Computational Challenges in Reservoir Modeling

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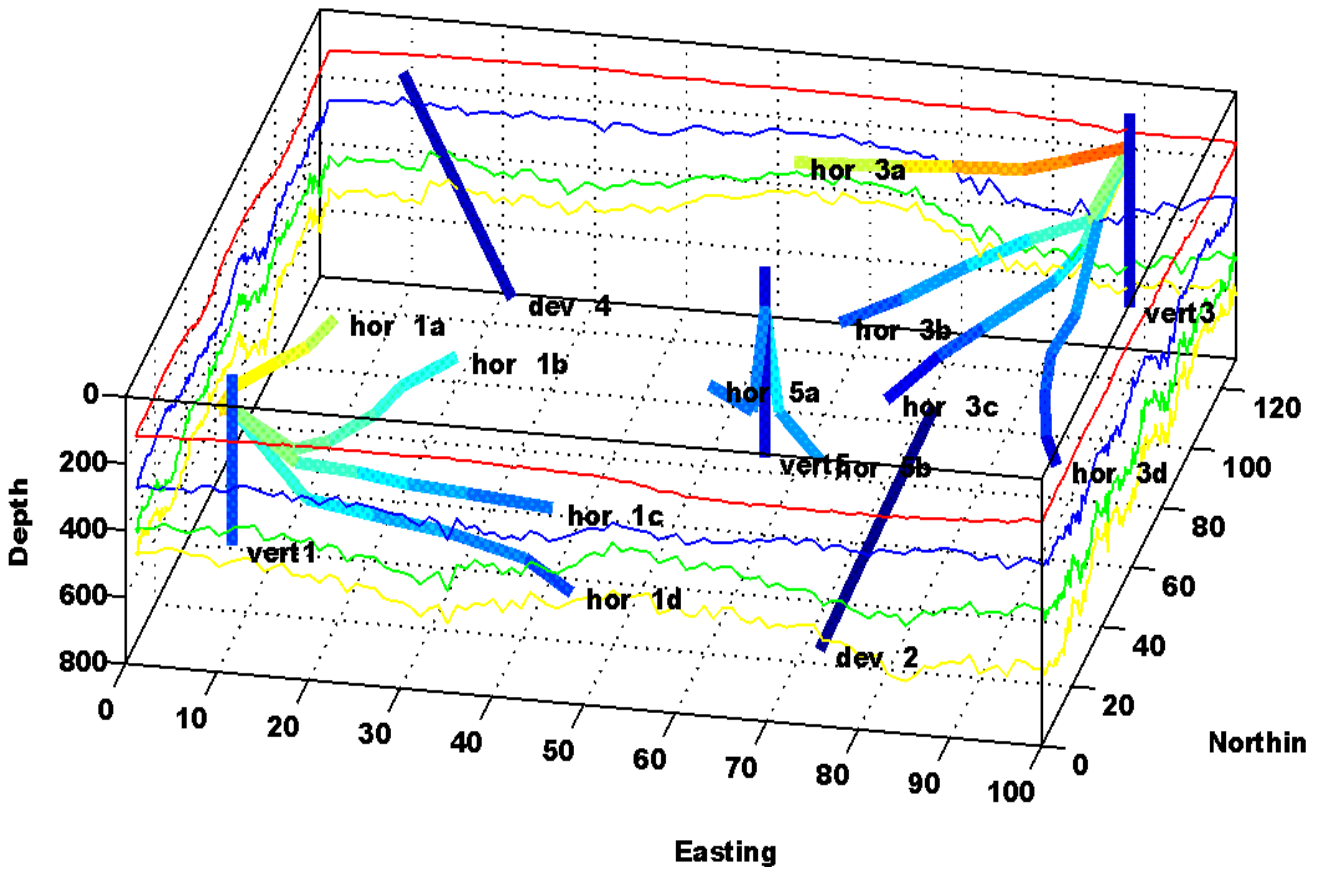
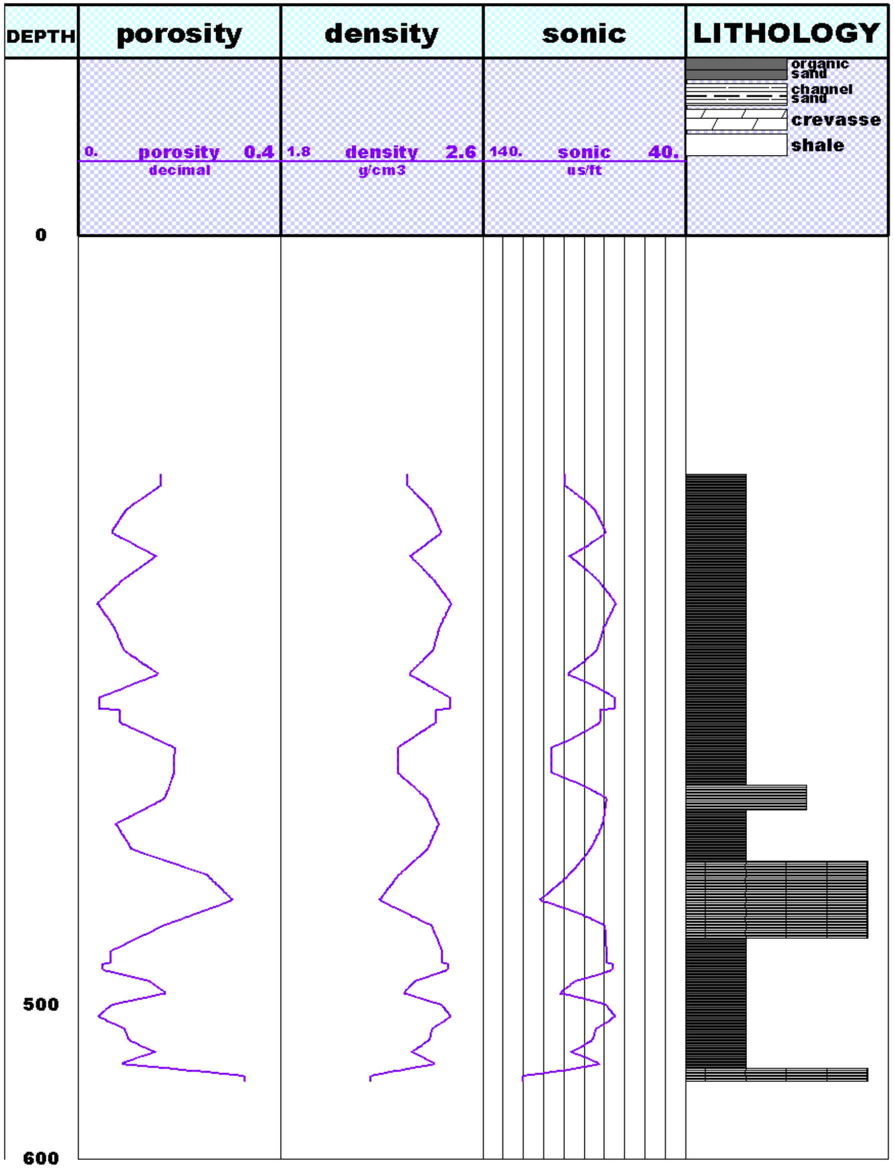
Well Data

3D view of well paths



- Inspired by an offshore development
- 4 platforms
- 2 vertical wells
- 2 deviated wells
- 8 horizontal wells

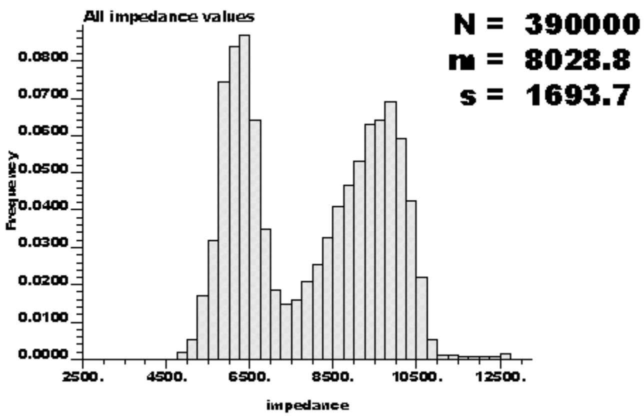
Sample well log

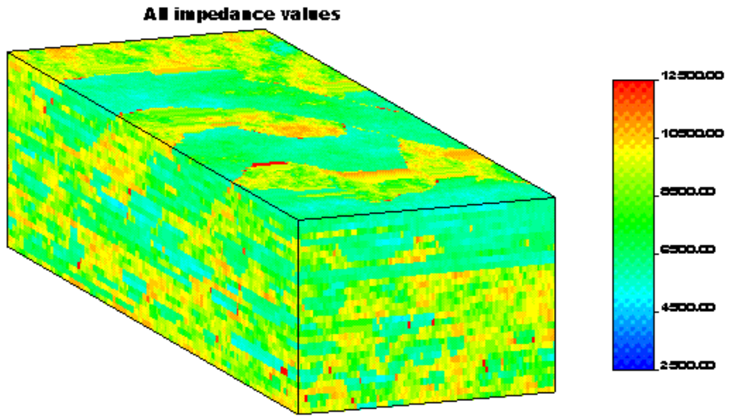


- Well data include:
  - horizontal location
  - true vertical depth
  - porosity
  - permeability
  - density
  - sonic log
  - lithology
  - layer indicator
- Well markers:

  true vertical depth when well intercepts a layer surface

Seismic impedance

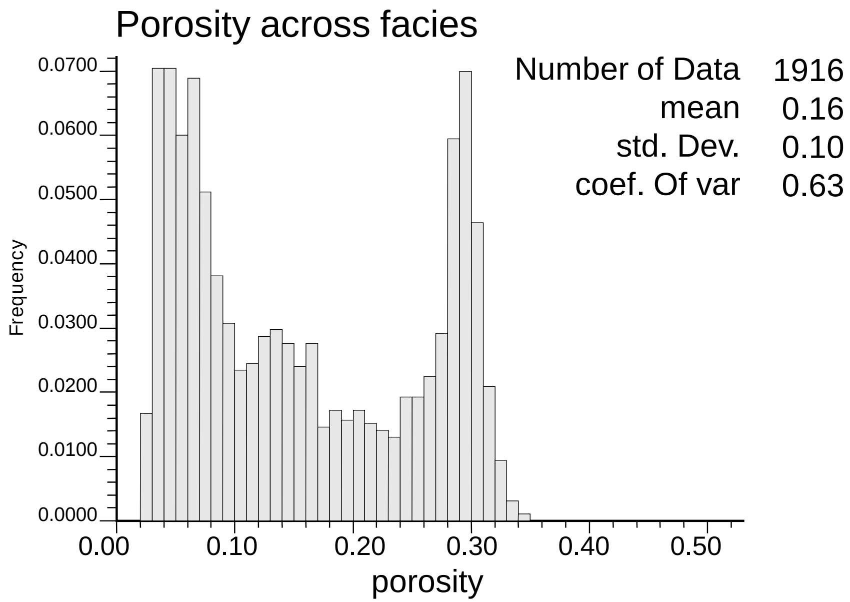




Impedance

Case Study Summary

- Reference reservoir is a fluvial reservoir
- Porosity and permeability histogram show distinct bi-modality

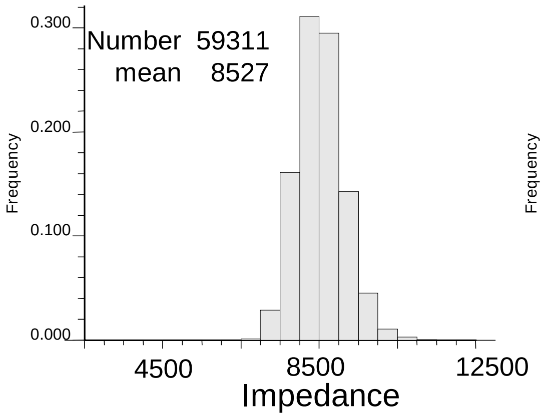


➡ Compartmentalization of reservoir into pay / non-pay – decision of stationarity

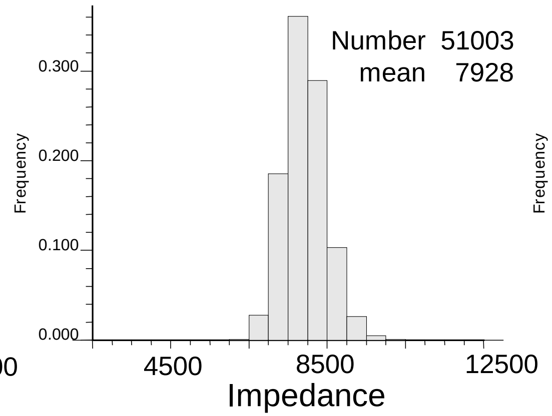
Facies classification basis

- Seismic impedance in channels, mudstone and border sands :

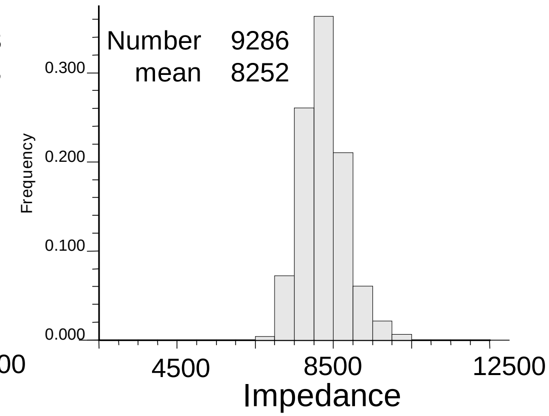
Impedance in mudstone



Impedance in channel



Impedance in border sand



Pay Facies: channel sands

Non-pay facies: mudstone + splays + levies

Work flow

- simulate multiple pay / non-pay facies models
- within non-pay, simulate border facies

