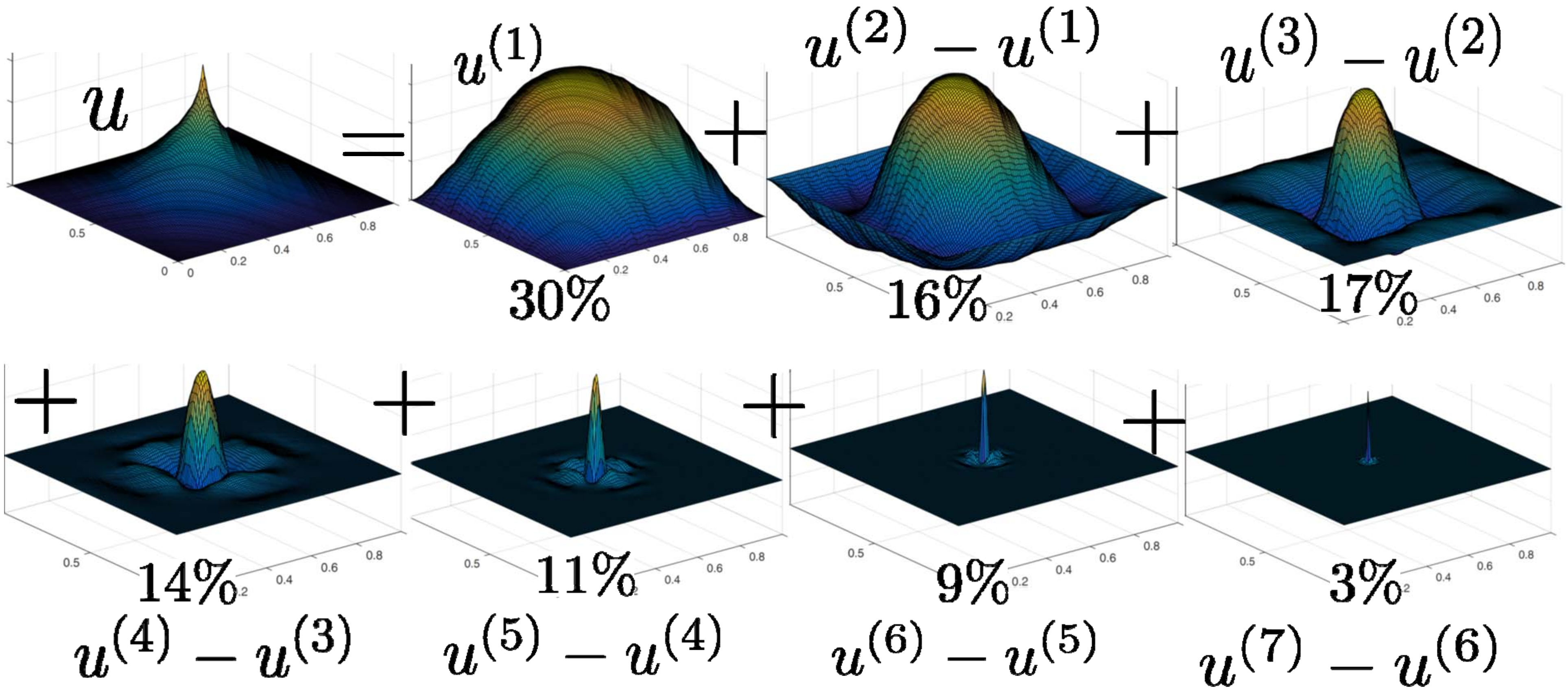
Flux norm Berlyand, Owhadi 2010; Symes 2012

Bayesian Numerical Homogenization Owhadi 2014

Gamblets – Operator compression [Owhadi, SIREV 2017]

[Owhadi, Zhang, 2016] [Hou, Qin, Zhang, 2016]

[Hou, Zhang, 2017]



Energy content

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$g = \delta(x - x_0)$$

Beyond numerical homogenization (gamblet mesh refinement)

Uniformly bounded condition numbers

$$A_{i,j}^{(k)} := \langle \psi_i^{(k)}, \psi_j^{(k)} \rangle_a$$

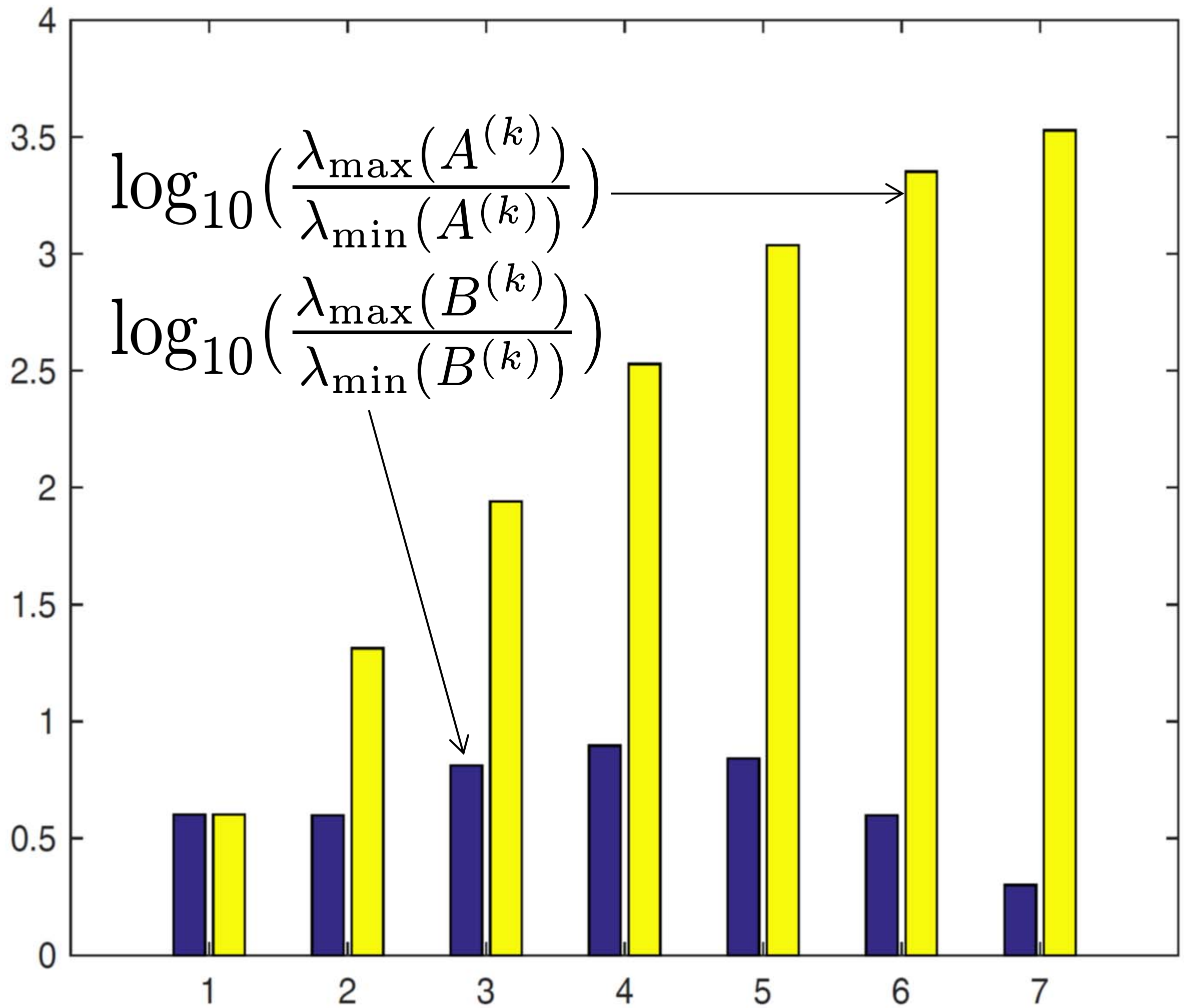
$$B_{i,j}^{(k)} := \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle_a$$

