GEOMETRIC AND FOURIER METHODS FOR NONLINEAR WAVE EQUATIONS

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ABSTRACT. The subject of nonlinear wave equations has been transformed in the last twenty years by three fundamental developments.

1. The emergence of sophisticated geometric techniques, i.e. techniques wich involve only geometric constructions in the physical space.
2. Systematic application of Littlewood-Paley decompositions and introduction of paradifferential calculus.
3. The development of Fourier spacetime techniques such as Strichartz type inequalities and Bilinear estimates.

The goal of these lectures is to discuss some of the technical issues concerning these advancements and illustrate them through the main new results which they led to. We also plan to discuss some recent developments which require a highly nontrivial cooperation of the geometric and Fourier methods and hint towards a future powerful fusion of these seemingly irreconcilable points of view.

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In the first lecture we shall review the basic estimates of solutions to the standard wave equation in Minkowski space and discuss various geometric and Fourier methods to prove them. In the second lecture we shall discuss applications of these techniques to semilinear equations such as Wave Maps, Maxwell-Klein -Gordon and Yang Mills. Finally, in the last lecture, we shall discuss applications to quasilinear equations related to the Einstein vacuum equations.

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1. Lecture I: Estimates for the standard wave equation

Consider the standard wave equation in Minkowski space $\mathbb{R}^{n+1}$

$$\Box\phi = 0. \tag{1}$$

The canonical, inertial, coordinates in $\mathbb{R}^{n+1}$ are denoted by x^μ, $\mu = 0, 1, \dots, n$ relative to which the Minkowski metric takes the diagonal form $m_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$. We have $x^0 = t$ and $x = (x^1, \dots, x^n)$ denote the spatial coordinates. We shall use throughout the notes the usual summation convention over repetead indices and the standard conventions concerning raising and lowering the indices of vectors and tensors. In particular, if $x_\mu = m_{\mu\nu}x^\nu$, we have $x_0 = -t$ and $x_i = x^i$, $i = 1, \dots, n$. We shall use the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial_t = \frac{\partial}{\partial x^0}$. We denote by Σ_{t_0} the spacelike hyperplanes $t = t_0$. The wave operator is defined by $\Box = m^{\alpha\beta}\partial_{\alpha\beta} = -\partial_t^2 + \sum_i \partial_i^2$.

In what follows we recall the basic known estimates for solutions of (1) which verify the initial value problem at $t = 0$,

$$\phi(0, x) = f(x), \quad \partial_t\phi(0, x) = g(x) \tag{2}$$

For convenience we denote $\phi[0] = (f, D^{-1}g)$ with D^{-1} the pseudodifferential operator with symbol $|\xi|^{-1}$.

Proposition 1.1 (Energy Identity, Inequality)**.** *The solutions* (1), (2) *verify,*

$$E[\partial\phi](t) = E[\partial\phi](0) \tag{3}$$

where

$$E[\partial\phi](t) = \int_{\Sigma_t} (|\partial_t\phi|^2 + \sum_i |\partial_i\phi|^2)dx \tag{4}$$

As a consequence we have the energy inequalities, for all $s \geq 0$,

$$\|\partial\phi(t)\|_{H^s(\mathbb{R}^n)} \leq \| \partial\phi(0) \|_{H^s(\mathbb{R}^n)}$$

Proof The energy identity can be proved both by geometric techniques, involving only integration by parts, and by Fourier techniques, using Plancherel formula together with the Fourier representation formula,

$$\phi(t, x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\big(\cos t|\xi| f\hat{}(\xi) + \frac{\sin t|\xi|}{|\xi|} g\hat{}(\xi)\big)d\xi \tag{5}$$

The higher energy inequalities are based on the commutation between $\Box$ and ∂_μ. ■

Remark 1.2. The standard Sobolev embedding $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, for $s > \frac{n}{2}$ allows us to get L^∞ bounds of solutions to (1) without using the explicit representation. This procedure generalizes to nonlinear equations in connection to the local existence theorem.

Proposition 1.3 (Dispersive inequality)**.** *The solutions to* (1), (2) *verify,*

$$|\phi(t)|_{L^\infty} \leq ct^{-\frac{n-1}{2}} \|D^{\frac{n+1}{2}}\phi[0]\|_{L^1} \tag{6}$$

Remark 1.4. In fact 6 is not quite right, the correct estimate holds if we replace the L^∞ norm on the left by the BMO-norm, or, the L^1 norm on the right by the Hardy norm $\mathcal{H}^1$. The inequality (6) is true however, as it stands, if the Fourier transform of the data $\phi(0) = f$, $\partial_t\phi(0) = g$ have their Fourier transform supported in a dyadic shell $\frac{\lambda}{2} \le |\xi| \le 2\lambda$ for some fixed $\lambda \in 2^{\mathbf{N}}$.

Proof Until very recently the only robust proof of (6) was based on the method of stationary phase applied to the representation (5). In odd dimensions one can prove a related form of the dispersive estimate using the spherical means representation of solutions. We shall later discuss a derivation of (6) which avoids any representation formulas. ■

Remark 1.5. The dispersive inequality provides two types of information. The first concerns the precise decay rate of $\|\phi(t)\|_{L^\infty}$ as $t \to \infty$ while the second provides information about the regularity properties of $\|\phi(t)\|_{L^\infty}$ for $t > 0$. As far as improved regularity is concerned the estimate (6) gains, for $t > 0$, $\frac{n-1}{2}$ derivatives when compared to the Sobolev embedding $L^\infty(\mathbb{R}^n) \subset W^{1,n}(\mathbb{R}^n)$.

It is well known that as far as the asymptotic behavior is concerned (6) is not very useful in applications to nonlinear wave equations. A more effective procedure to derive the asymptotic properties of solutions of the wave equation is based on generalized energy estimates, obtained by the commuting vectorfields method, together with global Sobolev inequalities. In what follows we shall give a very fast presentation of this point of view; we shall develop it in a more systematic manner in the Complement to Lecture I. In what follows we review the commuting vectorfields method for deriving the above decay rate estimate. The idea is to use the energy identity (3) together with commuting vectorfields and a global form of the classical Sobolev inequalities.