➡ multiple facies templates

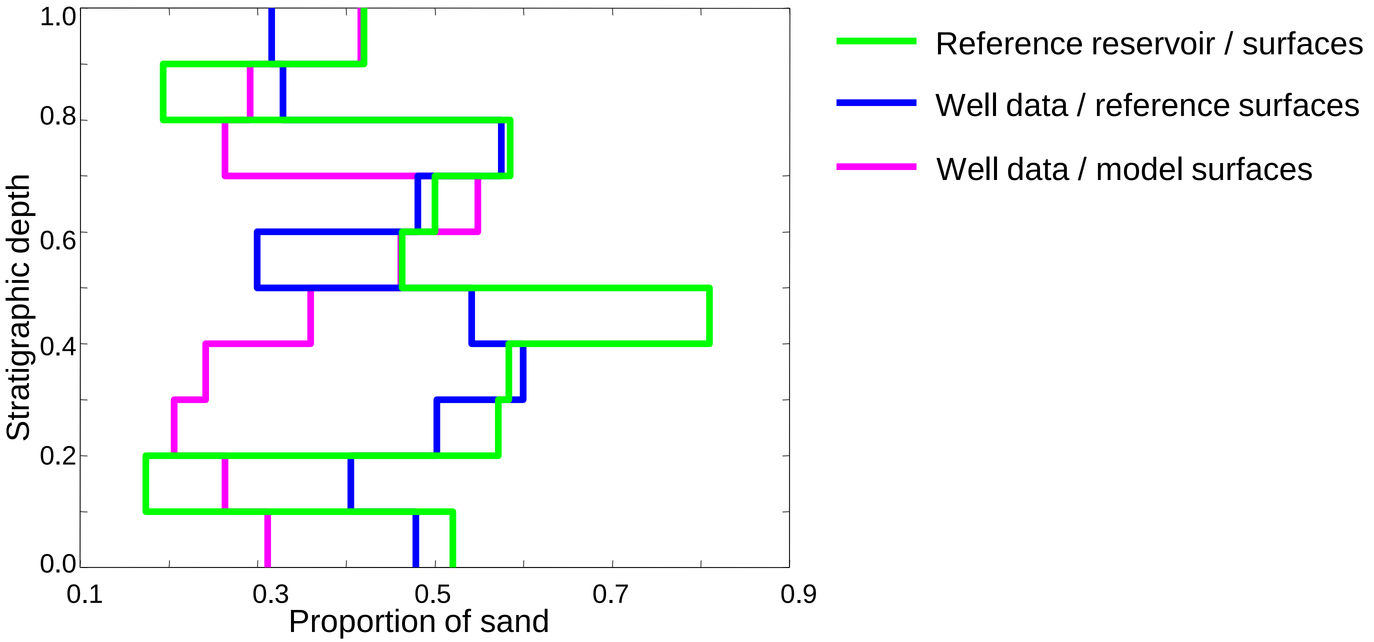
- simulate porosity models separately in pay and non-pay
- same for permeability
- merge pay / nonpay, porosity / permeability models using matching facies template

➡ multiple equi-probable reservoir models

Pay / Non-pay facies modeling

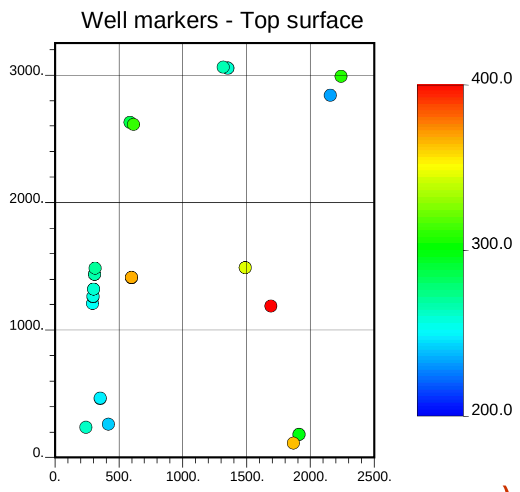
Available Data

1) Vertical proportion curve



uncertainty in modeled surfaces → uncertain vertical proportion curve

Semi-Variogram



Defined as: $$\gamma_{XY} = \frac{1}{2} E\left\{(X - Y)^2\right\}$$

In the spatial context, $$2\gamma(\mathbf{u}, \mathbf{u}+\mathbf{h}) = E\left\{\left(Z(\mathbf{u}) - Z(\mathbf{u}+\mathbf{h})\right)^2\right\}$$

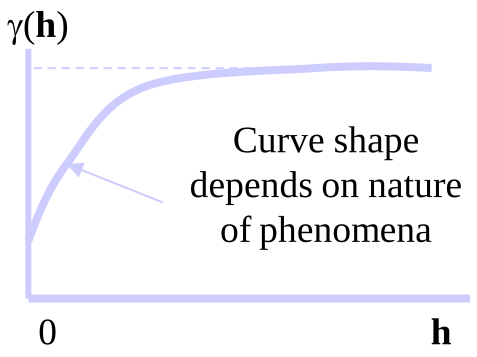
Under intrinsic stationarity – the RF $Z(\mathbf{u})$ itself might not be stationary, but increments of the RF $\{Z(\mathbf{u})-Z(\mathbf{u}+\mathbf{h})\}$ are presumed stationary. Thus:

$$\gamma(\mathbf{u}, \mathbf{u}+\mathbf{h}) = \frac{1}{2} E\left\{\left(Z(\mathbf{u}) - Z(\mathbf{u}+\mathbf{h})\right)^2\right\} = \frac{1}{2} Var\{Z(\mathbf{u}) - Z(\mathbf{u}+\mathbf{h})\}$$

$$= \gamma(\mathbf{h})$$ stationary

Variogram Inference

- Inference of the experimental variogram requires:
  - Plotting **h**-scatterplot by systematically varying h in particular spatial directions.
  - For each scatterplot compute the corresponding moment of inertia of the scatterplot.
  - Plot the moment of interia versus the lag **h** in specific directions



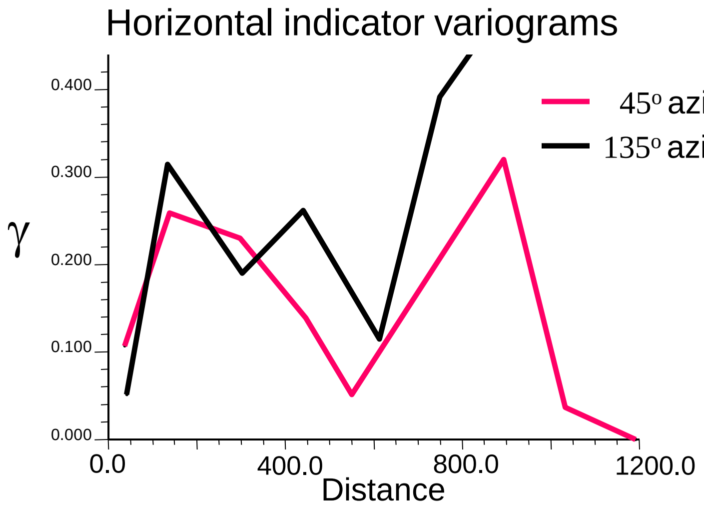
Some remarks

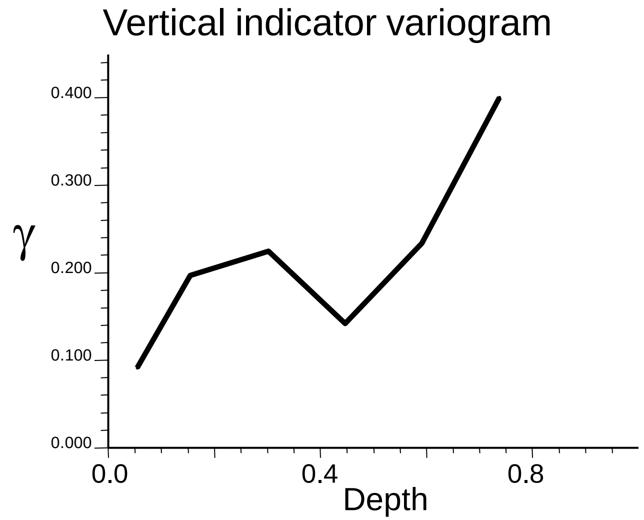
•There must be sufficient number of pairs in each scatterplot (statistical mass)

•Outliers (abnormal high or low values) can cause the moment of inertia to fluctuate wildly

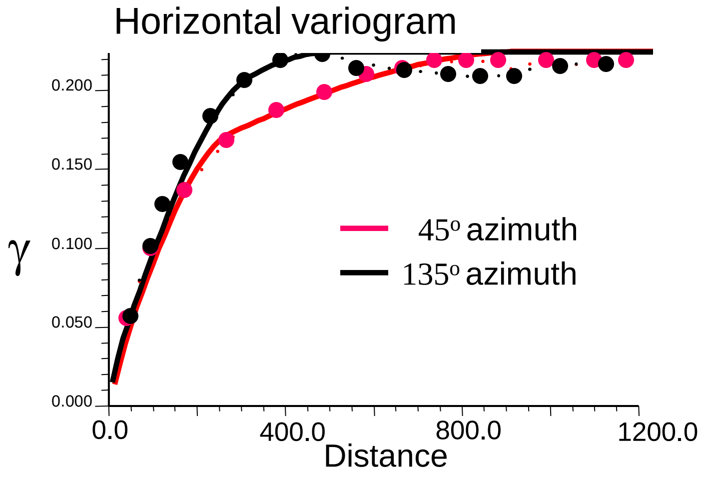
Facies modeling - Data

- Indicator variogram - “hard” data



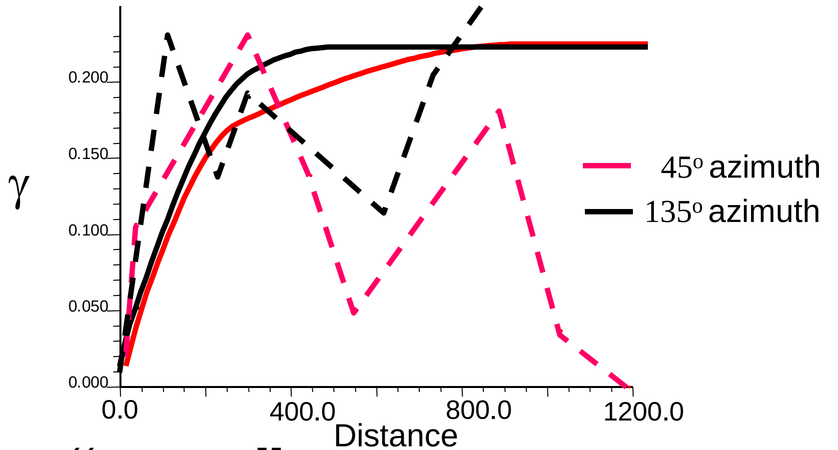


- Seismic impedance

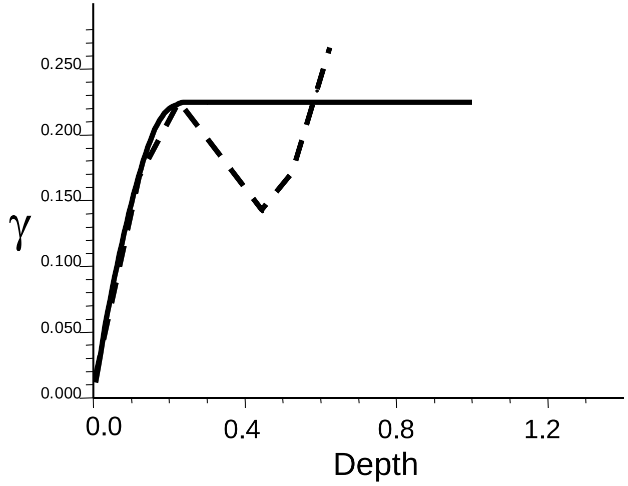


Facies modeling - variograms

Horizontal borrowed from seismic



Vertical from “hard” data



Spatial Estimation

Simple kriging (SK)

of the depth $d(\mathbf{x}_o)$ at unsampled locations $\mathbf{x}_o$
from marker data $d(\mathbf{x}_\alpha), \alpha = 1,..,n$

SK estimate:
$$D^*(\mathbf{x}_o) = \left(1 - \sum_\alpha \lambda_\alpha\right) \cdot m_A + \sum_{\alpha=1}^{n} \lambda_\alpha \cdot D(\mathbf{x}_\alpha)$$

SK system:
$$\sum_{\beta=1}^{n} \lambda_\beta \cdot C(\mathbf{x}_\alpha - \mathbf{x}_\beta) = C(\mathbf{x}_\alpha - \mathbf{x}_o)\ , \alpha = 1,\ldots,n$$

Note : $\mathbf{u} = (\mathbf{x}, \mathbf{d})$

Requisites

$E\{D(\mathbf{x})\} = m_A$ average depth-data over study A

$C(\mathbf{h})$ covariance modeled after a spatial average $C_A(\mathbf{h})$ over "stationary" area A

Indicator Paradigm

- **Consider the indicator RV:**

$$I(\mathbf{u}) = \begin{cases} 1, & \mathit{if}\ A(\mathbf{u}) = a \\ 0, & \text{otherwise} \end{cases}$$

- **Important property:**

$$E\{I(\mathbf{u})\} = 1 \times \text{Prob}(\text{A}(\mathbf{u}) = \text{a}) + 0 \times \text{Prob}(\text{A}(\mathbf{u}) \neq \text{a}) = \text{Prob}(\text{A}(\mathbf{u}) = \text{a})$$

or better still, given *n* data:

$$E\{I(\mathbf{u}) \,|\, (n)\} = \text{Prob}\left(\text{A}(\mathbf{u}) = \text{a} \,|\, (\text{n})\right)$$

Indicator basis function

- **Consider the expansion $\varphi(n)$ defined on the basis of *n* indicator random variables :** $I_\alpha, \alpha = 1, \ldots, n$

$$\varphi(n) = a + \sum_{j_1=1}^{n} b_{j_1}^{(1)} \cdot I(\mathbf{u}_{j_1}) + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} b_{j_1,j_2}^{(2)} \cdot I(\mathbf{u}_{j_1}) I(\mathbf{u}_{j_2}) + \cdots + b_{j_1,j_2,\cdots,j_n}^{(n)} \prod_{i=1,n} I(\mathbf{u}_i)$$

  - **Given the n indicator random variables, this expansion is the most complete possible.**
  - **There are a total of $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ terms in the above expansion**
  - **There are 2 outcomes for each RV there are 2^n outcomes possible for the function $\varphi(n)$.**
  - **These outcomes define the space $\varphi(n)$**

Indicator Expansion

- **The conditional expectation $E\{I(\mathbf{u})\,|\,(n)\}$ is precisely the projection of the unknown indicator event $I(\mathbf{u})$ on to $\varphi(n)$.**

- **If instead the projection function is defined on a reduced basis *L**$_k$**:**

$$E\{I\,|\,(n)\} \cong E_k\{I\,|\,(n)\} = I_k^* = a + \sum_{j_1=1}^{n} b_{j_1}^{(1)} \cdot I(\mathbf{u}_{j_1}) + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} b_{j_1,j_2}^{(2)} \cdot I(\mathbf{u}_{j_1}) I(\mathbf{u}_{j_2}) +$$

$$\cdots + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_k=1}^{n} b_{j_1,j_2,\ldots,j_k}^{(k)} \cdot I(\mathbf{u}_{j_1}) I(\mathbf{u}_{j_2}) \cdots I(\mathbf{u}_{j_k})$$

Indicator Expansion

- **Another way to write the expansion:**

$$I_k^* = \sum_{\ell=1}^{n_k} \lambda_\ell \cdot V_\ell \qquad n_k = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$$

basis function $V_1 = \{\mathbf{1}\}\ ;\ \{I_i\},\ \{I_i \cdot I_j\}$ etc.