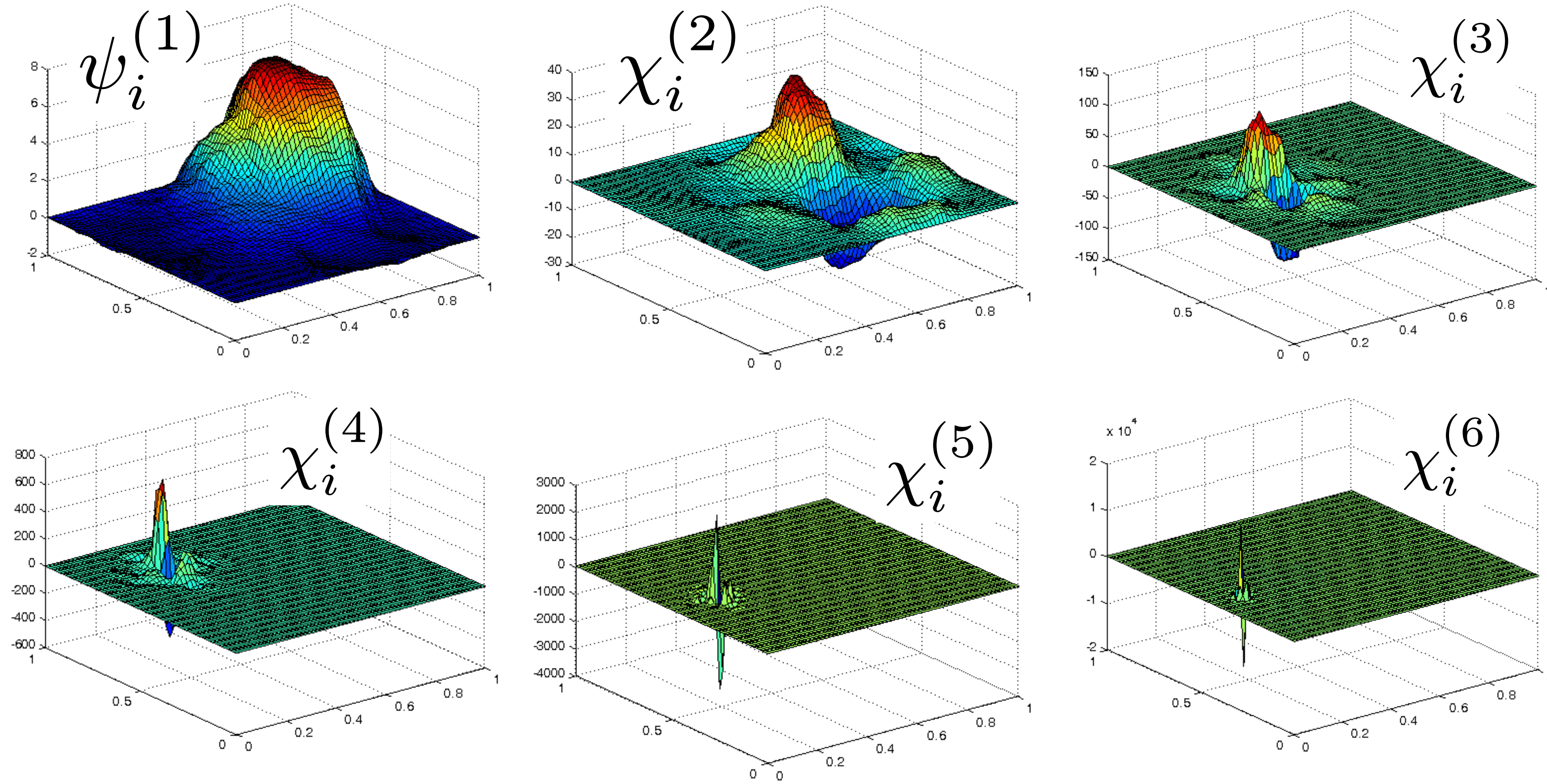
Theorem

$$\frac{\lambda_{\max}(B^{(k)})}{\lambda_{\min}(B^{(k)})} \leq C$$

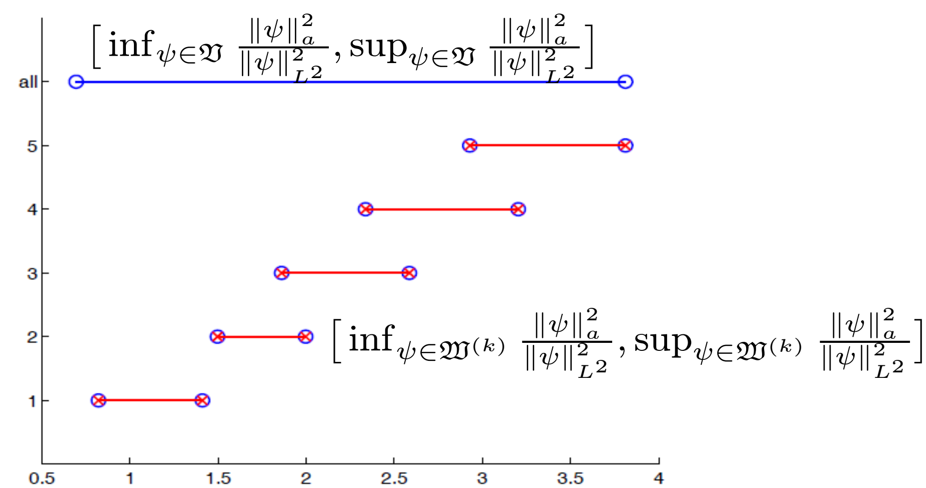
Just relax!

In $v \in \mathfrak{W}^{(k)}$
to get
$u^{(k)} - u^{(k-1)}$





Gamblets are not only localized in space and their linear combinations remain localized in frequency
They behave like wavelets and Wannier functions



Wannier functions

[Wannier. Dynamics of band electrons in electric and magnetic fields. 1962]

[Kohn. Analytic properties of Bloch waves and Wannier functions, 1959]

[Marzari, Vanderbilt. Maximally localized generalized Wannier functions for composite energy bands. 1997]

[E, Tiejun, Jianfeng. Localized bases of eigensubspaces and operator compression, 2010]

[Vidvuds, Lai, Caflisch, Osher, Compressed modes for variational problems in mathematics and physics, 2013]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

[Hou, Qin, Zhang, A sparse decomposition of low rank symmetric positive semi-definite matrices, 2016]

[Hou, Zhang, Sparse operator compression of elliptic operators. 2017]

Operator adapted wavelets

First Generation Wavelets: Signal and imaging processing

[Mallat, 1989] [Daubechies, 1990]

[Coifman, Meyer, and Wickerhauser, 1992]

First Generation Operator Adapted Wavelets (shift and scale invariant)

[Cohen, Daubechies, Feauveau. Biorthogonal bases of compactly supported wavelets. 1992]

[Beylkin, Coifman, Rokhlin, 1992] [Engquist, Osher, Zhong, 1992]

[Alpert, Beylkin, Coifman, Rokhlin, 1993] [Jawerth, Sweldens, 1993]

[Dahlke, Weinreich, 1993] [Bacry, Mallat, Papanicolaou. 1993]

[Bertoluzza, Maday, Ravel, 1994] [Vasilyev, Paolucci, 1996]

[Dahmen, Kunoth, 2005] [Stevenson, 2009]

Lazy wavelets (Multiresolution decomposition of solution space)

[Yserentant. Multilevel splitting, 1986]

[Bank, Dupont, Yserentant. Hierarchical basis multigrid method. 1988]

Operator adapted wavelets

Second Generation Operator Adapted Wavelets

[Sweldens. The lifting scheme, 1998] [Dorobantu - Engquist. 1998]

[Vassilevski, Wang. Stabilizing the hierarchical basis, 1997]

[Carnicer, Dahmen, Peña, 1996] [Lounsbery, DeRose, Warren, 1997]

[Vassilevski, Wang. Stabilizing hierarchical basis, 1997-1998]

[Barinka, Barsch, Charton, Cohen, Dahlke, Dahmen, Urban, 2001]

[Cohen, Dahmen, DeVore, 2001] [Chiavassa, Liandrat, 2001]

[Dahmen, Kunoth, 2005] [Schwab, Stevenson, 2008]

[Sudarshan, 2005] [Engquist, Runborg, 2009] [Yin, Liandrat, 2016]

We want (open problem solved here)

1. **Scale-orthogonal wavelets with respect to operator scalar product (leads to block-diagonalization)**
2. **Operator to be well conditioned within each subband**
3. **Wavelets need to be localized (compact support or exp. decay)**

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

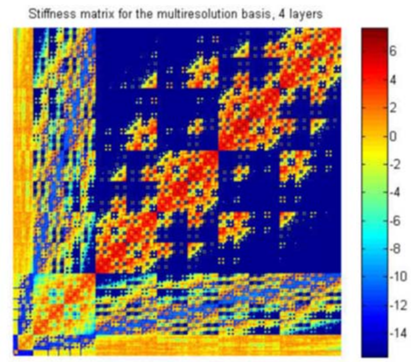
[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