The Minkowski space-time $\mathbb{R}^{n+1}$ is equipped with a family of Killing and conformal Killing vector fields, the translations $T_\mu = \partial_\mu$, Lorentz rotations $O_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, scaling $S = t\partial_t + x^i\partial_i$ and the inverted translations $K_\mu = -2x_\mu S + < x, x > \partial_\mu$. Recall that x^μ, denote the standard variables $x^0 = t, x^1, \dots, x^n$, and $x_\mu = m_{\mu\nu}x^\nu$. The Killing vector fields T_μ and $O_{\mu\nu}$ commute with $\Box$ while S preserves the space of solutions in the sense that $\Box\phi = 0$ implies $\Box S\phi = 0$ as $[\Box, S] = 2\Box$. We split the operators $O_{\mu\nu}$ into the angular rotation operators ${}^{(ij)}O = x_i\partial_j - x_j\partial_i$ and the boosts ${}^{(i)}L = x_i\partial_t + t\partial_i$, for $i, j, k = 1, \dots, n$. Recall the energy expression in (4), Based on the commutation properties described above we define the following "generalized energies "

$$E_k[\partial\phi] = \sum_{X_{i_1},..,X_{i_j}} E^2[\partial X_{i_1}X_{i_2}...X_{i_j}\phi] \tag{7}$$

with the sum taken over $0 \le j \le k$ and over all Killing vector fields $T, L_{\mu\nu}$ as well as the scaling vector field S. The crucial point of the commuting vectorfield method is that the quantities E_k, $k \ge 1$ are conserved by solutions to (1). Therefore, if for all $0 \le k \le s$ the data f, g verify,

$$\int (1+|x|)^{2k}\Big(|\nabla^{k+1} f(x)|^2 + |\nabla^k g(x)|^2\Big)dx \le C_s < \infty \tag{8}$$

then for all t, $E_s[\partial\phi](t) \le C_s$. The desired decay estimates of solutions to (1) can now be derived from the following global version of the Sobolev inequalities (see [27]):

Proposition 1.6 (Decay Estimates[27]). *Let ϕ be an arbitrary function in R^{n+1} such that $E_s[\phi]$ is finite for some integer $s > \frac{n}{2} + 1$. Then,*

$$|\partial\phi(t,x)| \lesssim (1+t+|x|)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}} E_s[u] \tag{9}$$

for all $t > 0$. Therefore if the data f, g in (1) satisfy 8 , for $0 \le k \le s$ with some $s > \frac{n}{2}$, then for all $t \ge 0$,

$$|\partial\phi(t,.)|_{L^\infty} \lesssim \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}} \tag{10}$$

Remark 1.7. Clearly this estimate, whose proof is purely geometric[1], implies the decay properties given by the dispersive inequality (6). In fact it provides more information outside the wave zone $|x| \sim t$ which fit very well with the expected propagation properties of the linear equation $\Box\phi = 0$. On the other hand, as (9) is really a global version of the Sobolev inequality, it seems that the estimates of the Proposition 1.6 have no bearing on the improved regularity features of (6). This is however not true as we shall see in the next proposition.

Remark 1.8. The commuting vectorfields([27], see also [46]) method was first developed as an alternate, vastly simplified, proof for the almost global existence result of [17]. The method is very versatile and applies to prove global existence in dimension $n > 3$ and almost global existence in dimension $n = 3$ to large classes of nonlinear wave equations, including fully nonlinear, for sufficiently small initial data. It also combines with the null condition(see discussion below following proposition 1.22) to prove global existence results in low dimensions $n = 3, 2$. The vectorfield method was further developed by many other authors, in particular H. Lindblad[48] and T. Sideris[63, 64, 65], see also Hormander[16] and references therein. A major modification of the vectorfields methods appears in [12], where the vectorfields had to be modified. This will be discussed to some extent in Lecture III.

Proposition 1.9 (see[30]). *The commuting vectorfields method implies the dispersive inequality* (6).

Proof Without loss of generality we may assume that $\partial_t\phi = g = 0$ and that the Fourier transform of $f = \phi(0)$ is supported in the shell $\frac{\lambda}{2} \le |\xi| \le 2\lambda$ for some $\lambda \in 2^{\mathbf{N}}$. By a simple scaling argument we may in fact assume $\lambda = 1$. Since $\hat{\phi}$, the Fourier transform of ϕ relative to the space variables x, is also supported in the same shell it suffices to prove the estimates for $\nabla\phi$ or $\nabla^k\phi$. Next we cover $\mathbb{R}^n$ by an union of discs D_I centered at points $I \in \mathbf{Z}^n$ with integer coordinates such that each D_I intersects at most a finite number c_n of discs D_J with c_n depending only on the

[1] In particular it does not require any explicit representation of solutions

dimension n. Consider a smooth partition of unity $(\chi_I)_{I\in\mathbf{Z}^n}$ with $\operatorname{supp}\chi_I \subset D_I$ and each χ_I positive. Clearly we can arrange to have, for all k,

$$\sum_{I\in\mathbf{Z}^n} |\nabla^k\chi_I(x)| \le C_{k,n} \tag{11}$$

uniformly in $x \in \mathbb{R}^n$. Now set, $f_I = \chi_I \cdot f$, and ϕ_I the corresponding solution to (1) with data $\phi_I(0) = f_I, \partial_t\phi_I(0) = 0$. Clearly $f = \sum_I f_I,\ \phi = \sum_I \phi_I$. It suffices to prove that for all I,

$$\|\nabla^k\phi_I(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \|D^j f_I\|_{L^1} \tag{12}$$

with a constant $C_{n,k}$ depending only on n and k. Indeed if (12) holds true we easily infer that, $\|\nabla^k\phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \|\sum_I \nabla^j\chi_I\|_{L^\infty}\|f\|_{L^1}$ and therefore, in view of (11) $\|\nabla^k\phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}}\|f\|_{L^1}$.

It therefore remains to check (12). Without loss of generality, by performing a space translation, we may assume that $I = 0$. Applying the proposition 1.6 to $\psi = \nabla\phi_0$ we derive, for s_* the first integer strictly larger than $\frac{n}{2}+1$,

$$\begin{aligned}\|\psi(t)\|_{L^\infty} &\le c(1+t)^{-\frac{n-1}{2}}E_{s_*}[\phi_0](t)\\ &\le c(1+t)^{-\frac{n-1}{2}}E_{s_*}[\phi_0](0).\end{aligned}$$

Since the support of ϕ_0 is included in in the ball of radius 1 centered at the origin we have,

$$E_{s_*}[\phi_0](0) \le C_n \sum_{j=0}^{s_*+1} \|D^j f_0\|_{L^2}.$$

Finally, according to the standard Sobolev inequality in $\mathbb{R}^n$, $\|f\|_{L^2} \le c\|\nabla^{\frac{n}{2}} f\|_{L^1}$, we conclude with,

$$\|\psi(t)\|_{L^\infty} \le c(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+2+1} \|D^j f_0\|_{L^1}$$

as desired.

∎

Intepolating between the energy and dispersive inequalities one derives the so called Strichartz-Brenner result,

$$\|\phi(t)\|_{L^r} \le c|t|^{-\gamma(r)}\|\nabla^\sigma\partial\phi(0)\|_{L^{r'}}$$

with $\gamma(r) = (n-1)(\frac{1}{2}-\frac{2}{r})$, $\frac{1}{r}+\frac{1}{r'} = 1$, $r \ge 2$ and scaling condition $\frac{n}{r} = -\gamma(r) - \sigma - 1 + \frac{n}{r'}$. This leads, by a standard TT^* argument, Hardy -Littlewood-Sobolev inequalities and an application of the Littlewood-Paley theory, to the generalized Strichartz type inequalities.