Projection Theorem

Let H be a Hilbert space and M a closed subspace of H, then corresponding to any $x \in H$ there exists a unique vector $m_o \in M$ such that:

$$\|x - m_o\| \leq \|x - m\| \quad \forall\ m \in M$$

Furthermore a necessary and sufficient condition that $m_o \in M$ be a unique minimizing vector is that $x - m_o$ be orthogonal to M (Luenberger 1997, p. 51).

Normal Equations

- **The implication of the projection theorem is that:**

$$\left\langle \left(I - I_k^*\right), V_\ell \right\rangle = 0 \quad \Leftrightarrow \quad \left\langle I \cdot V_\ell \right\rangle = \left\langle I_k^* \cdot V_\ell \right\rangle$$

or in terms of projections:

$$E\left\{I_k^* \;\; V_\ell\right\} = \;\; E\left\{I \;\; V_\ell\right\} \; , \; \forall \;\; \ell = 1 \; ,..., \;\; n_k$$

which is a system of n_k **normal equations**

Examples

$$V_1 = \mathbf{1} \;\Rightarrow\; E\{I_k^* \;\; \mathbf{1}\} = \sum_{\ell=1}^{n_k} \lambda_\ell \cdot E\{V_\ell\} = E\{I\}$$ **- unbiasedness**

$$V_\ell = I_j,\; j = 1,..,n \;\Rightarrow\; E\{I_k^* \;\; I_j\} = \sum_{\ell=1}^{n_k} \lambda_\ell \cdot E\{V_\ell \cdot I_j\} = E\{I \cdot I_j\}$$

- reproduction of indicator covariances

$$V_\ell = I_{j_1} \cdot I_{j_2},\; j_1,\; j_2 = 1,..,n \;\Rightarrow\; E\{I_k^* \;\; I_{j1} I_{j2}\} = \sum_{\ell=1}^{n_k} \lambda_\ell \cdot E\{V_\ell \cdot I_{j_1} I_{j_2}\} = E\{I \cdot I_{j_1} I_{j_2}\}$$

- reproduction of the 3rd order covariances

In general: $\boxed{\sum_{\ell=1}^{n_k} \lambda_\ell \cdot E\{V_\ell \cdot V_\ell'\} = E\{I \cdot V_\ell'\},\; \ell = 1,..,n_k}$

requires knowledge of up to order $k+1$ **indicator covariances**

Facies modeling - indicator approach

Indicator kriging (Simple IK)

$$I^*(\mathbf{u}) = p + \sum_{\alpha=1}^{n(\mathbf{u})} \lambda(\mathbf{u}_\alpha) \cdot \left[I(\mathbf{u}_\alpha) - p \right]$$

Interpretation

$$I^*(\mathbf{u}) = Prob\left\{ u \in \text{pay} \,|\, n \text{ data} \right\}$$

IK System

$$\sum_{\beta=1}^{n(\mathbf{u})} \lambda(\mathbf{u}_\beta) \cdot C_I(\mathbf{h}_{\alpha\beta}) = C_I(\mathbf{h}_{\alpha 0})$$

Facies modeling - indicator approach

Indicator kriging → local conditional probability
$$Prob\{u \in \text{pay} | n \text{ data}\}$$

Simulation

- Draw $r \in \text{Uniform}[0,1]$

  If $r \leq Prob\{u \in \text{pay} | n \text{ data}\}$

  then $I^{\ell}(\mathbf{u}) = 1 : \mathbf{u} \in \text{pay}$ else $I^{\ell}(\mathbf{u}) = 0 : \mathbf{u} \in \text{nonpay}$
- Add simulated $I^{\ell}(\mathbf{u})$ into data set : (n) → (n+1)
- Visit next node $\mathbf{u}'$

Facies modeling - indicator approach

Locally varying mean

$$I^*(\mathbf{u}) = p_V(d) + \sum_{\alpha=1}^{n(\mathbf{u})} \lambda(\mathbf{u}_\alpha) \cdot \left[I(\mathbf{u}_\alpha) - p_V(d_\alpha) \right]$$

Requisites

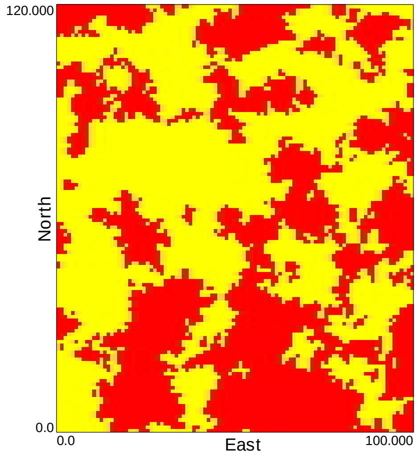
- vertical proportion of pay facies in stratigraphic layer d : $p_V(d)$

- Indicator variogram model, $\gamma_I(\mathbf{h})$

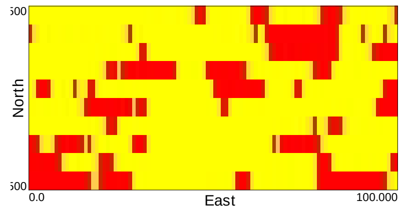
Indicator Simulation - Results

- Slices through the facies model

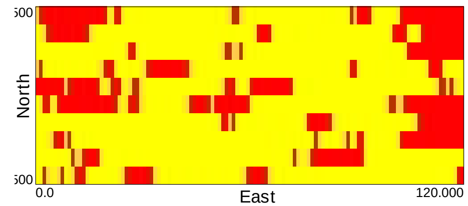
XY slice #7

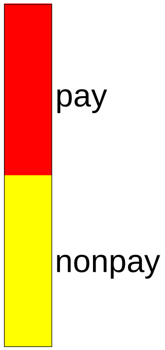


XZ slice #20



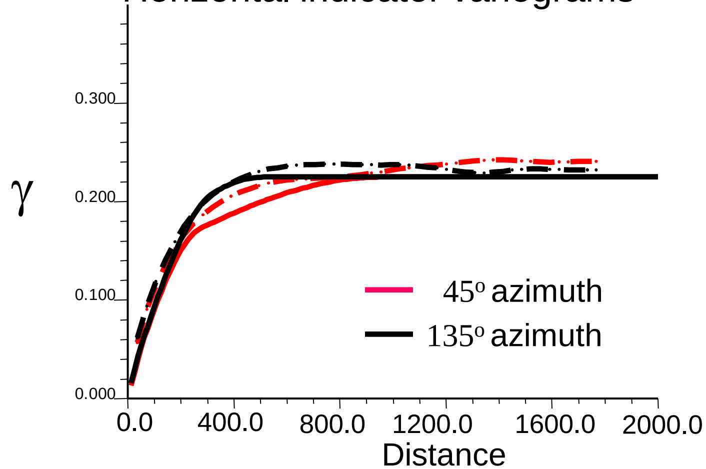
YZ slice #20





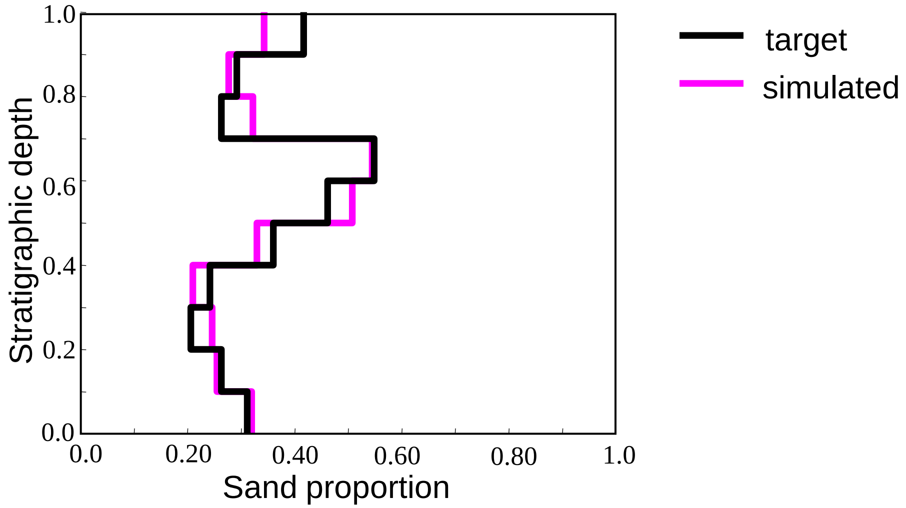
- Variogram reproduction

Horizontal indicator variograms



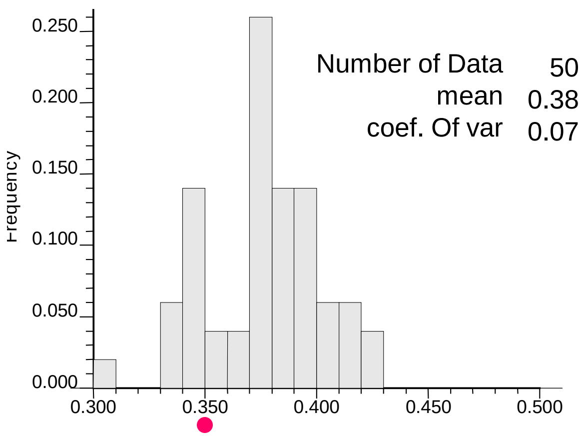
Indicator Simulation - Results

- Vertical proportion

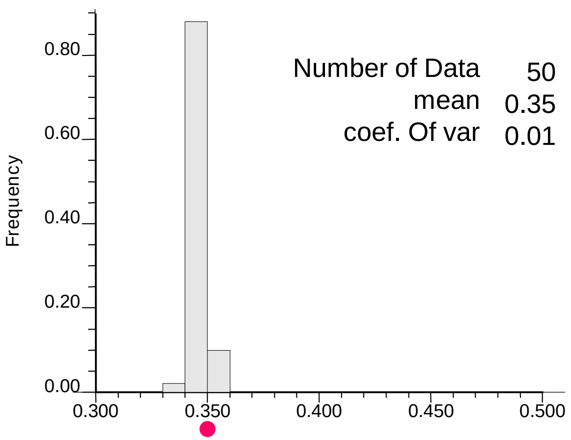


- Global pay proportion

before **trans**



after **trans**



Indicator Simulation - Results

- Perform a histogram transformation
- Simulation after **trans**

Facies modeling - using seismic info.

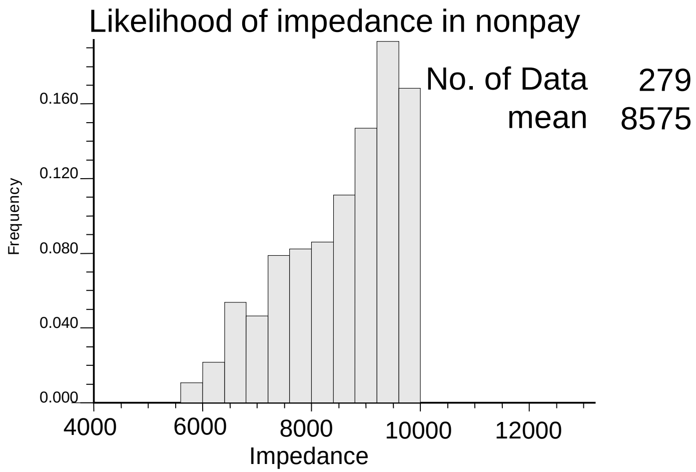
- Likelihood of seismic impedance | $u \in \text{pay}$

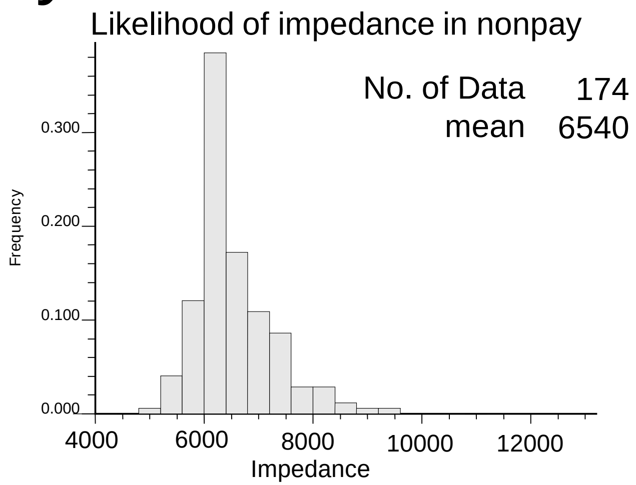
$$F(s\,|\,1) = Prob\{\, S(\mathbf{u}) \le s \;|\; \mathbf{u} \in \text{pay} \,\}$$

Similarly,

$$F(s\,|\,0) = Prob\{\, S(\mathbf{u}) \le s \;|\; \mathbf{u} \in \text{nonpay} \,\}$$

- If: $F(s\,|\,1) \neq F(s\,|\,0)$ then impedance discriminates pay / nonpay





Facies modeling - using seismic info.