Gamblet Transform

1. For $i,j \in \mathcal{I}^{(q)}$, $A^{\varphi}_{i,j} = \langle \varphi_i, \varphi_j \rangle$ // Stiffness matrix
2. For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q)} = \varphi_i$ // Level q gamblets
3. For $i,j \in \mathcal{I}^{(q)}$, $A^{(q)}_{i,j} = \langle \psi_i^{(q)}, \psi_j^{(q)} \rangle$
4. **For** $k = q$ to 2
5. For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$ // Level k, χ gamblets
6. $B^{(k)} = W^{(k)} A^{(k)} W^{(k),T}$ // $B^{(k)}_{i,j} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle$
7. $D^{(k,k-1)} = -B^{(k),-1} W^{(k)} A^{(k)} \bar{\pi}^{(k,k-1)}$ // $B^{(k),-1}$ =matrix inverse of $B^{(k)}$
8. $R^{(k-1,k)} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k)} W^{(k)}$ // Interpolation/restriction operator
9. For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1)} = \sum_{j \in \mathcal{I}^{(k)}} R_{i,j}^{(k-1,k)} \psi_j^{(k)}$ // Level $k-1$, ψ gamblets
10. $A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)}$ // $A^{(k-1)}_{i,j} = \langle \psi_i^{(k-1)}, \psi_j^{(k-1)} \rangle$
11. **End For**

Fast/Localized Gamblet Transform

1. For $i, j \in \mathcal{I}^{(q)}$, $A^{\varphi}_{i,j} = \langle \varphi_i, \varphi_j \rangle$
2. For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q),\mathrm{loc}} = \varphi_i$ // Localized basis at level q
3. For $i, j \in \mathcal{I}^{(q)}$, $A^{(q),\mathrm{loc}}_{i,j} = \langle \psi_i^{(q),\mathrm{loc}}, \psi_j^{(q),\mathrm{loc}} \rangle$
4. **For** $k = q$ to 2
5. $\quad B^{(k),\zeta,\mathrm{loc}} = W^{(k)} A^{(k),\mathrm{loc}} W^{(k),T}$
6. $\quad$ For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k),\mathrm{loc}} = \sum_{j \in \mathcal{I}^{(k)}} W^{(k)}_{i,j} \psi_j^{(k),\mathrm{loc}}$
7. $\quad \mathrm{Inv}(B^{(k),\mathrm{loc}} D^{(k,k-1),\mathrm{loc}} = -W^{(k)} A^{(k),\mathrm{loc}} \bar{\pi}^{(k,k-1)}, \rho_{k-1})$
8. $\quad R^{(k-1,k),\mathrm{loc}} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k),\mathrm{loc}} W^{(k)}$ // Localized restriction operator
9. $\quad A^{(k-1),\mathrm{loc}} = R^{(k-1,k),\mathrm{loc}} A^{(k),\mathrm{loc}} R^{(k,k-1),\mathrm{loc}}$
10. $\quad$ For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1),\mathrm{loc}} = \sum_{j \in \mathcal{I}^{(k)}} R^{(k-1,k),\mathrm{loc}}_{i,j} \psi_j^{(k),\mathrm{loc}}$ // Localized basis at level k
11. **End For**



Theorem

The number of operations to compute gamblets and achieve accuracy ϵ is $\mathcal{O}\big(N \ln^{3d}\big(\max(\frac{1}{\epsilon}, N^{1/d})\big)\big)$ (and $\mathcal{O}\big(N \ln^{d}(N^{1/d}) \ln \frac{1}{\epsilon}\big)$ for subsequent solves)

Complexity

Gamblet Transform $\boxed{\mathcal{O}(N \ln^{3d} N)}$ Linear Solve $\boxed{\mathcal{O}(N \ln^{d+1} N)}$

To achieve grid-size accuracy in H^1-norm

Can we design a universal scalable solver?

Sparse matrix Laplacians

Sparsified Cholesky and Multigrid Solvers for Connection Laplacians: [Kyng, Lee, Peng, Sachdeva, Spielman , 2016]

Approximate Gaussian Elimination: [Kyng and Sachdeva, 2016]

$$N\,\mathrm{polylog}(N) \text{ complexity}$$

Structured sparse matrices (SDD matrices)

Graph sparsification: [Spielman and Teng , 2004]

Diagonally dominant linear systems: [Spielman and Teng , 2014]

[Koutis, Miller, Gary and Peng , 2014]

[Cohen, Kyng, Miller, Pachocki, Peng, Rao, and Xu, 2014]

[Kelner, Orecchia, Sidford, Zhu, 2013]

The problem

$\mathcal{T}$: Continuous linear bijection

$$\mathcal{B} \xrightarrow{\quad \mathcal{T} \quad} \mathcal{B}^*$$

We want to approximate $\mathcal{T}^{-1}$ and
all its eigen-subpaces in near-linear complexity

For $u, v \in \mathcal{B}$,

- $[\mathcal{T}u, v] = [\mathcal{T}v, u]$,
- $[\mathcal{T}u, u] \geq 0$

$$\|u\|^2 := [\mathcal{T}u, u]$$

$(\mathcal{B}, \|\cdot\|)$: separable Banach space

Example

$$\begin{cases} -\operatorname{div}(a\nabla u) = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

$$\mathcal{T} = -\operatorname{div}(a\nabla \cdot)$$

$$(H_0^1(\Omega), \|\cdot\|_{H_0^1(\Omega)}) \xrightarrow{-\operatorname{div}(a\nabla \cdot)} (H^{-1}(\Omega), \|\cdot\|_{H^{-1}(\Omega)})$$

$$\mathcal{B} := H_0^1(\Omega)$$

$$\|u\|^2 := \int_\Omega (\nabla u)^T a \nabla u$$

Example

$$\mathcal{L}u = g$$

$\mathcal{L}$: arbitrary continuous linear bijection

$$\left(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}\right) \xrightarrow{\quad\mathcal{L}\quad} \left(H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)}\right)$$

$\mathcal{L}$: Symmetric and positive

- $[\mathcal{L}u, v] = [\mathcal{L}v, u]$,
- $[\mathcal{L}u, u] \geq 0$

$$\mathcal{B} := H_0^s(\Omega)$$

$$\mathcal{T} = \mathcal{L}$$

$$\|u\|^2 := [\mathcal{L}u, u]$$

Example

$$\mathcal{L}u = g \quad \Longleftrightarrow \quad \mathcal{L}^*\mathcal{L}u = \mathcal{L}^*g$$

$\mathcal{L}$: arbitrary continuous linear bijection

$$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}) \xrightarrow{\quad \mathcal{L} \quad} (L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$$

$$\mathcal{B} := H_0^s(\Omega)$$

$$\mathcal{T} = \mathcal{L}^*\mathcal{L}$$

$$\|u\| := \|\mathcal{L}u\|_{L^2(\Omega)}$$

Example

$$Ax = b$$

A: $N \times N$ symmetric postive definite matrix

$\mathcal{B} := \mathbb{R}^N$

$\mathcal{T} = A$

$$\|x\|^2 := x^T A x$$

Example

$$Ax = b \Longleftrightarrow A^T Ax = A^T b$$

A: $N \times N$ invertible matrix

$$\mathcal{B} := \mathbb{R}^N$$

$$\mathcal{T} = A^T A$$

$$\|x\|^2 := |Ax|^2$$

$$\mathcal{B} \xrightarrow{\quad\mathcal{T}\quad} \mathcal{B}^*$$

$$\|u\|^2 := [\mathcal{T}u, u]$$

Hierarchy of measurement functions

$$\phi_i^{(k)} \in \mathcal{B}^* \text{ with } k \in \{1,\ldots,q\}$$

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$

Hierarchy of gamblets

$$\psi_i^{(k)} = \sum_{j\in\mathcal{I}^{(k)}} \Theta_{i,j}^{(k),-1} \mathcal{T}^{-1} \phi_j^{(k)}$$

$$\Theta_{i,j}^{(k)} := [\phi_i^{(k)}, \mathcal{T}^{-1}\phi_j^{(k)}]$$

Biorthogonal system

$$[\phi_j^{(k)}, \psi_i^{(k)}] = \delta_{i,j}$$

$$\mathfrak{V}^{(k)} := \mathrm{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

Theorem

The $\langle \cdot, \cdot \rangle$ orthogonal projection of $u \in \mathcal{B}$ onto $\mathfrak{V}^{(k)}$ is

$$u^{(k)} = \sum_{i \in \mathcal{I}^{(k)}} [\phi_i^{(k)}, u] \psi_i^{(k)}$$

Measurement functions are nested

$$\phi_i^{(k)} = \sum_j \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)}$$

Gamblets are nested

$$\psi_i^{(k)} = \sum_{j\in\mathcal{I}^{(k+1)}} R_{i,j}^{(k,k+1)} \psi_j^{(k+1)}$$

Orthogonalized gamblets

$$\chi_i^{(k)} := \sum_{j\in\mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$$