Definition 1.10. A pair of real positive numbers (q,r) is said to be wave admissible if,

$$\frac{2}{q} \le \gamma(r) = (n-1)(\frac{1}{2} - \frac{2}{r}), \qquad q \ge 2, \qquad (q,r,n) \neq (\infty, 1, 3)$$

Proposition 1.11 (Strichartz Type Inequalities). *The solutions to the initial value problem* (1)-(2) *verify,*

$$\|\phi\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^n)} \quad \le \quad c\|\phi[0]\,\|_{H^\sigma}, \qquad \sigma = n(\frac{1}{2} - \frac{2}{r}) - \frac{1}{q} \tag{13}$$

Remark 1.12. Strichartz himself has proved only estimates in the isotropic case $q = r$. The other estimates are due to Pecher, Ginibre-Velo, Lindblad-Sogge and Keel-Tao. For the final version of these estimates, and precise references, we refer to Keel-Tao[20].

Remark 1.13. Strichartz type inequalities play a crucial role in many recent advances of the theory of nonlinear wave equations. Observe that the steps involved in deriving (13), at fixed frequency, from the energy identity and dispersive inequality are quite soft, they can be traced back to the Duhamel's principle and uniqueness of the initial value problem[2]. Both apply to general linear wave equations with variable coefficients and require very little regularity of the coefficients. Thus the main building blocks of the Strichartz type inequalities are (3) and (6).

All estimates discussed so far refer to individual solutions of the wave equation (1). They turn out to be of limited value in application to wave equations which contain derivatives in the nonlinear terms. A new point of view has developed in the last ten years, according to which one obtains far more flexibility by estimating directly bilinear or even multilinear expressions involving multiple solutions of (1).

Proposition 1.14 (Bilinear Strichartz[47]). *Let* ϕ, ψ *be two solutions of* (1). *Assume* $n \ge 2$ *and that the pair* (q,r) *is wave admissible. Then,*

$$\|D^\sigma(\phi\psi)\|L^{q/2}_t L^{r/2}_x \lesssim \|\phi[0]\|_{\dot{H}^{s_1}} \|\psi[0]\|_{\dot{H}^{s_2}} \tag{14}$$

where σ, s_1, s_2 *verify,*

$$0 < \sigma < n - \frac{2n}{r} - \frac{4}{q},$$
$$0 < s_1, s_2 < \frac{n}{2} - \frac{n}{r} - \frac{1}{q},$$
$$s_1 + s_2 + \sigma = n - \frac{2n}{r} - \frac{2}{q}.$$

Proof The proof is based on the following microlocal version of the dispersive inequality,

[2]There exists a straightforward derivation of (13) from (3) and (6) without using explicit representations, see [**?**]. The uniqueness of the I.V.P. is a consequence of the basic energy inequality.

Proposition 1.15 (Improved Dispersive Inequality[47]). *Consider a solution to* (1), (2) *whose initial data have their Fourier transform supported in a ball of small radius μ inluded in the unit dyadic ring $\frac{1}{2} \le |\xi| \le 2$. Then*

$$|\phi(t)|_{L^\infty} \le c\mu\, t^{-\frac{n-1}{2}} \|\phi[0]\,\|_{L^1}. \tag{15}$$

The improved dispersive inequality follows easily from the standard stationary phase method. We don't have yet a satisfactory geometric derivation. ■

Now consider more general bilinear estimates, of the form

$$\|D^\gamma D_+^{\gamma_+} D_-^{\gamma_-}(\phi\psi)\|_{L_t^{q/2} L_x^{r/2}} \lesssim \|\phi[0]\|_{\dot{H}^{s_1}} \|\psi[0]\|_{\dot{H}^{s_2}}, \tag{16}$$

where ϕ and ψ solve (1). Here $D^\gamma, D_+^{\gamma_+}, D_-^{\gamma_-}$ are multipler operators with symbols $|\xi|^\gamma$, $\big||\tau| + |\xi|\big|^{\gamma_+}$ and $\big||\tau| - |\xi|\big|^{\gamma_-}$. In the case $q, r = 4$ all such estimates are known. Special cases of the following theorem have appeared first in [31] and later in [37, 44, 39]. The complete solution was carried out recently by Foschi-Klainerman [14], see also [69].

Proposition 1.16 (Bilinear Estimates[14]). *Let $n \ge 2$ and $\gamma, \gamma_-, \gamma_+, s_1, s_2 \in \mathbb{R}$. The estimate*

$$\|D^\gamma D_+^{\gamma_+} D_-^{\gamma_-}(\phi\psi)\|_{L_t^2 L_x^2} \lesssim \|\phi[0]\|_{\dot{H}^{s_1}} \|\psi[0]\|_{\dot{H}^{s_2}}$$

is satisfied by the solutions of (1) *for all initial data iff the following conditions hold:*

$$\begin{aligned}
\gamma + \gamma_+ + \gamma_- &= s_1 + s_2 - \frac{n-1}{2}, \\
\gamma_- &\ge -\frac{n-3}{4}, \\
\gamma &> -\frac{n-1}{2}, \\
s_i &\le \gamma_- + \frac{n-1}{2}, \quad i = 1, 2, \\
s_1 + s_2 &\ge \frac{1}{2}, \\
(s_i, \gamma_-) &\ne \left(\frac{n+1}{4}, -\frac{n-3}{4}\right), \quad i = 1, 2, \\
(s_1 + s_2, \gamma_-) &\ne \left(\frac{1}{2}, -\frac{n-3}{4}\right).
\end{aligned}$$

These estimates are intimately tied to null quadratic forms.

Definition 1.17. Let ϕ, ψ be two smooth functions on $\mathbb{R}^{n+1}$. We define,

$$\begin{aligned}
Q_0(\phi, \psi) &= m^{\alpha\beta} \partial_\alpha \phi \partial_\beta \psi \\
Q_{ij}(\phi, \psi) &= \partial_i \phi \partial_j \psi - \partial_i \psi \partial_j \phi \\
Q_{0i}(\phi, \psi) &= \partial_i \phi \partial_t \psi - \partial_i \psi \partial_t \phi
\end{aligned}$$

Definition 1.18. If ϕ, ψ be distributions in $\mathcal{S}'(\mathbb{R}^{n+1})$ whose spacetime Fourier transform $\widehat{\phi}, \widehat{\psi}$ are functions. We write $\phi \precsim \psi$ iff $|\widehat{\phi}| \le C\widehat{\psi}$.