- Want $Prob\{I(\mathbf{u})=1 \mid s(\mathbf{u}) \in [s_l, s_{l+1}]\}$

- Apply Bayes' inversion:

$$Prob\{A \mid B\} = \frac{Prob\{B \mid A\}}{Prob\{B\}} \cdot Prob\{A\}$$

$$Prob\{I(\mathbf{u})=1 \mid s(\mathbf{u}) \in [s_l, s_{l+1}]\}$$

$$= \frac{Prob\{s(\mathbf{u}) \in [s_l, s_{l+1}] \mid I(\mathbf{u})=1\}}{Prob\{s(\mathbf{u}) \in [s_l, s_{l+1}]\}} \cdot Prob\{I(\mathbf{u})=1\}$$

$$= \frac{F(s_{l+1} \mid 1) - F(s_l \mid 1)}{Prob\{s(\mathbf{u}) \in [s_l, s_{l+1}]\}} \cdot p(\mathbf{u})$$

Facies modeling - Bayes' inversion

Requisites

- Prior vertical proportion, $p(d)$
- Likelihood, $F(s\,|\,1)$
- Seismic impedance distribution, $F(s)$

Implementation

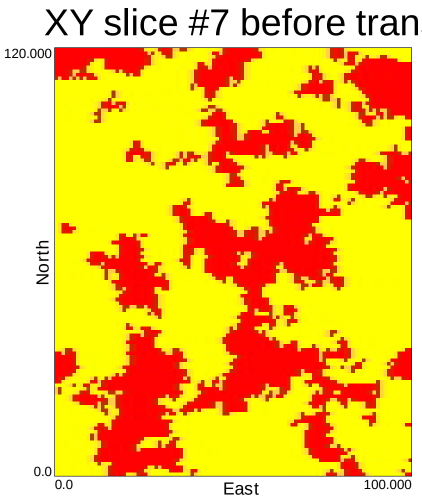
$$p'(\mathbf{u}) = Prob\{\, I(\mathbf{u}) = 1 \mid s(\mathbf{u}) \in [s_l, s_{l+1}] \}$$

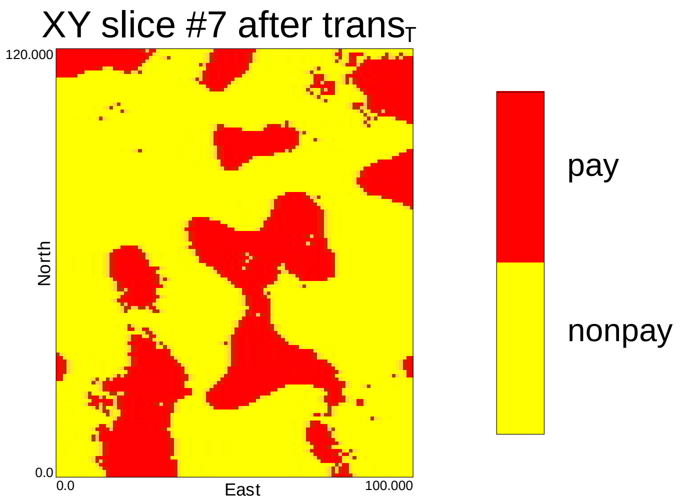
used as locally varying mean after standardization

$$\frac{1}{N_d}\sum_x p'(x,d) = \underbrace{p_V(d)}_{\text{layer d pay proportion}} \quad \text{where} \quad \mathbf{u} = (\mathbf{x}, \mathbf{d})$$

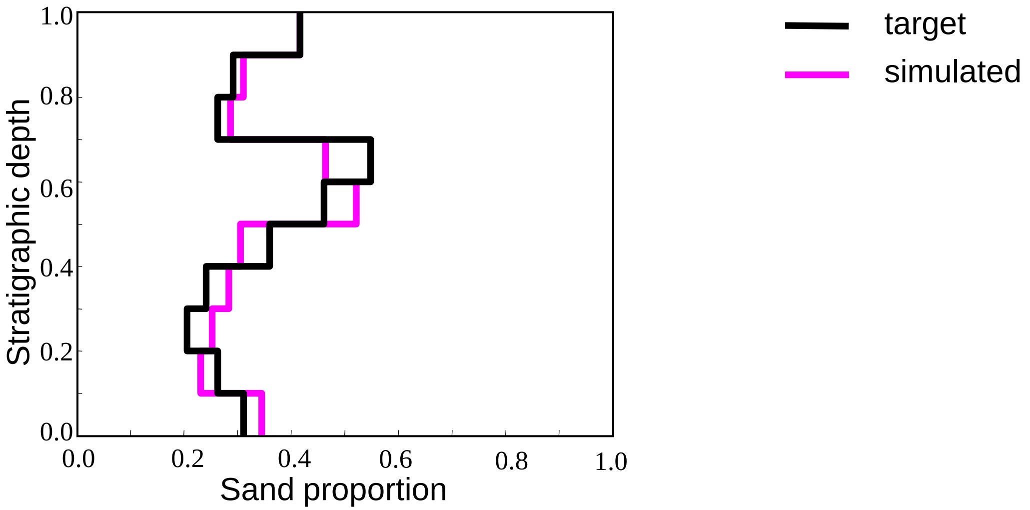
Bayes' inversion - Results

- Slices through facies model





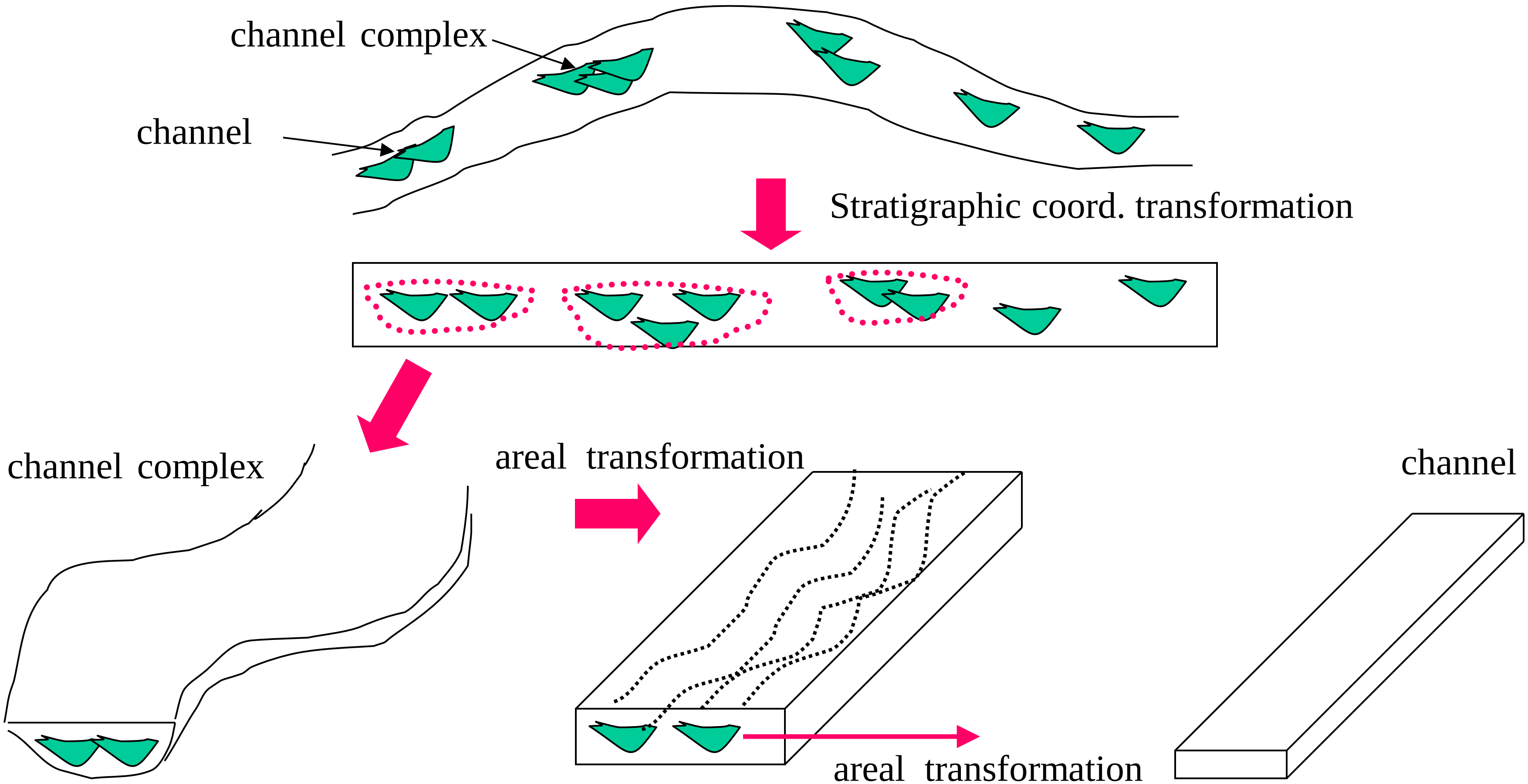
- Vertical proportion



Facies modeling- object based

fluvsim (Deutsch and Wang (1996))

Modeling using reversible coord. transforms



Facies modeling - object based

Simulation procedure in transformed space

– annealing simulation of channel complex

$$O_{cc} = \underbrace{|p_G - p_G^*|}_{\text{global proportion}} + \sum_{n_z} \underbrace{|p_V(d) - p_V(d)^*|}_{\text{vertical proportion}} + \sum_{n} \underbrace{|I(\mathbf{u}_\alpha) - I_{CC}(\mathbf{u}_\alpha)}_{\text{well data}})$$

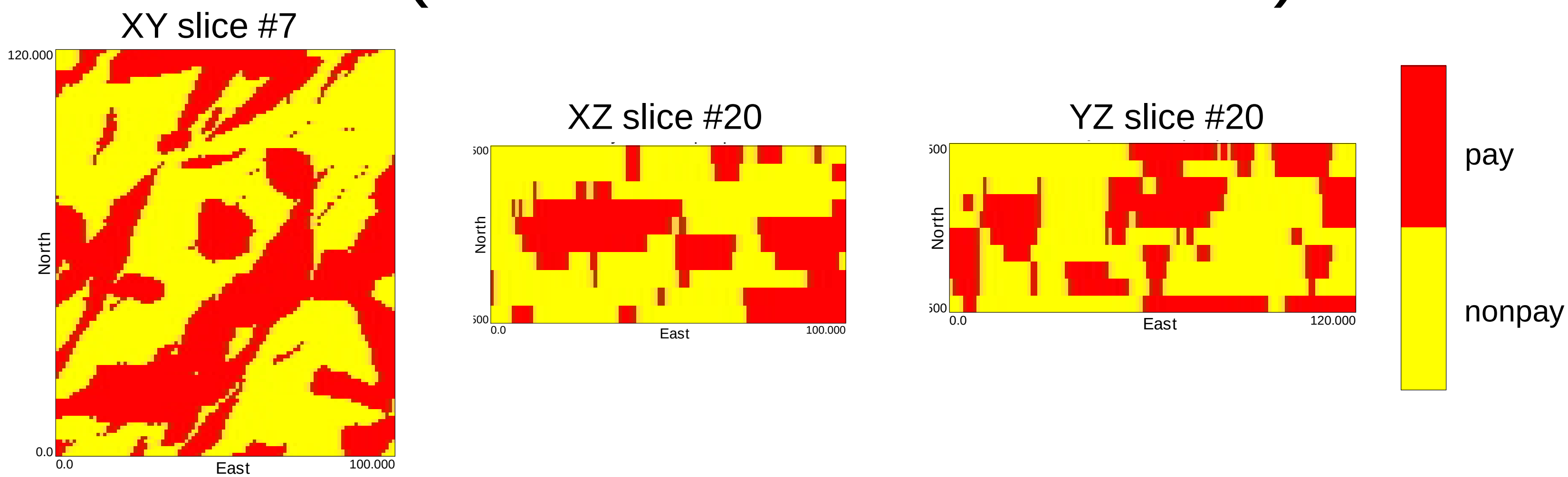
where $I(\mathbf{u}) = 1$ if simulated $\mathbf{u} \in$ channel

– perturb channel complex until convergence

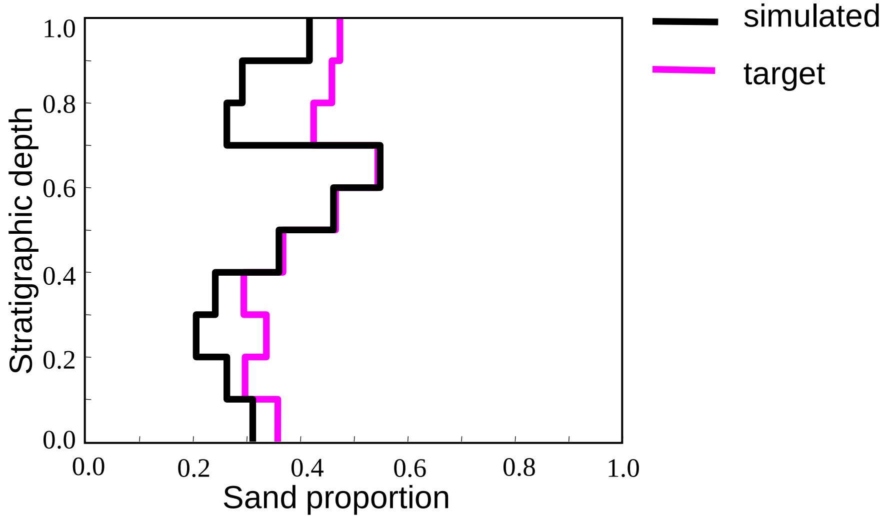
– annealing simulation of channel
– perturb channel until convergence
– back coord. transform to obtain reservoir model

Fluvsim - results

Facies model (well mismatch: 11.2 %)

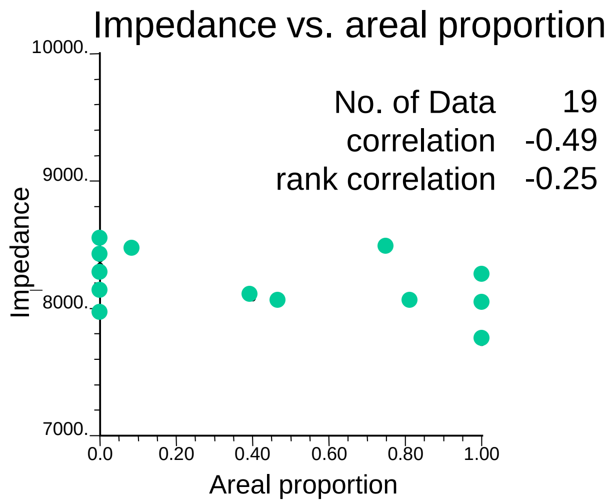


Vertical proportion curve



Fluvsim model - using seismic info.