For $k \geq 2$ $W^{(k)}$: $\mathcal{J}^{(k)} \times \mathcal{I}^{(k)}$ matrix such that $\mathrm{Img}(W^{(k),T}) = \mathrm{Ker}(\pi^{(k-1,k)})$ and $W^{(k)}(W^{(k)})^T = J^{(k)}$

Operator adapted MRA

$$\mathfrak{V}^{(k)} := \mathrm{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

$$\mathfrak{W}^{(k)} := \mathrm{span}\{\chi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

Theorem

$$\mathfrak{V}^{(k)} = \mathfrak{V}^{(k-1)} \oplus \mathfrak{W}^{(k)}$$

$$\mathcal{B} = \mathfrak{V}^{(1)} \oplus \mathfrak{W}^{(2)} \oplus \mathfrak{W}^{(3)} \oplus \cdots$$

$u^{(k)} - u^{(k-1)}$: The $\langle \cdot, \cdot \rangle$ orthogonal projection of $u \in \mathcal{B}$ onto $\mathfrak{W}^{(k)}$

$$\mathcal{B} \xrightarrow{\mathcal{T}} \mathcal{B}^*$$

$$u \longrightarrow g$$

$$\boxed{\mathcal{T} u = g}$$

Theorem $u = v^{(1)} + \cdots + v^{(k)} + \cdots$

$$v^{(k)} = \sum_{i \in \mathcal{I}^{(k)}} w_i^{(k)} \chi_i^{(k)}$$

$$\boxed{B^{(k)} w^{(k)} = g^{(k)}}$$

$$g_i^{(k)} = [g, \chi_i^{(k)}] \qquad B_{i,j}^{(k)} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle$$

Eigenspace adapted MRA

$$A_{i,j}^{(k)} = \langle \psi_i^{(k)}, \psi_j^{(k)} \rangle \qquad B_{i,j}^{(k)} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle$$

Theorem Under regularity of measurement functions

$$\frac{1}{C} H^{-2(k-1)} J^{(k)} \leq B^{(k)} \leq C H^{-2k} J^{(k)}$$

$$\boxed{\mathrm{Cond}(B^{(k)}) \leq C H^{-2}}$$

$$\frac{1}{C} I^{(1)} \leq A^{(1)} \leq C H^{-2} I^{(1)}$$

$$\mathrm{Cond}(A^{(1)}) \leq C H^{-2}$$

Regularity Conditions

$$(\mathcal{B}, \|\cdot\|) \xrightarrow{\quad\mathcal{T}\quad} (\mathcal{B}^*, \|\cdot\|_*)$$

$$\cup$$

$$\phi_i^{(k)} \in \mathcal{B}_0 \qquad (\mathcal{B}_0, \|\cdot\|_0)$$

$(\mathcal{B}_0, \|\cdot\|_0)$: Separable Banach subspace of $(\mathcal{B}^*, \|\cdot\|_*)$

Unit ball of $(\mathcal{B}_0, \|\cdot\|_0)$ compactly embedded in $(\mathcal{B}^*, \|\cdot\|_*)$

The method

Multi-resolution decomposition of $(\mathcal{B}_0, \|\cdot\|_0) \to (\mathcal{B}^*, \|\cdot\|_*)$

↓ Gamblet transform

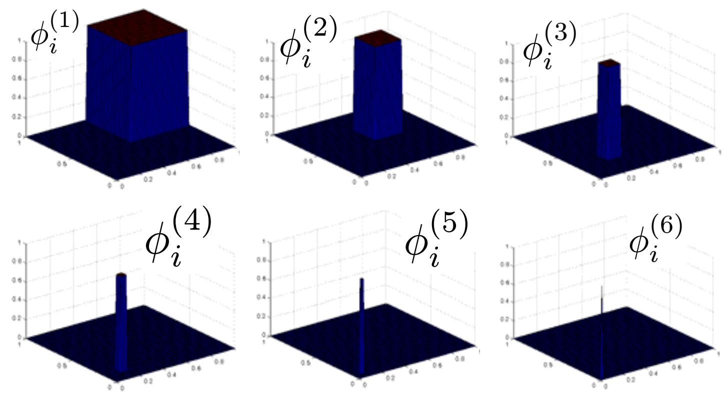
Multi-resolution decomposition of $(\mathcal{B}, \|\cdot\|) \to (\mathcal{B}^*, \|\cdot\|_*)$

$$(H^1_0(\Omega), \|\cdot\|_a) \xrightarrow{\;-\operatorname{div}(a\nabla\cdot)\;} (H^{-1}(\Omega), \|\cdot\|_{H^{-1}(\Omega)})$$

$$\bigcup$$

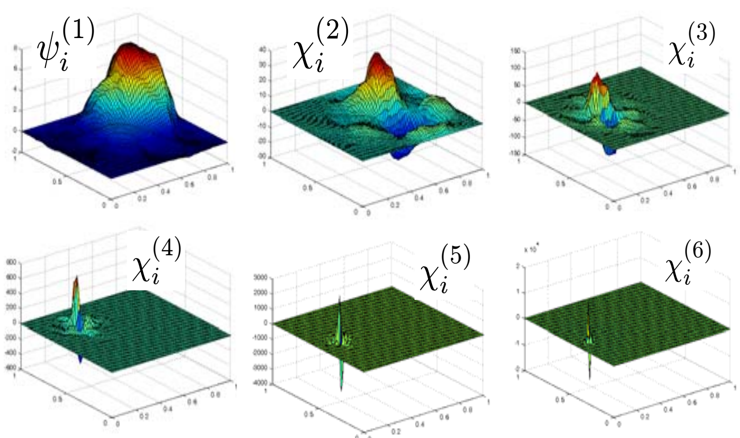
$$(L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$$

The method



Haar-wavelet decomposition of
$(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}) \to (H^{-1}(\Omega), \|\cdot\|_{H^{-1}(\Omega)})$

Gamblet transform



Multi-resolution decomposition of
$(H^1_0(\Omega), \|\cdot\|_{H^1_0(\Omega)}) \to (H^{-1}(\Omega), \|\cdot\|_{H^{-1}(\Omega)})$

$$\Phi^{(k)} := \operatorname{span}\{\phi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}$$

Regularity Conditions

For some $H \in (0,1)$ and $C_\Phi > 0$

1. $\frac{1}{C_0}|x|^2 \leq \| \sum_{i\in\mathcal{I}^{(k)}} x_i\, \phi_i^{(k)} \|_0^2 \leq C_0 |x|^2$ for $x \in \mathbb{R}^{\mathcal{I}^{(k)}}$.

2. $\|\phi\|_0 \leq C_0 H^{-k} \|\phi\|_*$ for $\phi \in \Phi^{(k)}$.

3. $\inf_{\phi\in\Phi^{(k)}} \|\varphi - \phi\|_* \leq C_0 H^k \|\varphi\|_0$ for $\varphi \in \mathcal{B}_0$

4. $\|\phi\|_* \leq C_0 H^k \|\phi\|_0$
   for $\phi \in \{\sum_{i\in\mathcal{I}^{(k+1)}} x_i \phi_i^{(k+1)} \mid x \in \operatorname{Ker}(\pi^{(k,k+1)})\}$

Conditions are covariant under norm equivalence

Example $\mathcal{B}^* = H^{-s}(\Omega)$ $\mathcal{B}_0 = L^2(\Omega)$

Regularity Conditions

For some $H \in (0,1)$ and $C > 0$

1. $\frac{1}{C}|x|^2 \leq \| \sum_{i\in\mathcal{I}^{(k)}} x_i \, \phi_i^{(k)} \|^2_{L^2(\Omega)} \leq C|x|^2$ for $x \in \mathbb{R}^{\mathcal{I}^{(k)}}$

2. $\|\phi\|_{L^2(\Omega)} \leq CH^{-k}\|\phi\|_{H^{-s}(\Omega)}$ for $\phi \in \Phi^{(k)}$