Lemma 1.19 (see [44] for references)**.** *The following hold true*

$$Q_0(\phi, \psi) \precsim D_+ D_-(\phi\psi) \tag{17}$$

$$Q_{ij}(\phi, \psi) \precsim D^{\frac{1}{2}} D_-^{\frac{1}{2}} (D^{\frac{1}{2}} \phi D^{\frac{1}{2}} \psi) \tag{18}$$

Remark 1.20. Similar formulas can be derived for Q_{0i}. One can use the Lemma above together with the bilinear estimates of proposition 1.16 to derive nontrivial L^2- spacetime estimates for the null quadratic forms.

Remark 1.21. Though the gain of regularity manifest in the bilinear estimates of proposition 1.16 is a development of the last ten years, the importance of null quadratic forms, for applications to nonlinear equations, was known before in connection with gain of decay. Here is a typical fact.

Proposition 1.22 (Decay for null forms[28])**.** *Let ϕ, ψ be two solutions of* (1) *and assume that their initial data verify the weighted Sobolev norm bounds* (8)*, for sufficiently large s. Then, for any of the null forms Q_0, Q_{ij}, Q_{0i} we have the decay estimates,*

$$\|Q(\phi, \psi)(t)\|_{L^2(\mathbb{R}^n)} \le Ct^{-n} \tag{19}$$

as $t \to \infty$.

The proof is an immediate consequence of the following refinement of the decay estimates of Proposition 1.6.

Proposition 1.23 (Peeling Decay Properties)**.** *Let ϕ be an arbitrary function in R^{n+1} such that $E_s[\phi]$ is finite for some integer $s > \frac{n}{2} + 2$. Let $E_\pm = \partial_r \pm \partial_r$, with $\partial_r = \sum_i \frac{x^i}{|x|} \partial_i$. Consider also*[3]*, at every point of $\mathbb{R}^{n+1}$, vectors $(E_A)_{A=1}^{n-1}$, such that $< E_A E_\pm >= 0$ and $< E_A, E_B >= \delta_{AB}$. Set $|\nabla\!\!\!/\phi|^2 = \sum_{i=1}^{n-1} E_A(\phi)^2$. We have,*

$$\|E_+\phi(t)\|_{L^\infty} \lesssim \big(\frac{1}{t}\big)^{\frac{n+1}{2}}$$
$$\|\nabla\!\!\!/\phi(t)\|_{L^\infty} \lesssim \big(\frac{1}{t}\big)^{\frac{n+1}{2}}$$
$$\|(1+|u|)E_-\phi(t)\|_{L^\infty} \lesssim \big(\frac{1}{t}\big)^{\frac{n-1}{2}}$$

Remark 1.24. The improved decay of null quadratic forms as appears in proposition 1.22 was used, see [28], to get a small data global existence, in $\mathbb{R}^{3+1}$, for a general class of quasilinear wave equations verifying the null condition. These vectorfield techniques used in the proof of the result in [28] have been extended by Sideris in [63] and [64],[65] in applications to Wave Maps and nonlinear elasticity.

So far we have only considered general bilinear estimates of type (16) for $q = r = 4$. For the case of "wave admissible " exponents q and r and γ_- the estimates given by proposition 1.14 are sharp, if $s_1 = s_2$. There exists a general bilinear conjecture

[3] E_+, E forms the canonical null pair in $\mathbb{R}^{n+1}$; together with the orthonormal vectors E_A they form a null frame.

which states that under suitable restrictions on the exponents $\gamma, \gamma_+, \gamma_-, s_1, s_2$ the estimates (16) are true. Some very important cases of this conjecture were proved true by T. Tao [68] using ideas developed by T. Wolff[76] in connection to his recent important work on the cone restriction conjecture. I want however to end this first lecture with one important, very simple, ingredient in these recent developments. This is the idea that a general wave can be decomposed into tubes.

Definition 1.25. We say that a wave ϕ is $\pm$wave if its initial data verify

$$\partial_t \phi(0) = \pm i\sqrt{-\Delta}\phi(0)$$

or in Fourier representation $\partial_t \hat{\phi}(0,\xi) = \pm i|\xi|\hat{\phi}(0,\xi)$. Observe that any wave can be decomposed between into $\pm$waves according to the formula

$$\phi = \phi^+ + \phi^-$$

where the initial data for $\phi^\pm$ is given by $\phi^\pm[0] = P^\pm \phi[0]$,

$$P_\pm \phi[0] := \Big(\frac{\phi(0) \mp i(-\Delta)^{-1/2}\phi_t(0)}{2}, \frac{\phi_t(0) \pm i(-\Delta)^{1/2}\phi(0)}{2}\Big).$$

Remark 1.26. Observe that the solutions ϕ^+, ϕ^- are uniquely defined by their data. In other words we only had to decompose the data of ϕ to define them. They can however be represented by Fourier transform as follows,

$$\phi^\pm(t,x) = \int e^{\pm it|\xi|} e^{ix\cdot\xi} \widehat{P^\pm \phi[0]}(\xi) d\xi. \tag{20}$$

Proposition 1.27. *Consider $\phi = \phi^-$ a minus wave with $\phi^-[0]$ (essentially) supported in the ball $B = B(0,\Lambda_0)$, $\Lambda_0 > 1$ and its Fourier transform supported in the region $\{\xi/ \frac{1}{2} \le \xi_1 \le 2,\ |\xi'| \le \frac{1}{\Lambda_0}\}$ where $|\xi'| = \sqrt{\xi_2^2 + \cdots \xi_n^2}$. For any $t \ge 0$ we consider the region $T^+(t,\Lambda) = \{x/\ |x| \le t + \Lambda_0;\ \big|x_1 - t\big|^2 + |x'|^2 \le \Lambda^2\}$. Then, for all t verifying $\Lambda_0 \le \Lambda \le t \le \Lambda_0^2$ we have:*

$$\|\partial\phi^-(t)\|_{L^2(\Sigma_t \setminus T^+(t,\Lambda))} \le C_N \Big(\frac{\Lambda_0}{\Lambda}\Big)^N E(\phi)^{\frac{1}{2}}$$

Proof The standard proof of the proposition is based on a simple integration by parts in the integral representation formulas (20). One can also give a purely geometric proof based on commuting vectorfields[43]. ■

The proof by vectorfields methods of the preceding proposition combined with the null estimates of proposition 1.22 and the important idea of "induction on scales" pioneered by T. Wolff[76] and T. Tao[68] allows us[4] to give a purely geometric proof of the following bilinear estimate,

Proposition 1.28. *Let ϕ, ψ be two solutions of* (1) *and Q one of the null forms Q_0, Q_{ij} or Q_{0i}. Then,*

$$\|Q(\phi,\psi)\|_{L^2(\mathbb{R}^{n+1})}\| \lesssim \|D\phi[0]\|_{L^2(\mathbb{R}^n)} \|D^{\frac{n+1}{2}}\phi[0]\|_{L^2(\mathbb{R}^n)} \tag{21}$$

[4]This is work on progress in collaboration with I Rodnianski and T. Tao.