- Average impedance vs. areal pay proportion

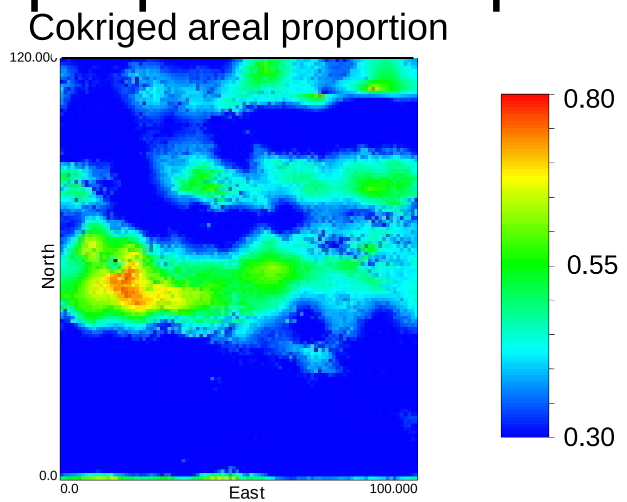


➡ Cokrige areal proportion map using:

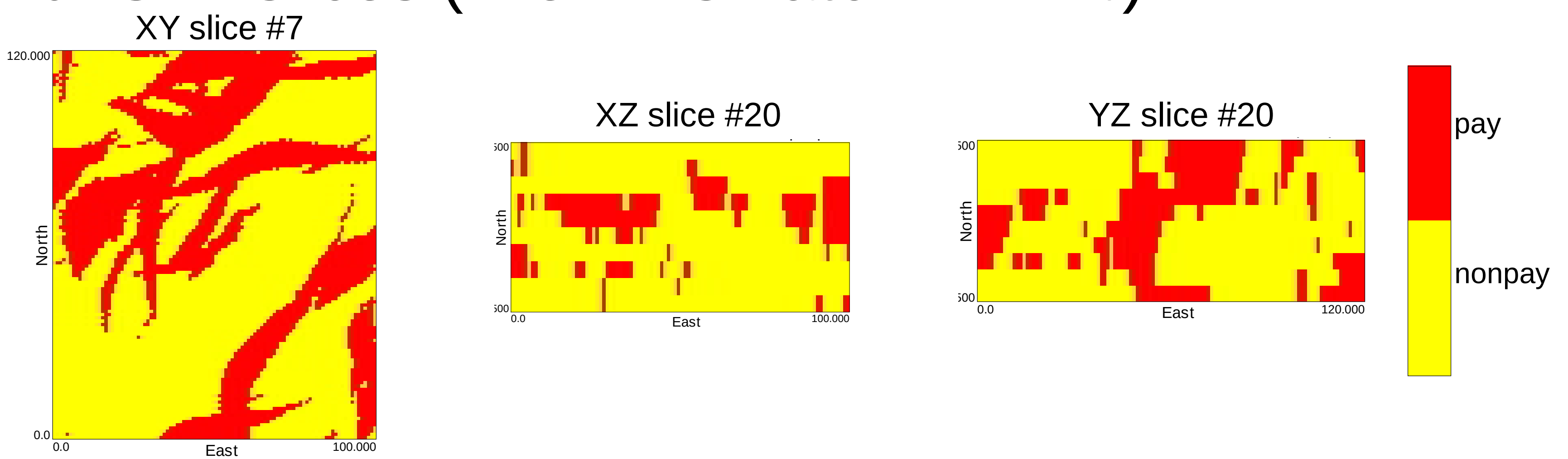
  – vertically averaged proportion data
  – exhaustive 2-D seismic impedance
  – variogram of facies proportion
  – co-located cokriging under MMI

Fluvsim - using seismic info.

- Cokriged areal proportion map

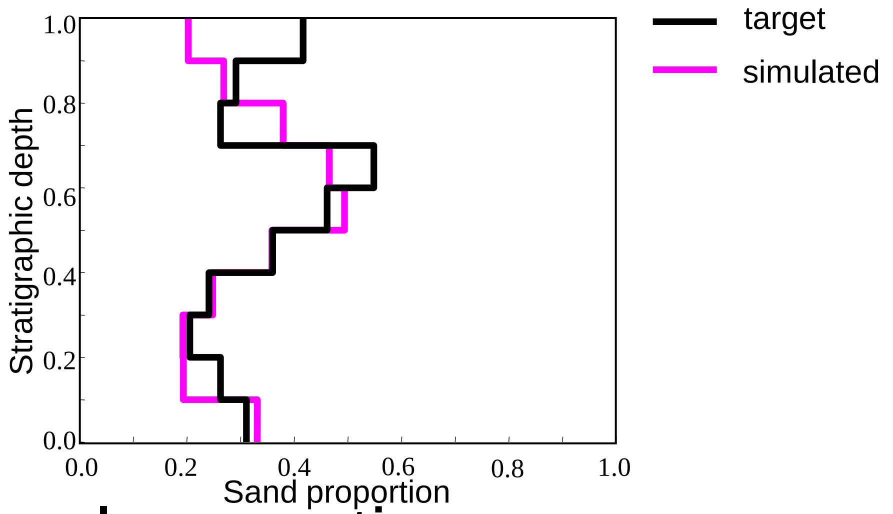


- **fluvsim** slices (well mismatch - 14 %)

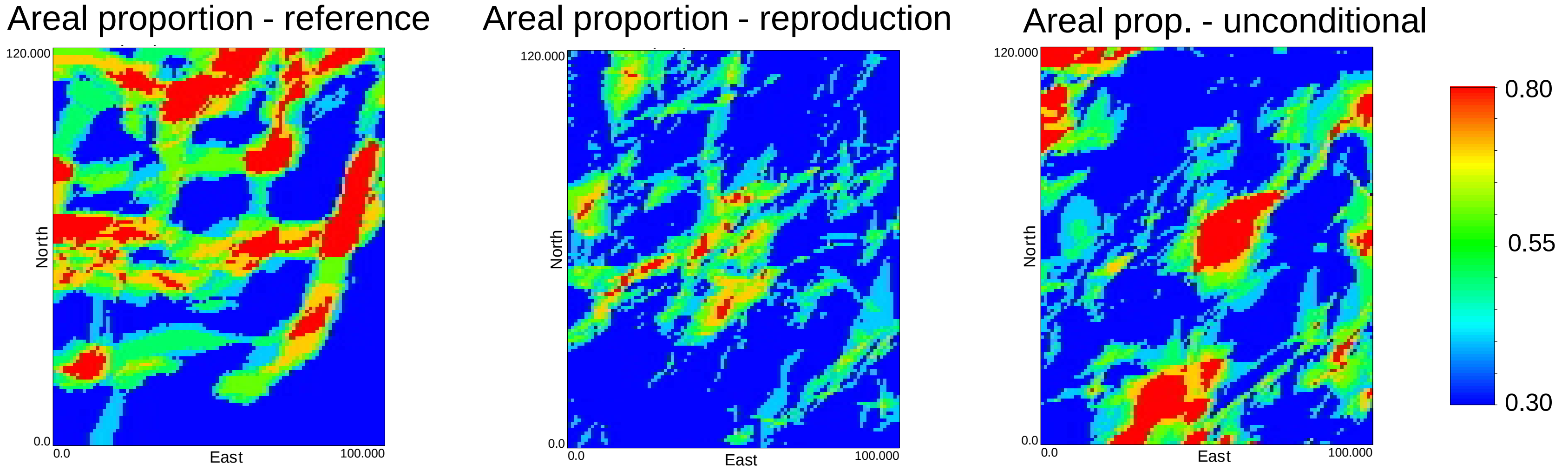


Fluvsim - using seismic info.

- **fluvsim** vertical proportion



- impact of areal proportion



Facies modeling - Some remarks

- Object based models geologically realistic but require numerous shape parameters
- **trans** post-processing of indicator simulations essential
- Integration of seismic has significant impact

Indicator basis function

- Consider the expansion $\varphi(n)$ defined on the basis of *n* indicator random variables : $I_\alpha, \alpha = 1,\ldots,n$

$$\varphi(n) = a + \sum_{j_1=1}^{n} b_{j_1}^{(1)} \cdot I(\mathbf{u}_{j_1}) + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} b_{j_1,j_2}^{(2)} \cdot I(\mathbf{u}_{j_1}) I(\mathbf{u}_{j_2}) + \cdots + b_{j_1,j_2,\cdots,j_n}^{(n)} \prod_{i=1,n} I(\mathbf{u}_i)$$

  - Given the n indicator random variables, this expansion is the most complete possible.
  - There are a total of $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ terms in the above expansion
  - There are 2 outcomes for each RV there are 2^n outcomes possible for the function $\varphi(n)$.
  - These outcomes define the space $\varphi(n)$

Indicator Expansion – Another Interpretation

In general the extended equations are for all *2*^{n} realizations of the *n* data.

If instead, the estimate I_o^* **corresponding to a specific realization of the indicator RVs, say**

$D=\{i_1,i_2,\dots,i_n\}=1$ **, then:** $I_o^*=\varphi(D)=\lambda_o+\lambda_1\cdot D$

Two bases *1,D* – two equations to obtain two unknowns

Unbiasedness: $I_o^*-p_o=\lambda_1\cdot\left(D-E\{D\}\right)$

Orthogonality:

$$\langle I_o-I_o^*,\ D\rangle=\langle I_o-p_o-\lambda_1\cdot\left(D-E\{D\}\right),\ D\rangle=0$$

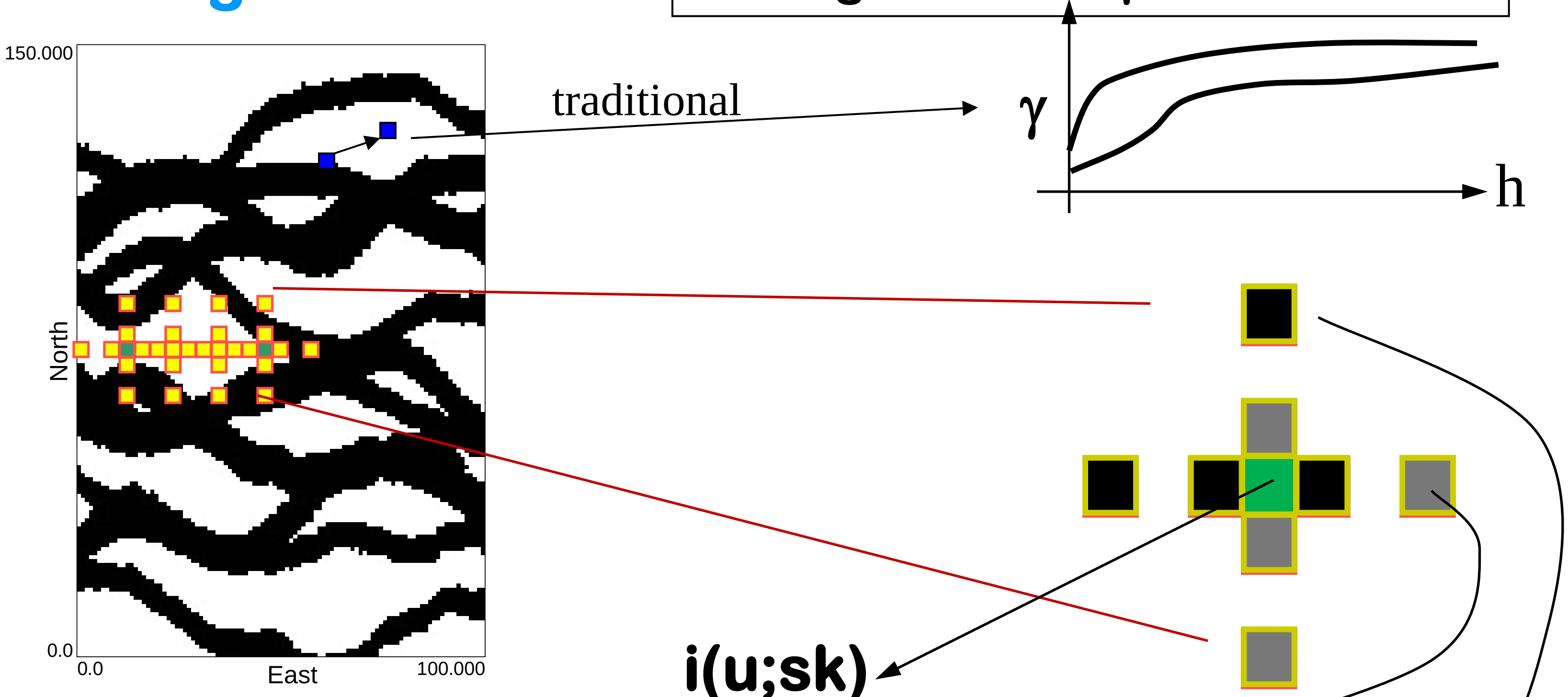
$$\Rightarrow\lambda_1=\frac{E(I_oD)-p_oE(D)}{Var\{D\}}=\frac{E(I_oD)-E(I_o)E(D)}{Var\{D\}}$$

Single Normal Equation

Calibration of geologic information

Geologic model

Variogram – two-point statistics

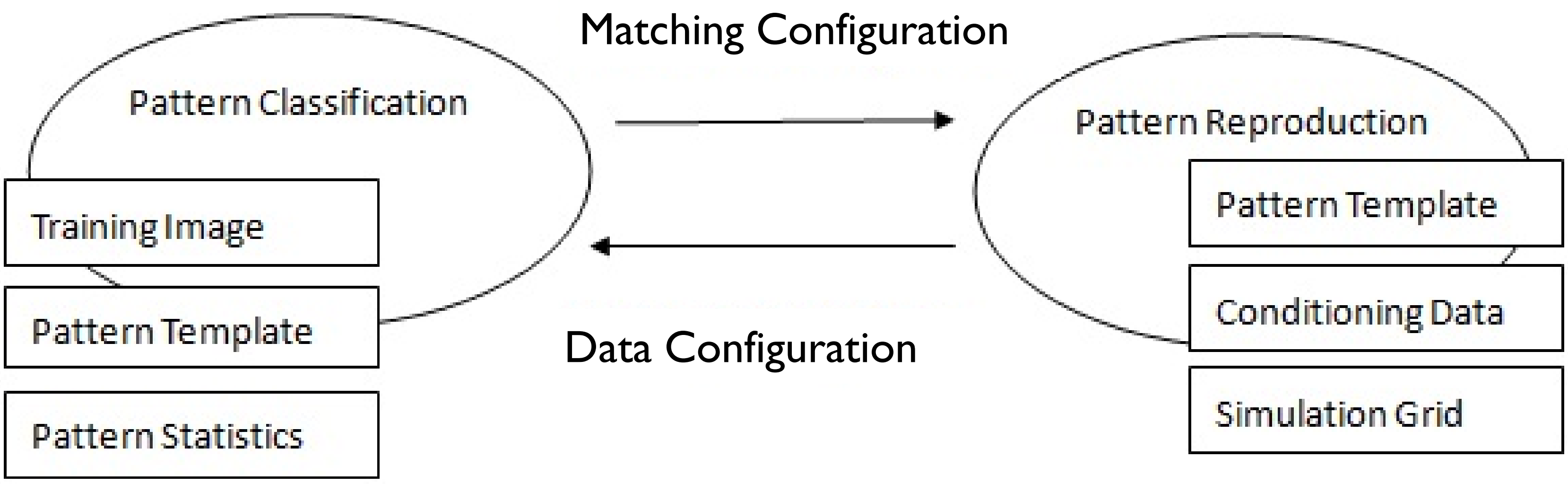


mp statistics $\mathbf{s_t(u) = \{\, s(u+h_\alpha),\ \alpha = 1, \ldots, n_t \,\}}$