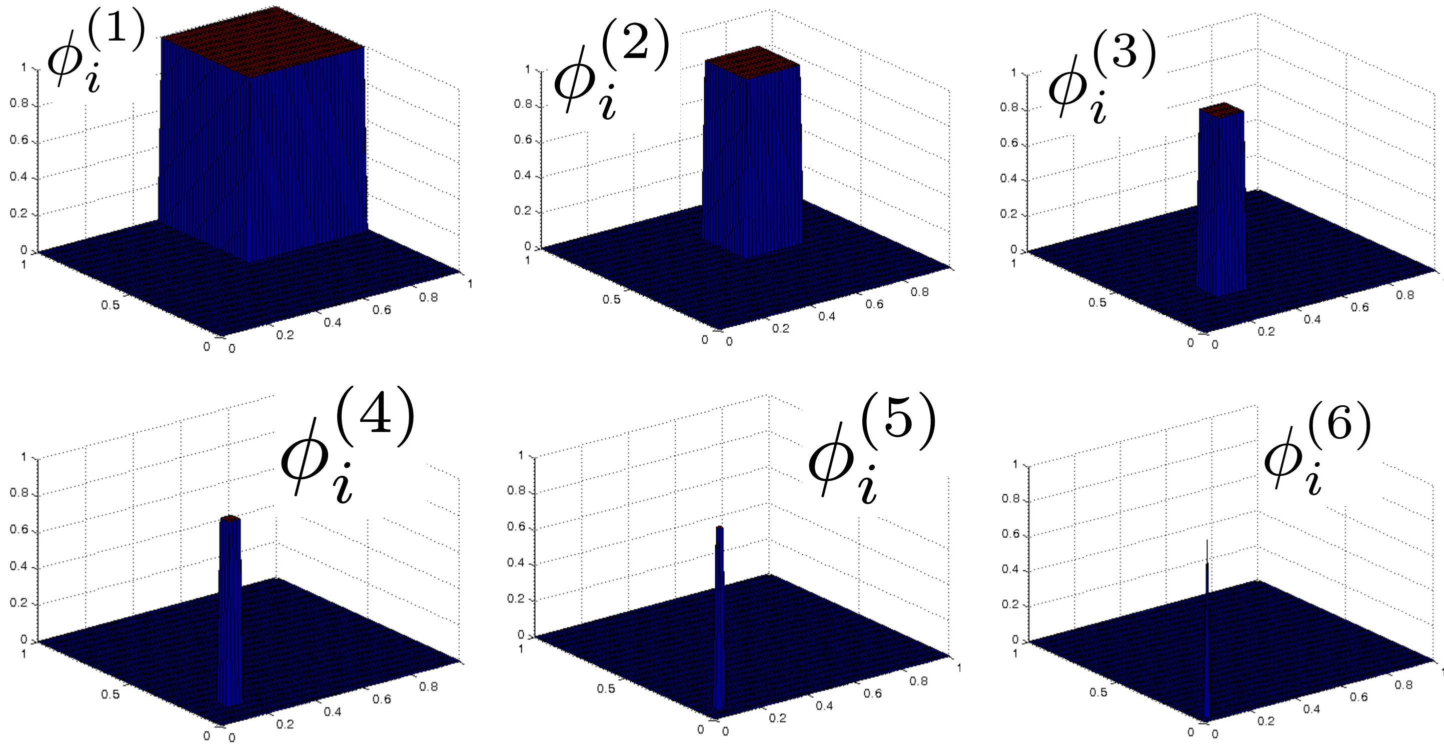
3. $\inf_{\phi\in\Phi^{(k)}} \|\varphi - \phi\|_{H^{-s}(\Omega)} \leq CH^k\|\varphi\|_{L^2(\Omega)}$ for $\varphi \in L^2(\Omega)$

4. $\|\phi\|_{H^{-s}(\Omega)} \leq CH^k\|\phi\|_{L^2(\Omega)}$
   for $\phi \in \{\sum_{i\in\mathcal{I}^{(k+1)}} x_i\phi_i^{(k+1)} \mid x \in \mathrm{Ker}(\pi^{(k,k+1)})\}$

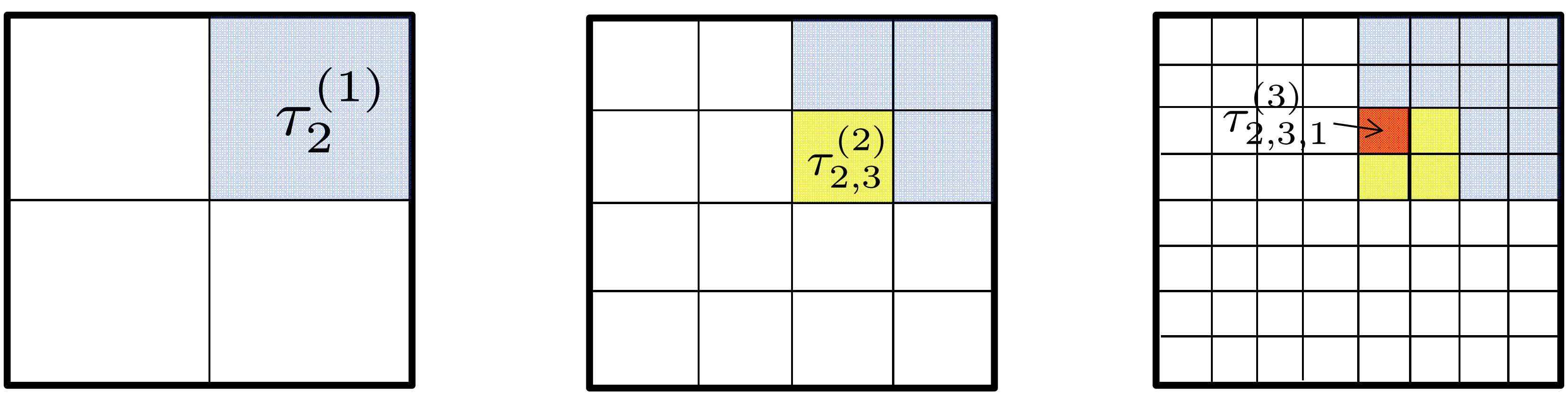
Example

$s = 1$

$H = \frac{1}{2}$



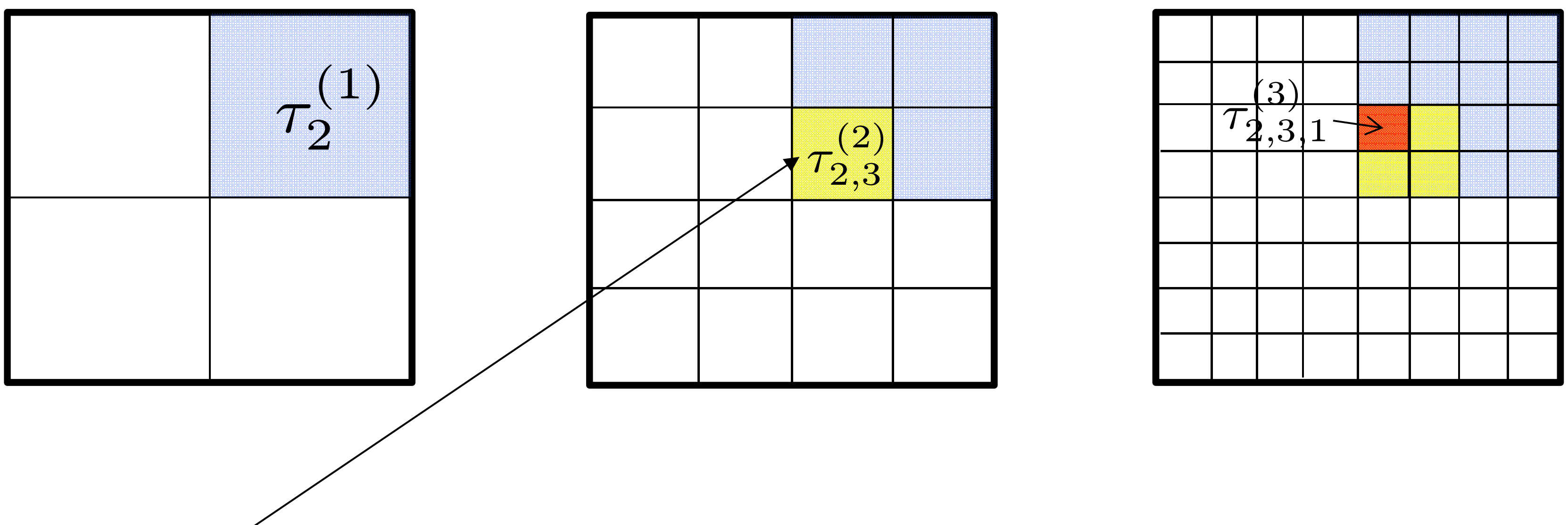
$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of Ω of resolution 2^{-k}



Example $s \geq 2$ $H = \frac{1}{2^s}$

$(\phi_{i,\alpha}^{(k)})_{\alpha \in \beth}$: orthonormal basis functions of $\mathcal{P}_{s-1}(\tau_i^{(k)})$