This estimate, which is a particular case of Proposition 1.16, was first proved by Machedon and I, see [31], by relying heavily on the spacetime Fourier transform of $Q(\phi,\psi)$.It has been used in the proof of finite energy global existence of solutions to the Yang-Mills[34], Maxwell-Klein-Gordon[33] and critical power Yang- Mills-Higgs[19].

2. Lecture II: Applications to semilinear wave equations

We are interested in the Cauchy problem for important geometric systems such as Wave Maps, Yang Mills and Maxwell Klein-Gordon equations. In a first approximation all these equations can be put in the form

$$\Box\phi = \mathcal{N}(\phi),$$

where $\Box = -\partial_t^2 + \Delta$ is the standard wave operator on $\mathbb{R} \times \mathbb{R}^n$, $\Delta = \sum_1^n \partial_j^2$ is the Laplacian on $\mathbb{R}^n$, $\phi = \phi(t,x)$ takes values in $\mathbb{R}^N$ for some $N \geq 1$ and $\mathcal{N}$ is a nonlinear operator which is local with respect to the time variable(i.e., $\mathcal{N}(\phi)(t,\cdot)$ only depends on $\phi(t',\cdot)$ for t' in any neighborhood of t).

We prescribe Cauchy data on the initial hypersurface $t = 0$:

$$(\phi, \partial_t\phi)_{t=0} = (f, g) \in H^s \times H^{s-1}$$

where $H^s = \{f : (I - \Delta)^{s/2} f \in L^2\}$. For simplicity we write $\phi[0] \in H^s$.

We shall in fact concentrate on systems of the following types:

1. *Wave Maps :*

$$\Box\phi^I + \sum_{J,K} \Gamma^I_{JK}(\phi) Q_0(\phi^J, \phi^K) = 0. \qquad \text{(WM)}$$

   Here, ϕ^I denotes the I-th component function of ϕ, the Γ^I_{JK} are smooth functions from $\mathbb{R}^N$ into $\mathbb{R}$ and Q_0 is the null form

$$Q_0(\phi,\psi) = \sum_{\mu=0}^{n} \partial_\mu\phi\partial^\mu\psi = -\partial_t\phi\partial_t\psi + \sum_{j=1}^{n} \partial_j\phi\partial_j\psi.$$

2. *Yang-Mills Type:*

$$\Box\phi = D^{-1}Q(\phi,\phi) + Q(D^{-1}\phi,\phi), \qquad \text{(“YM”)}$$

   where $D^\alpha = (-\Delta)^{\alpha/2}$ and Q stands for any bilinear operator of the following type: Given vector-valued functions ϕ and ψ, the I-th component function of $Q(\phi,\psi)$ is a linear combination, with constant, real coefficients, of $Q_{ij}(\phi^J, \psi^K)$ for all $1 \leq i < j \leq n$ and all J, K, where Q_{ij} is the null form

$$Q_{ij}(\phi,\psi) = \partial_i\phi\partial_j\psi - \partial_i\phi\partial_j\psi.$$

   (The two Q's on the right hand side of (“YM”) may represent two different such operators.)

3. *Maxwell-Klein-Gordon Type:*

$$\begin{cases} \Box\phi = D^{-1}Q(\psi,\psi), \\ \Box\psi = Q(D^{-1}\phi,\psi), \end{cases} \qquad \text{(“MKG”)}$$

   where $\phi = (\phi^1, \ldots, \phi^{N_1})$, $\psi = (\psi^1, \ldots, \psi^{N_2})$, $N = N_1 + N_2$ and Q has the same meaning as before. Thus (“MKG”) is a special case of (“YM”).

The following theorem summarizes the main well-posedness results known today.

Main Theorem

i. ([35, 44].) *If* $n \geq 2$ *and* $s > s_c = \frac{n}{2}$, *then* (WM) *is locally well-posed for initial data in* $H^s(\mathbb{R}^n)$.

ii. ([72],[73],) *The system* (WM) *is locally well posed for small data in the Besov space* $B_2^{\frac{n}{2},1}(\mathbb{R}^n)$ *for all* $n \geq 2$.

iii. ([70],[41], [71],) *If* $n \geq 5$ *and the target manifold admits a bounded parallelizable structure the wave map system is "weakly well posed" for small data in* $H^{\frac{n}{2}}(\mathbb{R}^n)$. *The same result holds true for* $n \geq 2$ *if the target manifold is a standard sphere* $\mathbf{S}^k$.

iv. ([33], [34], [19]) *If* $n = 3$ *the full MKG and YM systems are globally well poesed for large data in the energy norm,* $H^2(\mathbb{R}^n)$. *The result is also true for the Yang-Mills -Higgs system with critical power nonlinearities.*

v. ([13], [21].) *If* $n = 3$ *the full MKG system is locally well posed if* $s > \frac{3}{4}$, *and globally well posed if* $s > \frac{7}{8}$.

vi. ([39], [47])*If* $n = 4$ *the reduced MKG and YM type systems are locally well posed for* $s > s_c = 1$.

2.1. **Motivation for the Main Theorem.** Consider the system

$$\Box u = F(u, \partial u), \tag{22}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{R}^N$, $\partial u = (\partial_t u, \partial_1 u, \ldots, \partial_n u)$ and F is a smooth $\mathbb{R}^N$-valued function satisfying $F(0) = 0$. For this equation one has the following standard existence and uniqueness result:

Theorem 2.2 (Classical Local Existence)**.** *Equation* (22) *is locally well-posed for initial data in* $H^s \times H^{s-1}(\mathbb{R}^n)$ *for all* $s > \frac{n}{2} + 1$. *This means that for any initial data* $u[0] \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$, *there exists a* $T > 0$ *and unique solution*

$$u \in C^0([0, T], H^s(\mathbb{R}^n)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^n)).$$

Moreover the solution depends continuously(in fact analytically) on the data.

Remark 2.3. The proof of this very classical result is based only on simple energy estimates, the $L^\infty \subset H^s$ Sobolev embedding, and some simple interpolation inequalities. Moreover the proof extends without major difficulties to very general classes of evolution equations including quasilinear systems of wave equations.

Remark 2.4. The result is far from being sharp insofar as the regularity assumption on the initial data is concerned. The issue of optimal well posednes is of fundamental importance and has generated a large amount of work in the last 10 years.

To understand better the issue of optimal local well-posedness, in the context of our examples (wave maps, Maxwell-Klein-Gordon and Yang-Mills equations), we need to define the critical well-posedness (henceforth abbreviated WP) exponent s_c. All our equations have a natural scaling associated to them, and s_c is the unique value of s for which the $\dot{H}^s \times \dot{H}^{s-1}$-norm of the initial data is invariant under this scaling. For example, if u solves (WM), then so does

$$u_\lambda(t,x) = u(\lambda t, \lambda x),$$

for any $\lambda > 0$. Since $\|u_\lambda(t)\|_{H^s} = \lambda^{\frac{n}{2}-s}\|u(\lambda t)\|_{H^s}$, the critical WP exponent for (WM) is $s_c = \frac{n}{2}$.