Set of data events = training data set

GrowthSim

- components



Pattern Statistics

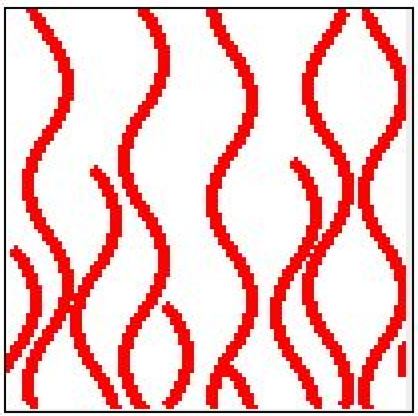
- pattern observation

Training Image

-100x100

-2-category data

Value	0	1
Color	white	red



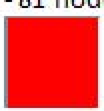
Pattern Configuration Statistics

- total configuration observed: 8464
- distinct configuration counts: 2331
- distribution:

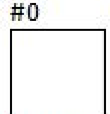
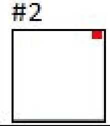
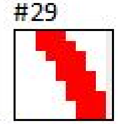
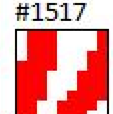
Occurrence	100+	51-100	21-50	11-20	6-10	2-5	1
# Configu-rations	1	3	9	84	346	392	1396

Pattern Template

- 9x9
- 81 nodes



Configurations Observation

Configuration	#0 	#2 	#29 	#1517 
Occurrence	1579	60	15	1

Multipoint Statistics

To condition realization to hard (static) data

Sample from distribution P(A | B)

Geological / Hard information
Well data

Use Bayes' Rule

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

MP Histogram

where A = Simulation pattern
B = Information from conditioning data & geology

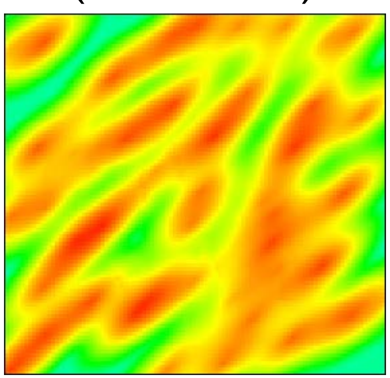
A – single point simulation event	A – multiple point simulation event
Strebelle (2000), Caers (1999)	GROWTHSIM, Arpat (2005)

Pattern Statistics

- optimized template
  - goal
    - reduce noises
    - reduce spurious pattern configurations

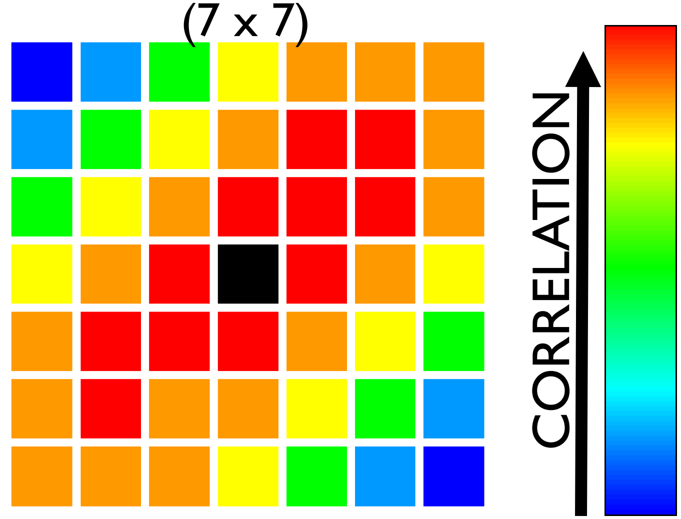
Training Image

(100 x 100)

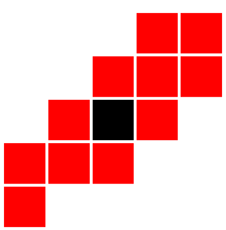


Template Domain

(7 x 7)



Spatial Correlation



Pattern Statistics

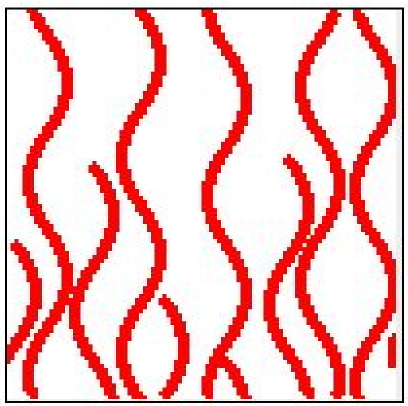
- effects of optimized template

Training Image

-100x100

-2-category data

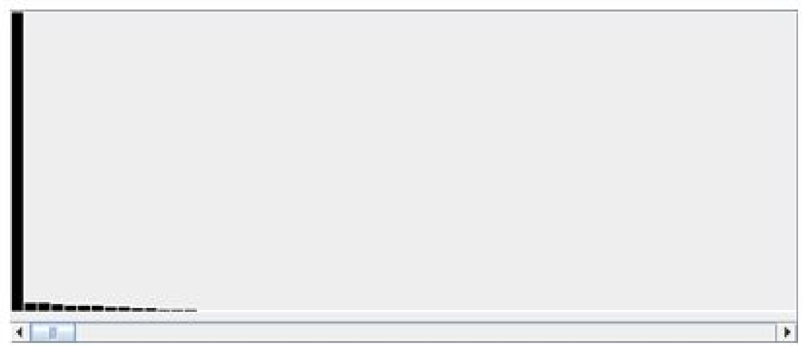
Value	0	1
Color	white	red



Pattern Configuration Statistics

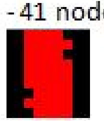
- total configurations: 8464
- distinct configurations: 960
- distribution:

Occurrence	3005	51-100	21-50	11-20	6-10	2-5	1
# Configu-rations	1	8	17	138	200	126	470

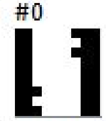
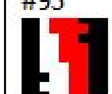


Pattern Template

- 9x9
- 41 nodes



Configurations Observation

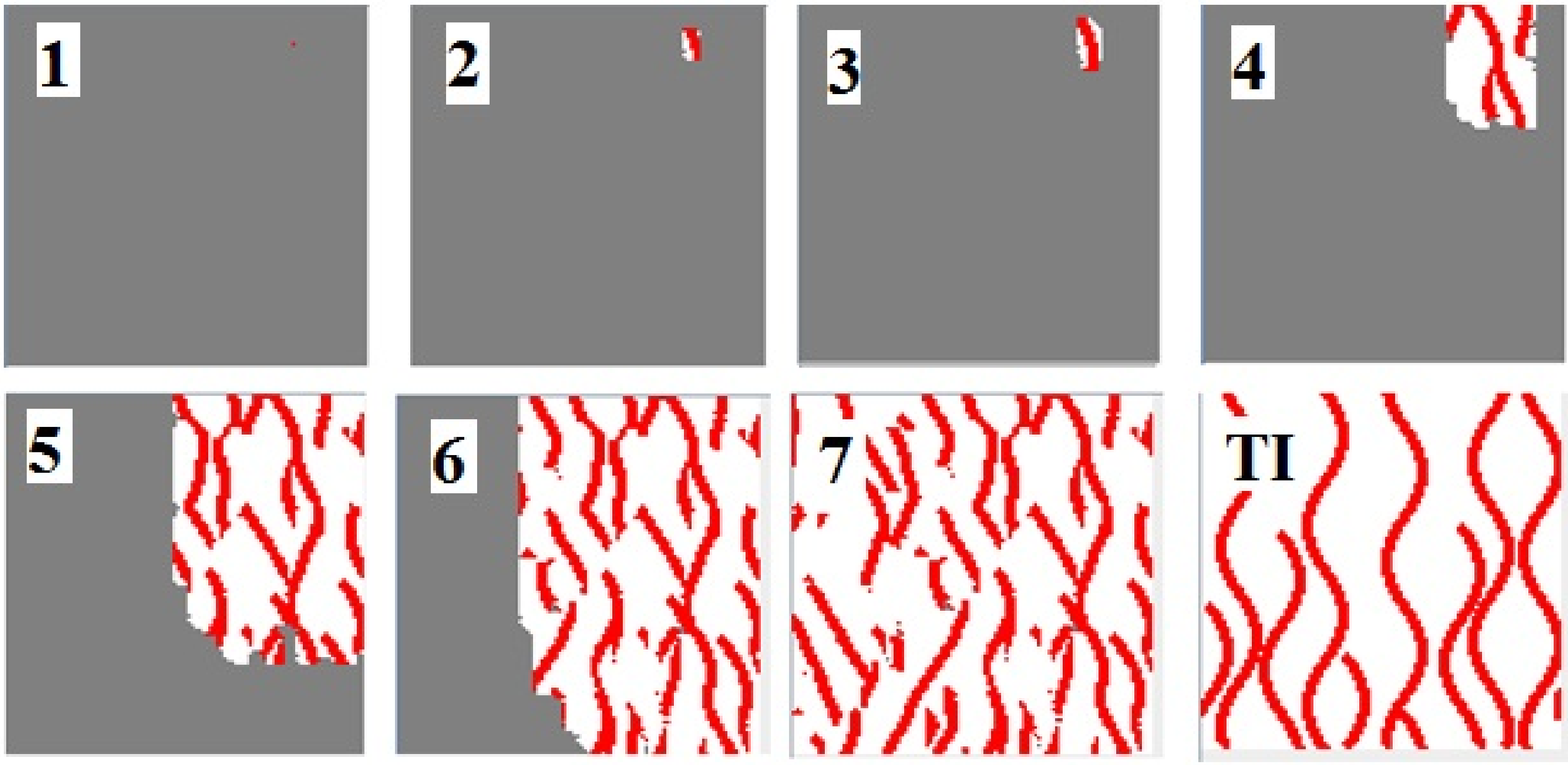
Configuration	#0 	#4 	#95 	#516 
Occurrence	3005	80	12	1

GrowthSim

- algorithm
  1. gather conditioning data around some node in the simulation grid
  2. look for matching pattern configurations to the conditioning data in the statistics
  3. apply one of the matching pattern configuration to the simulation grid
  4. repeat 1-3 around the newly simulated node, until the simulation grid is filled

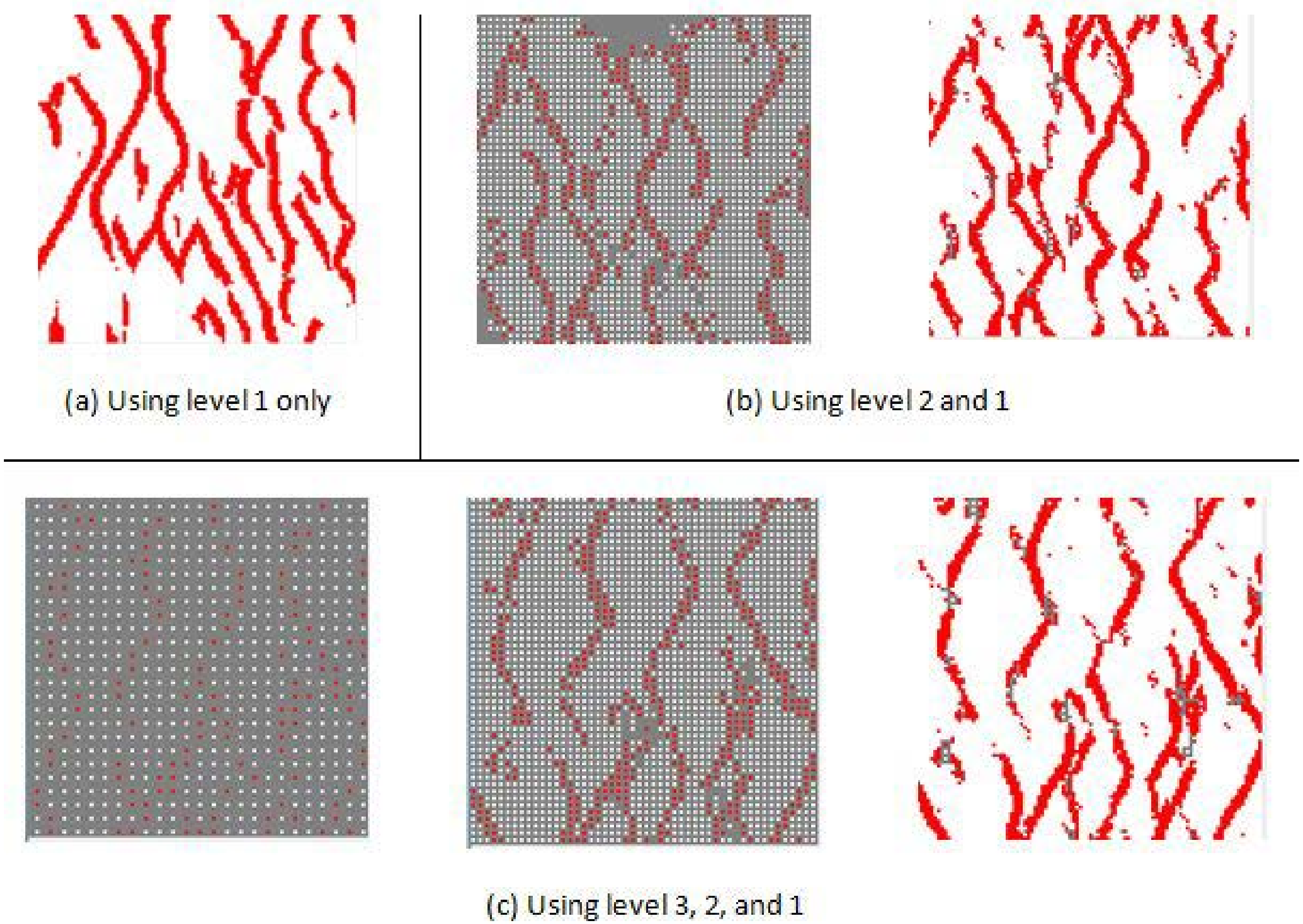
GrowthSim

- 100x100, 2-category data TI



Multiple Grid Simulation

- effects of multiple-level simulation



Calibration of geologic information

An exhaustive training image required for inference of statistics

Noisy spatial covariance (variogram) inferred using sparse data, modeled using smooth functions → restrictive, unrealistic

- Develop a geostatistical simulation method that does not rely on an exhaustive training model

- Perform inference and modeling of spatial connectivity functions in the spectral domain using sparse data.