$\mathcal{P}_{s-1}(\tau_i^{(k)})$: polynomials of degree at most $s-1$



$\tau_i^{(k)}$: Hierarchical nested partition of Ω of resolution 2^{-k}

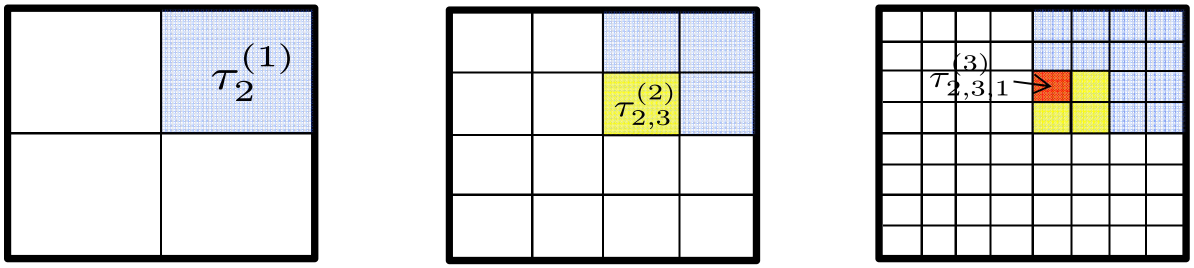
[Hou and Zhang, 2017]: Numerical homogenization of strongly elliptic PDEs
(h sufficiently small, and higher order polynomials as measurement functions)

Example

$$s \geq 2 \qquad H = \frac{1}{2^s}$$

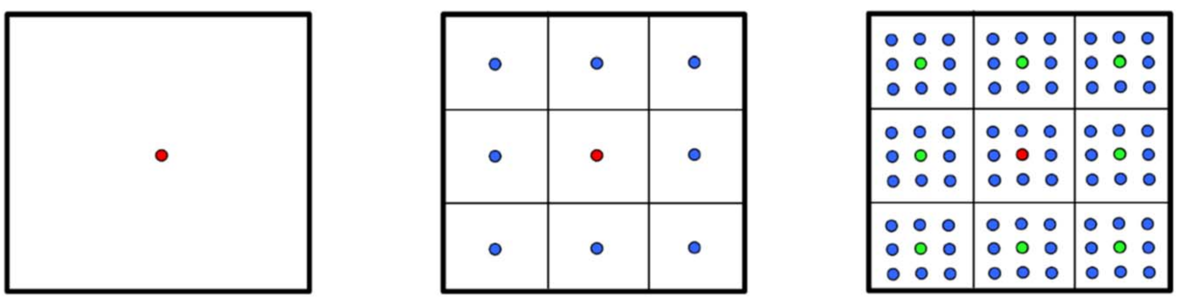
[Schäfer, Sullivan, Owhadi. 2017]: Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity.

$\phi_i^{(k)}$: Weighted indicator functions of a hierarchical nested partition of Ω of resolution 2^{-k}



$$s > d/2$$

$\phi_i^{(k)}$: Subsampled delta Dirac functions



Example

$$\mathcal{B} := \mathbb{R}^N$$

A: $N \times N$ symmetric postive definite matrix

$$\|x\|^2 := x^T A x$$

$$\|x\|_*^2 := x^T A^{-1} x$$

$$\|x\|_0^2 := x^T x$$

$$\phi_i^{(q)} = e_i$$

$$\pi^{(k,k+1)}\left(\pi^{(k,k+1)}\right)^T = I^{(k)}$$

Regularity Conditions

$$\pi^{(k,q)} = \pi^{(k,k+1)} \ldots \pi^{(q-1,q)}$$

For some $H \in (0,1)$ and $C > 0$

1. $\frac{1}{C\sqrt{\lambda_{\min}(A)}} H^k \leq \inf_{x \in \operatorname{Img}(\pi^{(q,k)})} \frac{\sqrt{x^T A^{-1} x}}{|x|}$

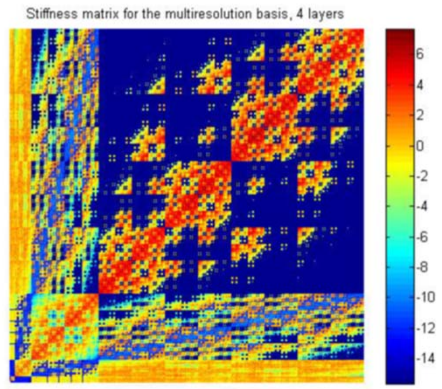
2. $\sup_{x \in \operatorname{Ker}(\pi^{(k,q)})} \frac{\sqrt{x^T A^{-1} x}}{|x|} \leq \frac{C}{\sqrt{\lambda_{\min}(A)}} H^k$

Conditions are covariant under quadratic form equivalence

Gamblet Transform/Solve

For $i \in \mathcal{I}^{(q)}$, $\psi_i^{(q)} = \varphi_i$
For $i \in \mathcal{I}^{(q)}$, $g_i^{(q)} = [g, \psi_i^{(q)}]$
For $i, j \in \mathcal{I}^{(q)}$, $A_{i,j}^{(q)} = \langle \psi_i^{(q)}, \psi_j^{(q)} \rangle$
for $k = q$ to 2 **do**
    $B^{(k)} = W^{(k)} A^{(k)} W^{(k),T}$
    $w^{(k)} = B^{(k),-1} W^{(k)} g^{(k)}$
    For $i \in \mathcal{J}^{(k)}$, $\chi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} W_{i,j}^{(k)} \psi_j^{(k)}$
    $u^{(k)} - u^{(k-1)} = \sum_{i \in \mathcal{J}^{(k)}} w_i^{(k)} \chi_i^{(k)}$
    $D^{(k,k-1)} = -B^{(k),-1} W^{(k)} A^{(k)} \bar{\pi}^{(k,k-1)}$
    $R^{(k-1,k)} = \bar{\pi}^{(k-1,k)} + D^{(k-1,k)} W^{(k)}$
    $A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)}$
    For $i \in \mathcal{I}^{(k-1)}$, $\psi_i^{(k-1)} = \sum_{j \in \mathcal{I}^{(k)}} R_{i,j}^{(k-1,k)} \psi_j^{(k)}$
    $g^{(k-1)} = R^{(k-1,k)} g^{(k)}$
end for
$U^{(1)} = A^{(1),-1} g^{(1)}$
$u^{(1)} = \sum_{i \in \mathcal{I}^{(1)}} U_i^{(1)} \psi_i^{(1)}$
$u = u^{(1)} + (u^{(2)} - u^{(1)}) + \cdots + (u^{(q)} - u^{(q-1)})$

Fast Gamblet Transform obtained by truncation/localization



Complexity Theorem $N = \mathrm{Card}(\mathcal{I}^{(q)})$

$N \log^{3d}(N)$: Computation of all gamblets

$N \log^{d+1}(N)$: Gamblet transform/solve of $u \in \mathcal{B}$ to accuracy H^q in $\|\cdot\|$ norm

Based on exponential decay of gamblets and locality of the operator

d: Hausdorff dimension of d^A.

d^A: Graph distance of A on $\mathcal{I}^{(q)}$

$A_{i,j} := \langle \varphi_i, \varphi_j \rangle$, stiffness matrix of the operator

$$\mathrm{Card}\{j | d^A_{i,j} \leq r\} \leq C\, r^d$$

Localization of Gamblets

Localization problem in Numerical Homogenization

[Chu-Graham-Hou-2010] (limited inclusions)

[Efendiev-Galvis-Wu-2010] (limited inclusions or mask)

[Babuska-Lipton 2010] (local boundary eigenvectors)

[Owhadi-Zhang 2011] (localized transfer property)

[Malqvist-Peterseim 2012] Local Orthogonal Decomposition

[Owhadi-Zhang-Berlyand 2013] (Rough Polyharmonic Splines)

[A. Gloria, S. Neukamm, and F. Otto, 2015] (quantification of ergodicity)

[Hou and Liu,DCDS-A, 2016] [Chung-Efendiev-Hou, JCP 2016]

[Owhadi, Multiresolution operator decomposition, SIREV 2017]

[Owhadi, Zhang, gamblets for hyperbolic and parabolic PDEs, 2016]

[Hou, Qin, Zhang, 2016] [Hou, Zhang, 2017]

[Hou and Zhang, 2017]: Higher order PDEs (localization under strong ellipticity, h sufficiently small, and higher order polynomials as measurement functions)