The same principle works for (MKG), (YM). In fact, they both have critical WP exponent $s_c = \frac{n-2}{2}$.

With this definition we formulate the following, see [29]:

General WP Conjecture

1. *For all basic field theories the initial value problem is locally well posed for initial data in H^s, $s > s_c$.*
2. *The basic field theories are weakly*[5] *globally well-posed for all initial data with small $H^{s_c} \times H^{s_c-1}$-norm.*
3. *The basic field theories are ill posed for initial data in $H^s \times H^{s-1}$, $s < s_c$.*

To prove local posedness for $s > s_c$ one proceeds by Picard iteration in a suitable Banach space. Consider the Cauchy problem

$$\Box u = \mathcal{N}(u), \qquad (u, \partial_t u)_{t=0} = (f, g)$$

The 0-th iterate u_0 is just the homogeneous part of the solution:

$$\Box u_0 = 0, \qquad (u, \partial_t u)_{t=0} = (f, g).$$

The subsequent iterates are given inductively by

$$u_{j+1} = u_0 + \Box^{-1}\mathcal{N}(u_j)$$

for $j \geq 0$, where $\Box^{-1}$ is the operator which to any sufficiently regular F assigns the solution v of $\Box v = F$ with $(v, \partial_t v)_{t=0} = 0$.

If we are to prove existence of a local solution of $\Box u = \mathcal{N}(u)$ with initial data in $H^s \times H^{s-1}$ by iteration, we must be able to prove that the iterates remain in the data space:

$$f \in H^s, \quad g \in H^{s-1} \implies u_j(t) \in H^s, \quad \partial_t u_j(t) \in H^{s-1} \tag{23}$$

[5]The solutions may fail to depend smoothly (analytically) on the data.

for all $j \geq 0$ and all t in some interval $(0,T)$. For $j=0$, (23) is trivial, but the case $j=1$ already offers valuable insights.

Definition 2.5. We will say that the first iterate is *WP for initial data in H^s* if (23) holds for $j=1$ and all $(f,g) \in H^s \times H^{s-1}$.

Example 1. Consider the model problem

$$\Box u = (\partial_t u)^2,$$

where u is real-valued. This equation has the same scaling properties as (WM), hence the WP-exponent is $s_c = \frac{n}{2}$. We want to find the lower bound for the set of s such that the first iterate u_1 is WP for initial data in H^s. A simple calculation involving Duhamel's principle, shows that this reduces to proving an estimate of the type

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(\xi,\eta) f(\xi) g(\eta) h(\xi+\eta)\, d\xi\, d\eta \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2} \tag{24}$$

for all $f,g,h \in L^2(\mathbb{R}^n)$, where

$$K(\xi,\eta) = \frac{<\xi+\eta>^{s-1}}{<\xi>^{s-1}<\eta>^{s-1}\big(1+\Delta_{\pm}(\xi,\eta)\big)} \tag{25}$$

$$\Delta_+ = |\xi| + |\eta| - |\xi+\eta|, \qquad \Delta_- = |\xi+\eta| - \big||\xi| - |\eta|\big|. \tag{26}$$

Here we use the notation $<\cdot>= 1 + |\cdot|$.

One can prove the following result concerning integral estimates of the type

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(\xi) g(\eta) h(\xi+\eta)}{<\xi>^a <\eta>^b \big(1+\Delta_{\pm}(\xi,\eta)\big)^c}\, d\xi\, d\eta \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}, \tag{27}$$

where $\Delta_{\pm}$ are given by (26).

Proposition 2.6. *Let $a,b,c \geq 0$. Then (27) holds if $a+b+c > \frac{n}{2}$ and $c < \frac{n-1}{4}$.*

Remark 2.7. It should be remarked that the estimate fails if $a+b+c < \frac{n}{2}$ or $c \geq \frac{n-1}{4}$, based on simple concentration type counterexamples.

Applying proposition 2.6 to the kernel (25) we need

$$s - 1 + \min\left(1, \frac{n-1}{4}\right) > \frac{n}{2},$$

i.e., $s > \max\left(\frac{n}{2}, \frac{n+5}{4}\right)$.

Thus, for the model equation $\Box u = (\partial_t u)^2$ in dimension $n=3$, the above example shows that the first iterate is WP for initial data in H^s if $s > 2 = s_c + \frac{1}{2}$; in fact, one can show that this fails to be true if $s \leq 2$. This should be compared to the counterexamples of Lindblad [49] in dimension $n=3$, which show that there are equations of the type $\Box u = q(\partial u)$, where q is a quadratic form on $\mathbb{R}^4$, which are ill posed for data in H^2. However, if the quadratic form q is of null form type, one can go almost all the way to the critical WP-exponent s_c. The next two examples verify this at the level of the first iterate.

Example 2. Consider the equation[6]

$$\Box u = Q_0(u,u),$$

where u is real-valued. Again the question of WP of the first iterate leads to the problem of proving an estimate of the type (24), but because of the special null structure of the operator Q_0, the singular factors $\Delta_\pm$ cancel out completely from the denominator of the kernel. In fact, K is given by

$$K(\xi,\eta) = <\xi>^{-s} + <\eta>^{-s},$$

so by Proposition 2.6, the first iterate is WP for data in H^s, $s > s_c = \frac{n}{2}$.

Example 3. Consider the equation

$$\Box u = Q(u,u),$$

where u is vector-valued and $Q(u,u)$ is a vector whose I-th component is a linear combination of $Q_{ij}(u^J, u^K)$ for all i, j, J and K. As in the preceding example, there is a cancellation due to the null strucure of Q_{ij}, but in this case we only get rid of half a power of $\Delta_\pm$. In fact, K is now given by

$$K(\xi,\eta) = \left(<\xi>^{-s+\frac{1}{2}} + <\eta>^{-s+\frac{1}{2}}\right)\left(1+\Delta_\pm(\xi,\eta)\right)^{-\frac{1}{2}}$$

so the first iterate is WP for data in H^s, $s > \max\left(\frac{n}{2}, \frac{n+3}{4}\right)$.

By an obvious modification, if we consider instead the equation

$$\Box u = Q(D^{-1}u, u),$$

we find that the first iterate is WP for data in H^s, $s > \max\left(\frac{n-2}{2}, \frac{n-1}{4}\right)$.

Higher Iterates Once the analysis of the first iterate is completed the really hard work starts. In order to show that the iterate u_{k+1} verfies (23) it does not suffice[7] to use (23) for all previous u_j, $j \leq k$. We need a much stronger functional space intimately tied to the Strichartz and bilinear estimates described in the first lecture.