Moments and Cumulants

Moment Generating Function

$$M[\omega] = E[e^{\omega z}] = \int_{-\infty}^{\infty} e^{\omega z} f_{Z}(z)\, dz$$

Expanding the function $e^{\omega z}$ as a Taylor series about the origin:

$$M[\omega] = E[e^{\omega z}] = E\left[\sum_{r=0}^{\infty} \frac{\omega^r z^r}{r!}\right]$$

$$= \sum_{r=0}^{\infty} \frac{\omega^r E[z^r]}{r!} = \sum_{r=0}^{\infty} \frac{\omega^r \mathrm{Mom}[z, \ldots, z]}{r!}$$

Cumulant generating function of Z is the Neperian logarithm of the moments generating function M:

$$K[\omega] = \ln(E[e^{\omega z}]) = ln\left(E\left[\sum_{r=0}^{\infty} \frac{\omega^r z^r}{r!}\right]\right)$$

$$= \sum_{r=0}^{\infty} \frac{\omega^r \mathrm{Cum}[z, \ldots, z]}{r!}$$

Moments and Cumulants

Relation between the first few moments and cumulants

$$\mathrm{Cum}[z] = \mathrm{Mom}[z]$$

$$\mathrm{Cum}[z,z] = \mathrm{Mom}[z,z] - \mathrm{Mom}[z]^2$$

$$\mathrm{Cum}[z,z,z] = \mathrm{Mom}[z,z,z] - 3\mathrm{Mom}[z,z] \cdot \mathrm{Mom}[z] + 2\mathrm{Mom}[z]^3$$

Moments can be calculated from the cumulants by

$$\mathrm{Mom}[z,z] = \mathrm{Cum}[z,z] + \mathrm{Cum}[z]^2$$

$$\mathrm{Mom}[z,z,z] = \mathrm{Cum}[z,z,z] + 3\mathrm{Cum}[z,z]\mathrm{Mom}[z] + \mathrm{Cum}[z]^3$$

Three-point moment is a measure of similarity between three spatial locations. High value implies three locations jointly have high values of the spatial attribute Z   Spatial cumulant is also a measure of spatial connectivity.

Polyspectra

Assuming $C_{k,z}(\boldsymbol{h_1}, \boldsymbol{h_2}, \dots, \boldsymbol{h_{k-1}})$ is absolutely summable (i.e. $\sum_{\boldsymbol{h_1}=-\infty}^{\infty} \dots \sum_{\boldsymbol{h_{k-1}}=-\infty}^{\infty} \left| C_{k,z}(\boldsymbol{h_1}, \boldsymbol{h_2}, \dots, \boldsymbol{h_{k-1}}) \right|$ exists and is finite), the *k*-th order polyspectrum is defined as the (*k*-1)-dimensional discrete Fourier transform of the *k*-th order cumulant, i.e.,

$$S_{k,z}(\boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \dots, \boldsymbol{\omega_{k-1}}) = \sum_{\boldsymbol{h_1}=-\infty}^{\infty} \dots \sum_{\boldsymbol{h_{k-1}}=-\infty}^{\infty} C_{k,z}(\boldsymbol{h_1}, \boldsymbol{h_2}, \dots, \boldsymbol{h_{k-1}}) \times \exp\left[-j \sum_{i=1}^{k-1} \boldsymbol{\omega_i} \boldsymbol{h_i}\right]$$

$S_{2,z}(\boldsymbol{\omega})$ is the Fourier transform of covariance function $\Rightarrow$ power spectrum

$S_{3,z}(\boldsymbol{\omega_1}, \boldsymbol{\omega_2})$: bispectrum (also noted as $B(\boldsymbol{\omega_1}, \boldsymbol{\omega_2})$)

$S_{4,z}(\boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \boldsymbol{\omega_3})$: trispectrum (also noted as $T(\boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \boldsymbol{\omega_3})$)

Power Spectrum

The power spectrum is the Fourier transform of the covariance function - a measure of the energy of a signal. For a deterministic signal $z(t)$, energy:

$$E = \int_{-\infty}^{\infty} |z(t)|^2 dt = \int_{-\infty}^{\infty} P(\omega) d\omega$$

$P(\omega)$ is the power spectrum of the signal

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-i\omega t} dt$$ and the inverse $$z(t) = \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega$$

$$P(\boldsymbol{\omega}) = Z(\boldsymbol{\omega})\, Z^*(\boldsymbol{\omega}) = |Z(\boldsymbol{\omega})|^2$$

$Z^*(\boldsymbol{\omega})$ -conjugate of Fourier transform $Z(\boldsymbol{\omega})$

For a stochastic process Z(t), energy is defined as

$$E = E\{|Z(t)|^2\} = \int_{-\infty}^{\infty} |Z(t)|^2 dz = \int_{-\infty}^{\infty} P(\omega) d\omega$$

$$P(\boldsymbol{\omega}) = \lim_{T \to \infty} E\{Z_T(\boldsymbol{\omega})\, {Z_T}^*(\boldsymbol{\omega})\} = \lim_{T \to \infty} E\{|Z_T(\boldsymbol{\omega})|\}^2$$

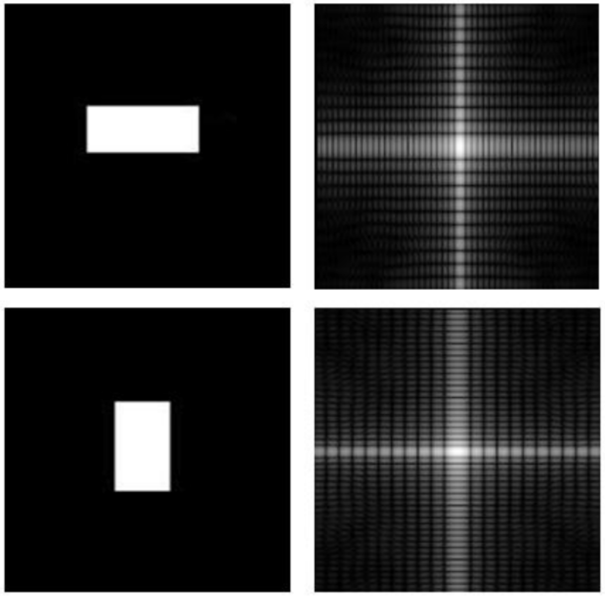
$Z_T(\boldsymbol{\omega})$ - Fourier transform of $z_T(t)$

Power Spectrum – Detecting orientations

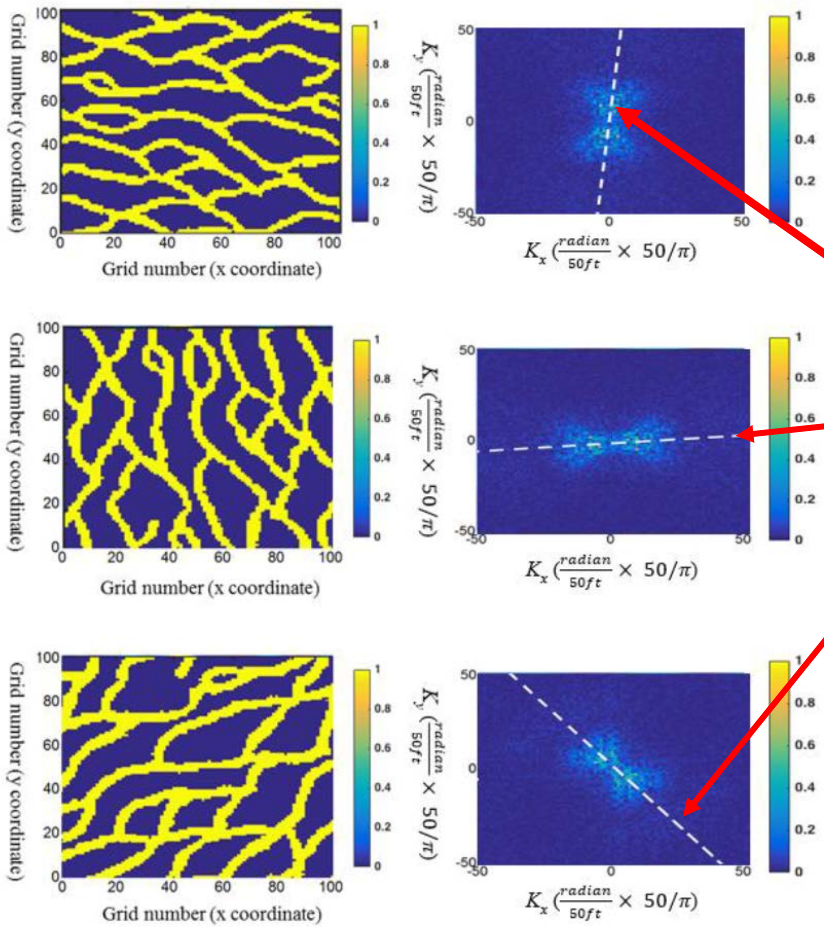
Property of Fourier transform - if an image is rotated, its Fourier transform also rotates

changes in direction of spatial continuity → rotation in Fourier transform space

orientation of objects can be detected from power spectrum



Power Spectrum – Detecting orientations



Orientation calculated using eigenvectors of the inertia matrix computed by thresholding the Fourier transform

Bispectrum

For a deterministic signal $z(t)$: $\int_{-\infty}^{\infty} |z(t)|^3 dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(\omega_1, \omega_2) d\omega_1 d\omega_2$

$$B(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = Z(\boldsymbol{\omega}_1) Z(\boldsymbol{\omega}_2) Z^*(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)$$ Bispectrum

For a stationary random process : $\mathrm{E}\{|z(t)|^3\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(\omega_1, \omega_2) d\omega_1 d\omega_2$

$$B(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \lim_{T \to \infty} E\{Z_T(\boldsymbol{\omega}_1) Z_T(\boldsymbol{\omega}_2) {Z_T}^*(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\}$$ Bispectrum

Power spectrum contribution of each individual frequency component independently,

Bispectrum includes frequency interactions between components $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$

Because it measures interaction between two frequency components – it retains information about the phase of the Fourier transform

Integrated Bispectrum

Phase of Fourier transform is a nonlinear function of frequency, extracted by biphase

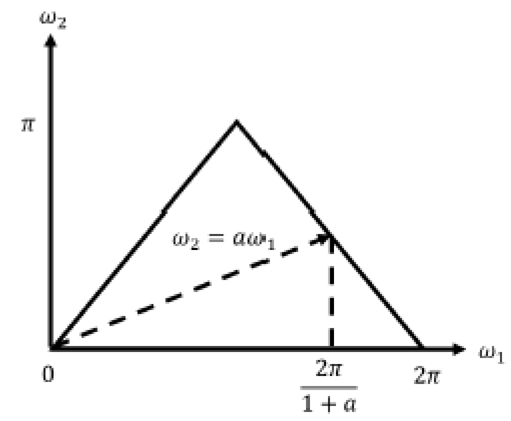
$$\psi(\boldsymbol{\omega_1}, \boldsymbol{\omega_2}) = \phi(\boldsymbol{\omega_1}) + \phi(\boldsymbol{\omega_2}) - \phi(\boldsymbol{\omega_1} + \boldsymbol{\omega_2})$$

Biphase Phase of Fourier spectrum

Need to find a feature identifier from bispectrum that can distinguish object shapes within the images - invariant under translation, scaling, rotation, and amplification

Define the integrated bi-spectrum:

$$I(a) = I_r(a) + i\, I_i(a) = \int_{\boldsymbol{\omega_1}=0}^{2\pi/(1+a)} B(\boldsymbol{\omega_1}, a\boldsymbol{\omega_1})d\boldsymbol{\omega_1} \quad \text{for } 0 < a < 1$$



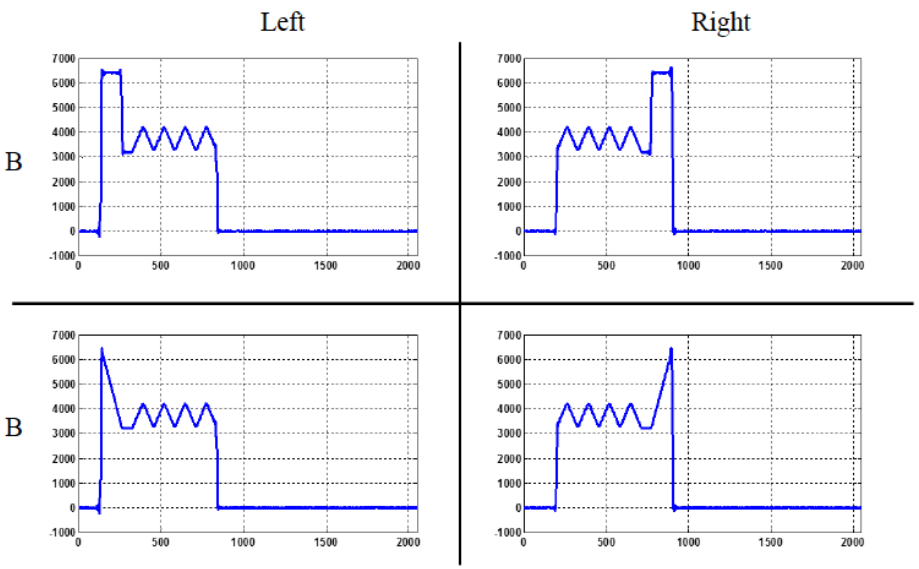
Define phase of integrated bi-spectrum

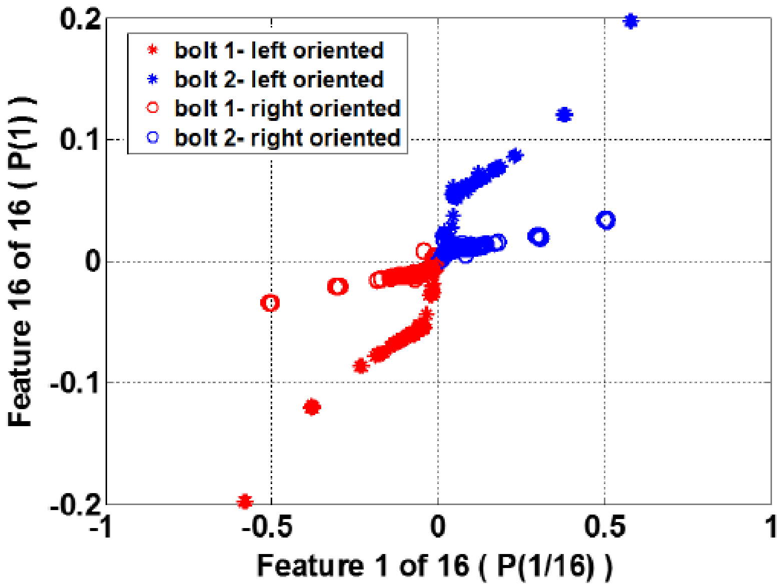
$$P(a) = \arctan\left(\frac{I_i(a)}{I_r(a)}\right)$$

Identifier $P(a)$ is invariant to translation, scaling, rotation and amplification

(Nikias and Raghuveer, 1987 and others)

Detecting Shapes





Bispectrum from sparse data

So far: Higher-order spectra can provide some interesting insights into spatial characteristics of reservoir models

However, FFT requires an exhaustive dataset or image (such as a training image)

What about when only scattered data is available?

Consider the inverse transform:

$$y_{N\times 1} = A_{N\times N}\, x_{N\times 1}$$

x is the Fourier transform coefficient, A is the inverse Fourier transform matrix, and y is the actual image. Because image exhibits spatial correlation, its Fourier transform is sparse.