[Kornhuber, Yserentant, 2016]: Subspace decomposition

Subspace decomposition/correction and Schwarz iterative methods

[J. Xu, 1992]: Iterative methods by space decomposition and subspace correction

[Griebel-Oswald, 1995]: Schwarz algorithms

Example

$$\mathcal{B} := H_0^s(\Omega) \qquad \|u\|^2 := [\mathcal{L}u, u]$$

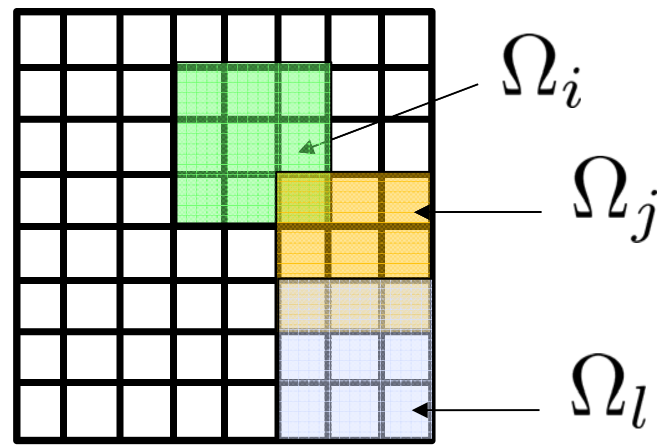
$\mathcal{L}$: arbitrary continuous positive symmetric linear bijection

$$\left(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}\right) \xrightarrow{\quad \mathcal{L} \quad} \left(H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)}\right)$$

$\mathcal{L}$ is local $\quad \langle u, v \rangle = 0$ if u and v have disjoint supports

$$H_0^s(\Omega) = \sum_{i \in \beth} H_0^s(\Omega_i)$$

Ω



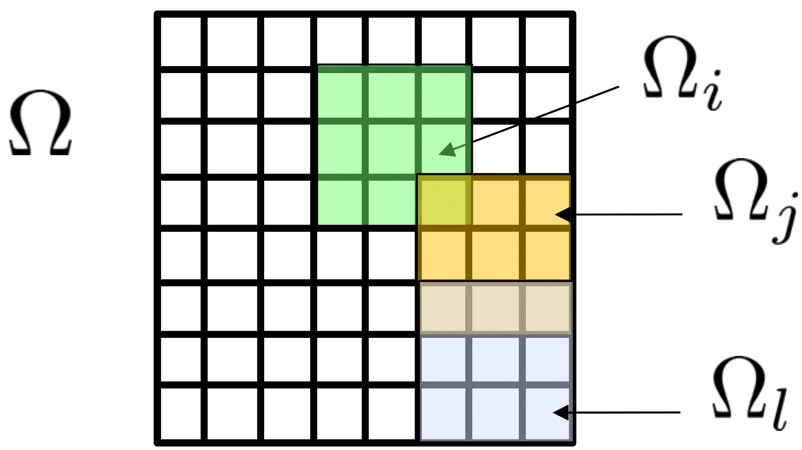
$$\Omega = \cup_i \Omega_i$$

Condition for localization

For $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \leq C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i,\alpha) \in \beth \times \aleph\}$$



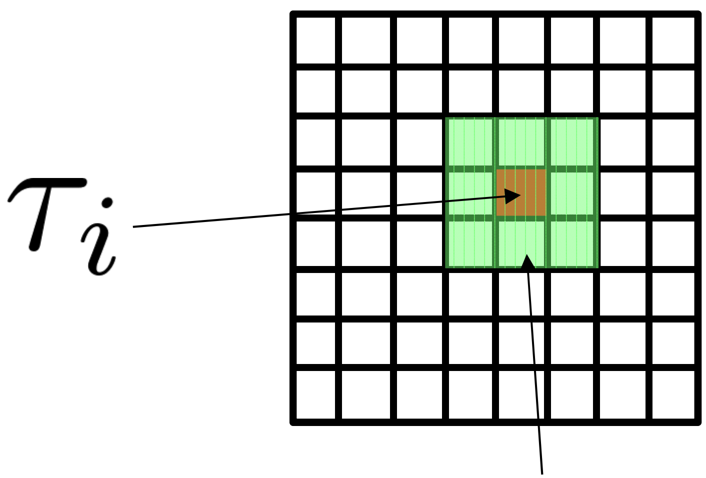
$$\Omega = \cup_i \Omega_i$$

Examples

$$\Omega = \cup_i \tau_i$$

$$B(x_i, \delta h) \subset \tau_i$$

$$\tau_i \subset B(x_i, h)$$



$$B(x_i, 2h) \subset \Omega_i$$

- $\phi_i = \frac{1_{\tau_i}}{\sqrt{|\tau_i|}}$.
- $\phi_i = \delta(\cdot - x_i)$, $\left(s > \frac{d}{2}\right)$
- $(\phi_{i,\alpha})_{\alpha \in \beth}$ forms an orthonormal basis of $\mathcal{P}_{s-1}(\tau_i)$

Theorem

Assume that there exists a constant C_0 such that $|\aleph| \leq C_0$,

- $\|D^t f\|_{L^2(\Omega)} \leq C_0 h^{s-t} \|f\|_{H_0^s(\Omega)}$ for $t \in \{0, 1, \ldots, s\}$, for $f \in H_0^s(\Omega)$ such that $[\phi_{i,\alpha}, f] = 0$ for $(i, \alpha) \in \beth \times \aleph$,
- $\sum_{i \in \beth, \alpha \in \aleph} [\phi_{i,\alpha}, f]^2 \leq C_0 \big(\|f\|^2_{L^2(\Omega)} + h^{2s} \|f\|^2_{H_0^s(\Omega)} \big)$, for $f \in H_0^s(\Omega)$, and
- $|x|^2 \leq C_0 h^{-2s} \| \sum_{\alpha \in \aleph} x_\alpha \phi_{i,\alpha} \|^2_{H^{-s}(\tau_i)}$, for $i \in \beth$ and $x \in \mathbb{R}^{\aleph}$.

Then for $\varphi \in H^{-s}(\Omega)$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega_i)}}{\inf_{\phi \in \Phi} \|\varphi - \phi\|^2_{H^{-s}(\Omega)}} \leq C_{\max}$$

Where $C_{\max}, C_{\min}$ depend only on C_0, d, δ and s

Straightforward generalization

$$\mathcal{B} := H_0^s(\Omega) \qquad \|u\| := \|\mathcal{L}u\|_{L^2(\Omega)}$$

$\mathcal{L}$: arbitrary continuous linear bijection

$$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}) \xrightarrow{\quad \mathcal{L} \quad} (L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$$

$\mathcal{L}$ is local $\quad \langle u, v \rangle = 0$ if u and v have disjoint supports

$$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)}) \xrightarrow{\quad \mathcal{T} = \mathcal{L}^*\mathcal{L} \quad} (H^{-s}(\Omega), \|\cdot\|_{H^{-s}(\Omega)})$$

Localization of gamblets

$$\mathcal{B} = \sum_{i \in \gimel} \mathcal{B}_i$$

$\|\cdot\|_i$ and $\|\cdot\|_{i,*}$ norms induced by $\|\cdot\|$ on $\mathcal{B}_i$ and $\mathcal{B}_i^*$

Operator connectivity distance

C: $\gimel \times \gimel$ connectivity matrix

$C_{i,j} = 1$ if $\exists (\chi_i, \chi_j) \in \mathcal{B}_i \times \mathcal{B}_j$ s.t. $\langle \chi_i, \chi_j \rangle \neq 0$

$C_{i,j} = 0$ otherwise

d: Graph distance on $\gimel$ induced by C

$(\phi_{i,\alpha})_{(i,\alpha)\in \beth\times\aleph}$: Measurement functions

$\phi_{i,\alpha} \in \mathcal{B}_i^*$

$(\psi_{i,\alpha})_{(i,\alpha)\in \beth\times\aleph}$: Gamblets

$\psi_{i,\alpha}^n$: Localization of $\psi_{i,\alpha}$ to $\mathcal{B}_i^n$

$$\mathcal{B}_i^n = \cup_{j:\mathbf{d}(i,j)\leq n} B_i$$

Theorem Under localization conditions

$$\|\psi_{i,\alpha} - \psi_{i,\alpha}^n\| \leq Ce^{-n/C}$$

Measurement functions

$$(\phi_{i,\alpha})_{(i,\alpha)\in \beth \times \aleph}$$

Condition for localization

For $\varphi \in \mathcal{B}^*$

$$C_{\min} \leq \frac{\sum_i \inf_{\phi\in\Phi} \|\varphi - \phi\|_{i,*}^2}{\inf_{\phi\in\Phi} \|\varphi - \phi\|_*^2} \leq C_{\max}$$

$$\Phi = \{\phi_{i,\alpha} \mid (i,\alpha) \in \beth \times \aleph\}$$

Conditions are covariant under norm equivalence

Game theoretic origin/interpreation

To compute fast we need to compute with partial information

$$u \in \mathcal{B}$$

Restriction ↓ ↑ **Interpolation**

$$u_m = ([\phi_1, u], \ldots, [\phi_m, u])$$
$$\phi_1, \ldots, \phi_m \in \mathcal{B}^*$$

$$u_m \xrightarrow{\boxed{\text{Missing information}}} u$$

Problem

$$\text{Given } ([\phi_1, u], \ldots, [\phi_m, u]) \text{ recover } u$$

Repeated adversarial information games

Player I

Chooses $u^I \in \mathcal{B}$

Player II

Sees $([\phi_i, u^I])_{i \in \mathcal{I}}$

Chooses $u^{II} \in \mathcal{B}$

Max

Min

$$\frac{\|u^I - u^{II}\|}{\|u^I\|}$$

Motivations: loss in relative error translates into loss in CPU time and total CPU time is the sum of these losses