Notation The Fourier transform of a tempered distribution u in $\mathbb{R}^{n+1}$ is denoted by $\mathcal{F}u$ or $\widehat{u}$, in any space-dimension. In frequency space we use coordinates $(\tau,\xi) = \Xi = (\Xi^0,\ldots,\Xi^1)$, where $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ correspond to the time variable t and the space variable x respectively. The Lorentzian inner product on $\mathbb{R}^{1+n}$ is

$$<\Xi;\widetilde{\Xi}> = \sum_{\mu=0}^{n} \Xi_\mu \widetilde{\Xi}^\mu = -\Xi^0\widetilde{\Xi}^0 + \sum_{j=1}^{n} \Xi^j \widetilde{\Xi}^j,$$

and the symbol of the wave operator $\Box$ is $-<\Xi;\Xi> = \tau^2 - |\xi|^2$. By $|\Xi|$ we always mean the Euclidean norm $|\Xi|^2 = \tau^2 + |\xi|^2$.

[6]The equation below can in fact be trivially solved and analyzed, see the first page in the introduction of [31].

[7]Except in the proof of the classical local existence result for $s > s_c + 1$.

Definition 2.8. Let Λ^α, Λ_+^α and Λ_-^α be the multipliers given by

$$\begin{aligned}\widehat{\Lambda^\alpha f}(\xi) &= \left(1+|\xi|^2\right)^{\alpha/2}\widehat{f}(\xi),\\ \widehat{\Lambda_+^\alpha u}(\Xi) &= \left(1+|\Xi|^2\right)^{\alpha/2}\widehat{u}(\Xi),\\ \widehat{\Lambda_-^\alpha u}(\Xi) &= \left(1+\frac{<\Xi;\Xi>^2}{1+|\Xi|^2}\right)^{\alpha/2}\widehat{u}(\Xi).\end{aligned}$$

We define the space $H^{s,\theta}(\mathbb{R}^{n+1})$, which is adapted to the wave operator on $\mathbb{R}^{1+n}$ in the same way that $H^s(\mathbb{R}^n)$ is adapted to the Laplacian on $\mathbb{R}^n$.

Definition 2.9. For $s,\theta\in\mathbb{R}$, define

$$H^{s,\theta}=\left\{u\in\mathcal{S}'(\mathbb{R}^{n+1}):\Lambda^s\Lambda_-^\theta u\in L^2(\mathbb{R}^{n+1})\right\}$$

with norm $\|u\|_{s,\theta}=\|\Lambda^s\Lambda_-^\theta u\|_{L^2}$.

There is a remarkably simple connection between $H^{s,\theta}$ and the space of solutions of the homogeneous wave equation with data in H^s. In effect, every $u\in H^{s,\theta}$ is of the form

$$u(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{it\lambda}u_\lambda(t)}{(1+|\lambda|)^\theta}d\lambda \qquad (H^s\text{-valued integral}) \tag{28}$$

where $\{u_\lambda\}_{\lambda\in\mathbb{R}}$ is a one-parameter family of solutions of (1) with data in H^s; i.e., $\Box u_\lambda=0$ and $(u_\lambda,\partial_t u_\lambda)_{t=0}=(f_\lambda,0)$, where $\lambda\mapsto f_\lambda$ belongs to $L^2(\mathbb{R},H^s)$. Moreover, $\|u\|_{s,\theta}^2=\int\|f_\lambda\|_s^2\,d\lambda$.

An important consequence of (28) is the following:

Principle *A linear or multilinear space-time estimate for solutions of the homogeneous wave equation with data in H^s implies a corresponding estimate for elements of $H^{s,\theta}$, $\theta>\frac{1}{2}$.*

The spaces $H^{s,\theta}$ provide a rough framework for proving the Main Theorem. In fact they are applicable only to prove part i) of the Main Theorem. All further advancements have required the introduction of substantial refinements of these spaces. We refer the interested reader to our survey [45] for how the $H^{s,\theta}$ spaces, and their modifications, together with the bilinear and Strichartz estimates of the first lecture and the principle mentioned above are used to prove the well posedness results with $s>s_c$ of the Main Theorem. The case $s=s_c$ was entirely open until the recent work of T. Tao which will be described in the next section.

2.10. **Wave Maps with critical regularity.** In what follos I will discuss the breakthrough results on wave maps with critical regularity obtained recently by T.Tao. His results illustrate in a powerful way the the interaction between geometric ideas, in physical space, and Fourier analysis techniques such as paradifferential calculus, Strichartz and bilinear estimates. It thus fits very well within the framework of these lectures

His results apply to wave maps $\phi : \mathbb{R}^{n+1} \to S^N$ with S^N the unit sphere in the euclidian space $\mathbb{R}^{N+1}$, for any dimensions $n \geq 2$. The case of dimension $n \geq 5$ is a lot easier, however, allowing one to concentrate on Tao's main new idea of gauge renormalization. I will present in fact an extension of Tao's $n \geq 5$ result, due to Rodnianski and I[41], to a general class of target manifolds.

Let $\phi : \mathbb{R}^{n+1} \longrightarrow (\mathcal{N}, h)$ with $(\mathcal{N}, h)$ a Riemannian manifold of dimension N. Here $\mathbb{R}^{n+1}$ denotes the standard Minkowski space endowed with the metric $m = \text{diag}(-1, 1, \dots, 1)$. We denote by ∇ the Levi-Civitta connection on $T\mathcal{N}$, the tangent bundle of $\mathcal{N}$. In local coordinates $y^I, I = 1, \dots, N$ on $\mathcal{N}$ the wave maps equation takes the familiar form

$$\partial^\alpha \partial_\alpha \phi^I + \Gamma^I_{JK} \partial_\beta \phi^J \partial_\gamma \phi^K m^{\beta\gamma} = 0. \tag{29}$$

where Γ^I_{JK} are the Christoffel coefficients of the Levi-Civitta connection ∇ on $\mathcal{N}$ and $\phi^I, I = 1, \dots, N$ the components of the map ϕ in local coordinates on $\mathcal{N}$. Let $e_a = e^I_a \frac{\partial}{\partial y^I}$ be an orthonormal frame of vectorfields and $\omega^a = \omega^a_I dy^I$ be the corresponding dual basis of 1-forms $\omega^a(e_b) = \delta^a_b$. Since $h(e_a, e_b) = \delta_{ab}$ we infer that $h_{IJ} = \sum_a \omega^a_I \omega^a_J$. Define,

$$\phi^a_\alpha = \omega^a_I \partial_\alpha \phi^I \tag{30}$$

where ϕ^I are the components of the map ϕ relative to the local coordinates y^I on $\mathcal{N}$. Clearly, $\partial_\alpha \phi^I = e^I_a \phi^a_\alpha$. Given a function F on $\mathcal{N}$ we write $\partial_\alpha F(\phi) = \frac{\partial}{\partial y^I} F(\phi) \partial_\alpha \phi^I = \frac{\partial}{\partial y^I} F(\phi) e^I_d \phi^d_\alpha = F_{,d}(\phi) \phi^d_\alpha$ where $F_{,d} = e_d(F)$.