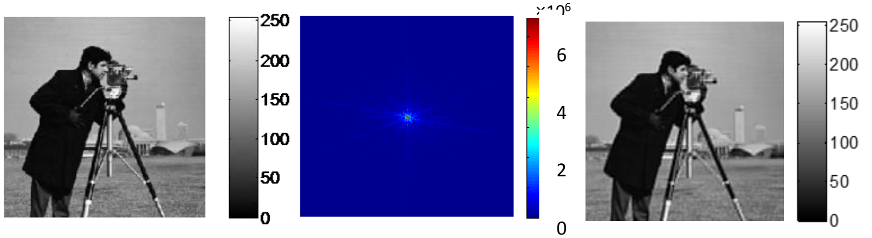


Image reconstructed using only highest 15% of Fourier coefficients

Compressed Sensing

In reservoir modeling sparse observations $y_{K\times 1}$ can be reconstituted as:

$$y_{K\times 1} = A_{K\times N}\, x_{N\times 1}$$

Problem is ill-posed and has multiple solutions. For a unique solution impose sparsity constraint on x

$$\min_{x\in R^N} \|x\|_1 \quad \text{subject to} \quad y_{K\times 1} = A_{K\times N}\, x_{N\times 1}$$

Spatial correlation of geologic models, leads to x having a sparse representation ➡ minimization problem

used for geostatistical simulation (Jafarpour, Goyal, McLaughlin, & Freeman, 2009) assuming a DCT

➡ unique solution – no uncertainty quantification

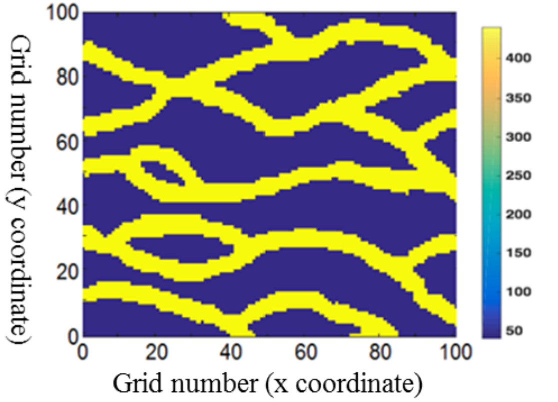
Sparse reconstruction

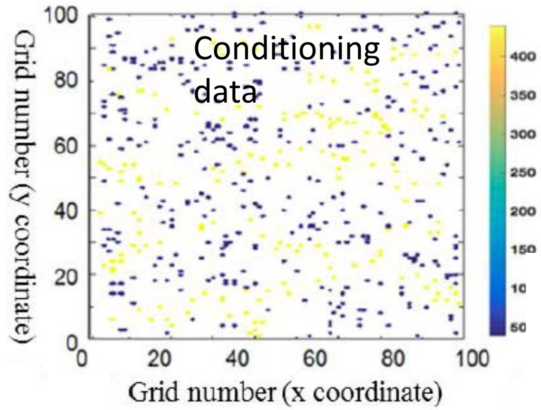
Estimate Fourier transform coefficients from scattered data solving a regularization problem

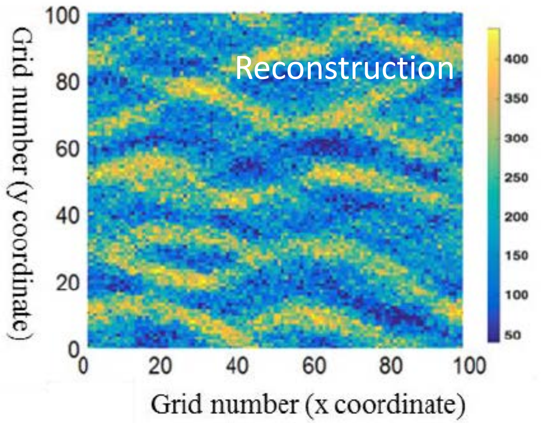
$$\min_{x \in R^N} \|y - Ax\|_p + \lambda \|x\|_q$$ least mixed norm (LMN) [p=2, q=1]

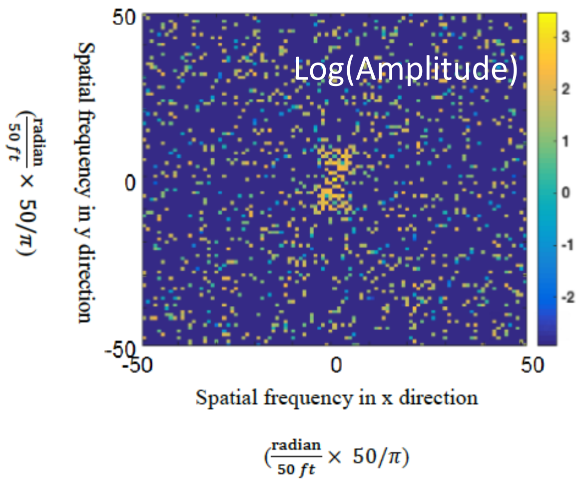
Optimum lambda established using jack-knife

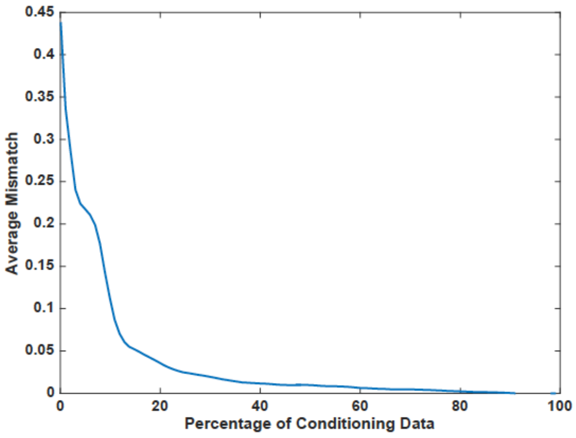
Sparse reconstruction – some results





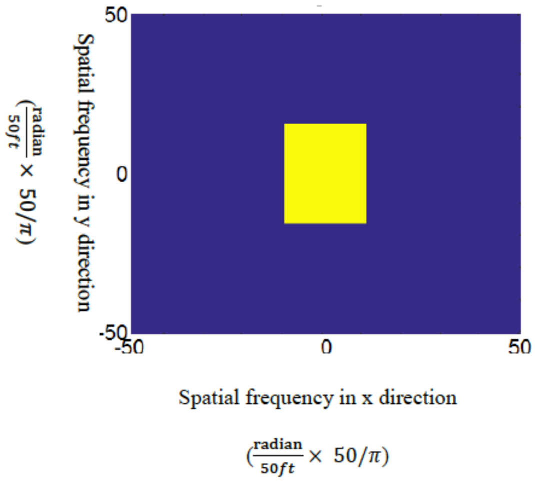


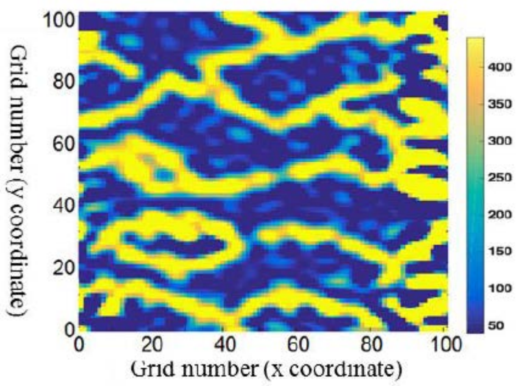


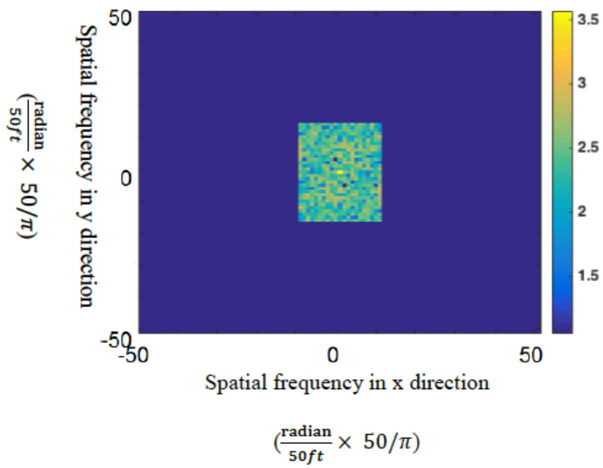


An improvement

One idea to improve quality of reconstruction - limit the search space to low frequency coefficients



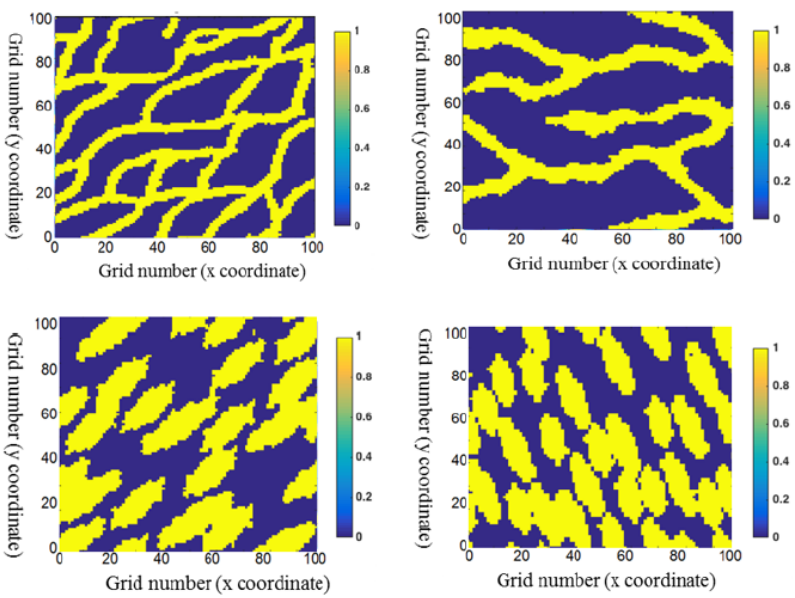




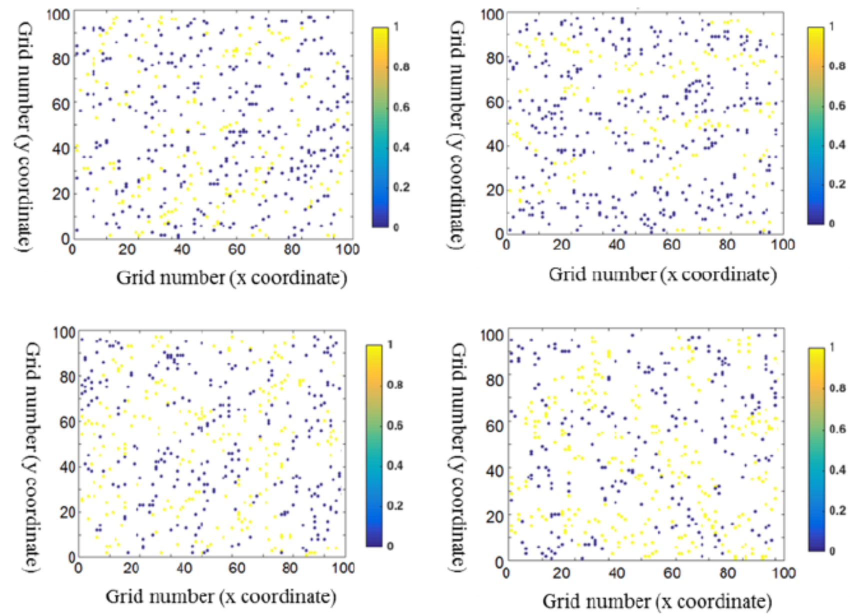
How to automatically select the search space? Apply a jackknife method. Randomly select 10% of the conditioning data (validation data) and use the remaining data for reconstruction

Can we detect shapes using sparse data?

Reference Models

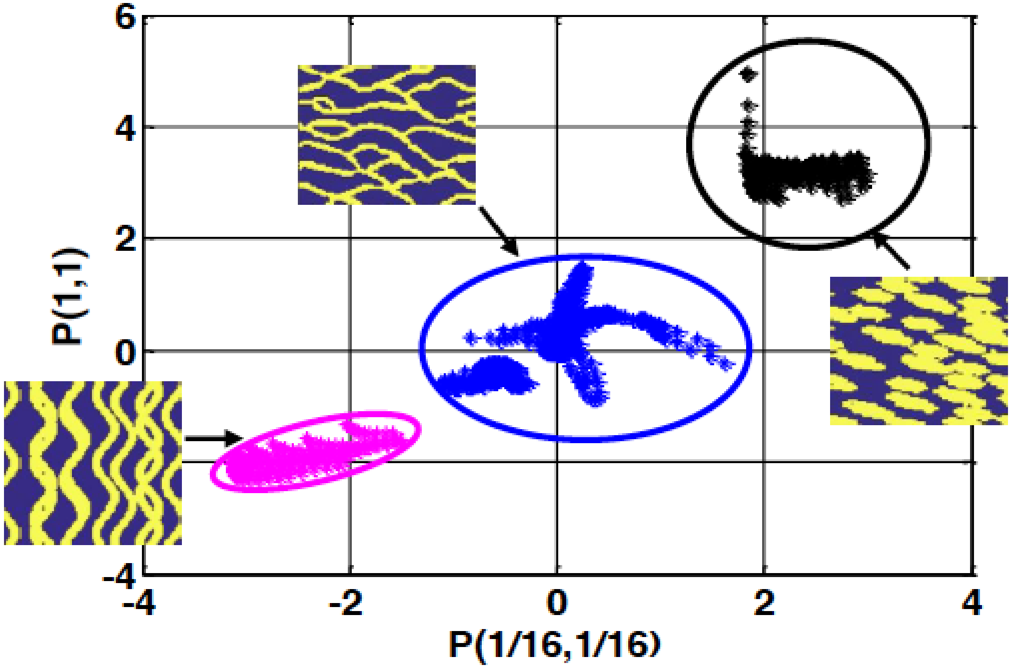


Sparse Conditioning Data



Can we detect shapes using sparse data?

Our goal is to use the integrated bi-spectrum computed using the Fourier transform from sparse data to classify the models reflecting different feature



Conclusions

- Performing inference and modeling of connectivity statistics in the Fourier space provides an avenue for modeling geological realism in a computationally efficient manner
- Computation of higher order moments, cumulant and polyspectra using sparse data appears feasible
- Detection of size, orientation and shapes of features consistent with available conditioning data is an interesting aspect of this research ➡ training image selection
- While a broad idea about reservoir structures is possible using the power spectrum, the location of objects and details of the shapes are only possible by identifying phase ➡ bispectrum, higher-order spectra