Theorem The optimal mixed strategy for Player I is

$$u^{I,\dagger} \sim \mathcal{N}(0, Q)$$

$\mathcal{N}(0, Q)$: Gaussian field on $\mathcal{B}$ with covariance operator Q.

$$Q = \mathcal{T}^{-1}$$

$\xi \sim \mathcal{N}(0, Q)$ ⇔ ξ: Linear isometry mapping $\mathcal{B}^*$ to a Gaussian Space

For $\phi, \varphi \in \mathcal{B}^*$,

- $[\phi, \xi] \sim \mathcal{N}(0, \|\phi\|_*^2)$,
- $\mathbb{E}\big[[\varphi, \xi][\phi, \xi]\big] = \langle \varphi, \phi \rangle_*$

$$\|\phi\|_* := \sup_{v \in \mathcal{B}} \frac{[\phi, v]}{\|v\|}$$

The optimal measure (mixed strategy) for Player I is solely determined by the norm $\|\cdot\|$

Universal Optimal Measure

Theorem The optimal strategy for Player II is

$$u^{II,\dagger} = \mathbb{E}\big[\xi \mid [\phi_i, \xi] = [\phi_i, u^I] \text{ for } i \in \mathcal{I}\big]$$

$$\xi \sim \mathcal{N}(0, Q)$$

The optimal measure (mixed strategy) for Player II is solely determined by the norm $\| \cdot \| = [Q^{-1} \cdot, \cdot]^{\frac{1}{2}}$

It is a universal optimal measure
(it does not depend on the measurements)

Gamblets

Theorem The optimal strategy for Player II is

$$u^{II,\dagger} = \sum_{i\in\mathcal{I}} \psi_i [\phi_i, u^I]$$

$$\psi_i = \mathbb{E}\Big[\xi \mid [\phi_j, \xi] = \delta_{i,j} \text{ for } j \in \mathcal{I}\Big]$$

$$\xi \sim \mathcal{N}(0, Q)$$

ψ_i: Best gamble/bet (gamblet) of Player II on the value of u^I given the information that $[\phi_j, u^I] = \delta_{i,j}$ for $j \in \{1, \ldots, m\}$.

Theorem ψ_i is the minimizer of

$$\begin{cases} \text{Minimize } \|v\| \\ \text{Subject to } v \in \mathcal{B} \text{ and } [\phi_j, v] = \delta_{i,j} \text{ for } j \in \mathcal{I} \end{cases}$$

$$\psi_i = \sum_j \Theta^{-1}_{i,j} Q \phi_j \qquad \Theta_{i,j} = [\phi_i, Q\phi_j]$$

Gamblets ψ_i are
optimal recovery splines
in the sense of (Micchelli & Rivlin 1977)

Optimal recovery splines

[Micchelli & Rivlin, 1972]

- Polyharmonic splines [Harder-Desmarais, 72] [Duchon 72]
- Variational Multiscale Methods [Hughes et al, 98]
- Rough Polyharmonic Splines [Owhadi-Zhang-Berlyand, 14]
- LOD basis [Malqvist-Peterseim, 14]
- Bayesian Inference interpretation of Numerical Homogenization [Owhadi, 15]
- Gamblets [Owhadi-15], [Owhadi-Zhang, 16]
- Numerical homogenization of higher order PDEs [Hou-Zhang, 17]

Theorem We have

$$u^{II,\dagger} = \operatorname*{argmin}_{u^{II}} \inf_{u^{II} \in L(\Phi,\mathcal{B})} \sup_{u^I \in \mathcal{B}} \frac{\|u^I - u^{II}(u^I)\|}{\|u^I\|}$$

$$u^{II} \in L(\Phi, \mathcal{B})$$

$$\Updownarrow$$

u^{II} is a (measurable) function of $[\phi_1, u^I], \ldots, [\phi_m, u^I]$

The optimal game theoretic solution
is equal to the optimal recovery solution

Theorem $u^{II,\dagger}$ is the minimizer of

$$\begin{cases} \text{Minimize } \|v\| \\ \text{Subject to } v \in \mathcal{B} \text{ and } [\phi_i, v] = [\phi_i, u^I] \text{ for } i \in \mathcal{I} \end{cases}$$

Link between numerical analysis and statistical inference

$$u^I \sim \mathcal{N}(0, Q) \Longleftrightarrow u^I - u^{II} \sim \mathcal{N}(0, Q^\Phi)$$

$$Q^\Phi = (I - P_{Q\Phi}) Q (I - P_{Q\Phi})^*$$

$$P_{Q\Phi} = \sum_{i \in \mathcal{I}} \psi_i \otimes \phi_i$$

$$P^*_{Q\Phi} = Q^{-1} P_{Q\Phi} Q$$

- Express numerical approximation errors as (posterior) probability distributions.
- Statistical inference approaches to numerical analysis: enables seamless coupling of numerical approximation errors with model uncertainty.

Coupling numerical approximation error with model uncertainty

Statistical inference approaches to numerical approximation

Pioneering work

[Henri Poincaré. Calcul des probabilités. 1896.]

[A. V. Sul'din, Wiener measure and its applications to approximation methods. Matematika 1959]

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[F.M. Larkin. Gaussian measure in Hilbert space and applications in numerical analysis. Rocky Mountain J. Math, 1972]

Statistical inference approaches to numerical approximation

Information based complexity

[H. Woźniakowski. Probabilistic setting of information-based complexity. J. Complexity, 1986.]

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Statistical inference approaches to numerical approximation

Bayesian Numerical Analysis

[P. Diaconis. Bayesian numerical analysis. In Statistical decision theory and related topics, 1988]

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[A. O'Hagan. Bayes-Hermite quadrature. J. Statist. Plann. Inference, 29(3):245 260, 1991.]

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Probabilistic Numerics

[Chkrebtii, O. A., Campbell, D. A., Girolami, M. A. and Calderhead, B. Bayesian uncertainty quantification for differential equations. arXiv:1306.2365. 2013]

[H. Owhadi. Bayesian Numerical Homogenization. SIAM MMS, 2015]

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Probabilistic Numerics

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[H. Owhadi and C. Scovel. Universal Scalable Robust Solvers from Computational Information Games and fast eigenspace adapted Multiresolution Analysis 2017.]

[Schäfer, Sullivan, Owhadi. Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity, 2017]

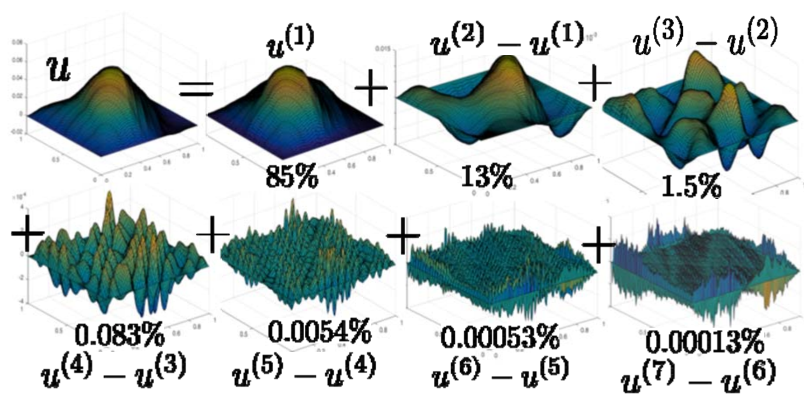
Game Theoretic approach to Numerical Analysis

- Here distributions are not arbitrary and are minimax optimal from both the numerical analysis (optimal recovery) and the decision theoretic perspectives.

- To compute posterior distribution we need to invert dense kernel (covariance) matrices (complexity bottleneck for Probabilistic Numerics)

- Can be done in near-linear complexity with gamblets
[Schäfer, Sullivan, Owhadi, 2017]: Compression, inversion, and approximate PCA of dense kernel matrices at near-linear computational complexity.

- Express numerical approximation errors as sums of of independent Gaussian fields (probabilistic version of mesh refinement).



Thank you

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