We easily check that the functions $\phi^a_\alpha = < \partial_\alpha \phi, e_a >$ associated to a wave map ϕ verify[8] the following divergence-curl system,

$$\partial_\beta \phi^a_\alpha - \partial_\alpha \phi^a_\beta = C^a_{bc}\, \phi^b_\alpha\, \phi^c_\beta \tag{31}$$

$$\partial^\alpha \phi^a_\alpha = -\Gamma^a_{bc}\, \phi^b_\beta\, \phi^c_\gamma\, m^{\beta\gamma} \tag{32}$$

where, C^a_{bc} and Γ^a_{bc} are respectively the structure and connection coefficients of the frame,

$$\begin{aligned} [e_b, e_c] &= C^a_{bc} e_a \\ \nabla_{e_b} e_c &= \Gamma^a_{bc} e_a \end{aligned}$$

In view of the formula $[e_b, e_c] = \nabla_{e_b} e_c - \nabla_{e_c} e_b$ we infer that,

$$C^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}.$$

Since the frame e_a is orthonormal, $\Gamma^a_{bc} = < \nabla_{e_b} e_c\,;\, e_a > = - < e_c\,; \nabla_{e_b} e_a >$ and therefore

$$\Gamma^a_{bc} = -\Gamma^c_{ba}. \tag{33}$$

Also,

$$\Gamma^a_{bc} = \frac{1}{2}\Big(C^a_{bc} + C^b_{ac} + C^c_{ab} \Big).$$

[8] Our description of wave maps expressed relative to an ortonormal frame follows closely that of [11]. A similar formalism has been used earlier by Helein [15] in his well known work on 2-dimensional weak harmonic maps

Definition 2.11. We say that a Riemannian manifold $\mathcal{N}$ has a " bounded parallelizable" structure if there exists an orthonormal frame $(e_a)_{a=1}^N$ on $\mathcal{N}$ relative to which the structure coefficients C_{bc}^a and their frame derivatives $C_{bc\,,\,d_1 d_2 \ldots d_k}^a$ are uniformly bounded on $\mathcal{N}$.

Remark 2.12. There are plenty of examples of bounded parallelizable manifolds. To start with on any Lie group we can construct an orthonormal basis of left invariant vectorfields e_a relative to which the structure constants C_{bc}^a are constant[9]. The constant negative curvature manifolds $\mathbf{H}^N$, $N > 2$ are bounded parallelizable. Moreover $\mathbf{H}^2$, i.e. the hyperbolic plane, is a Lie group, see the relevant discussion in section 3.1 of [11]. In addition any compact Riemannian manifold can be embedded as a totally geodesic submanifold in a bounded parallelizable Riemannian manifold, see [11].

Proposition 2.13. *Let $\mathcal{N}$ be a Riemannian manifold and $\phi : \mathbb{R}^{n+1} \longrightarrow \mathcal{N}$ a wave map. The 1-forms $\phi_\alpha^a =< \partial_\alpha \phi, e_a >$ verify the equations,* (31),(32) *as well as the system of wave equations,*

$$\Box \Phi = -2R_\mu \cdot \partial^\mu \Phi + E \tag{34}$$

with $\Phi = (\phi_\alpha^a)$, $R_\mu = (R_{b\mu}^a)_{a,b=1}^N$ and $R_{b\mu}^a = \Gamma_{cb}^a \phi_\mu^c$. The components of $E = (E_\alpha^a)$ are homogeneous polynomial of degree three relative to the components of $\Phi = (\phi_\alpha^a)$ with coefficients depending only on the structure functions C_{bc}^a and their derivatives $C_{bc,d}^a$ with respect to the frame.

Remark 2.14. It is essential to remark that the matrices R_μ are antisymmetric i.e.

$$R_{b\mu}^a = -R_{a\mu}^b \tag{35}$$

This is an immediate consequence of (**??**). This shows that the well known "Helein trick" of antisymmetrizing the form of the wave maps equations in the particular case when $\mathcal{N}$ is a standard sphere, a trick which plays a fundamental role in Tao's work for the standard sphere [70], [71], is due in fact to a general feature of the connection coefficients on *any Riemannian manifold*, expressed relative to *orthonormal frames.*

We study the evolution of wave maps subject to the initial value problem

$$\phi(0) = \varphi, \qquad \partial_t \phi(0) = \psi = \psi_0^a e_a \tag{36}$$

φ is an arbitrary smooth map defined from $\mathbb{R}^n$ with values in $\mathcal{N}$ and $\psi = \psi_0^a e_a$ and arbitrary smooth map from $\mathbb{R}^{n+1}$ to TN. Let $\varphi_i^a =< \partial_i \varphi, e_a >$.

Definition 2.15. We shall say that the initial data $\phi[0] = (\varphi, \psi)$ belongs to the Sobolev space $\dot{H}^s(\mathbb{R}^n)$, resp. $H^s(\mathbb{R}^n)$, if all components φ_i^a, ψ_i^a belong to the space $\dot{H}^{s-1}(\mathbb{R}^n)$, resp. $H^{s-1}(\mathbb{R}^n)$. We write

$$\|\phi[0]\|_{\dot{H}^s} = \sum_{a,i} \left(\|\varphi_i^a\|_{\dot{H}^{s-1}} + \|\psi_i^a\|_{\dot{H}^{s-1}} \right)$$

and similarly for $\|\phi[0]\|_{H^s}$.

We are now ready to state our theorem, see[41].

[9] We refer to these as "constant parallelizable".

Theorem 2.16. *Let $\mathcal{N}$ be a Riemannian manifold endowed with a bounded parallelizable structure. Assume $n \geq 5$ and that the initial data $\phi[0] = (\varphi, \psi = \psi_i^a e_a)$ is in H^s for some $\frac{n}{2} < s$. We make also the critical smallness assumption:*

$$\|\phi[0]\|_{\dot{H}^{\frac{n}{2}}} \leq \varepsilon$$

Then the wave map ϕ with initial data $\phi[0]$ can be uniquely continued in H^s norm globally in time.

Remark 2.17. The theorem provides an extension of the result in [70] from the case when the target manifold is a standard sphere to that of bounded, parallelizable manifolds. The restriction on the dimension, $n \geq 5$, is the same as in [70]; this allows us to rely only on Strichartz estimates. The dimensional restriction, for the case of the standard sphere, was removed in [71] with the additional help of bilinear estimates[10], $H^{s,\theta}$ spaces, and the refined methods of [73]. Even in light of [71] the extension of our result to two dimensions does not seem to be straightforward[11].

The proof of the Main Theorem relies on a local well-posedness result in H^s, $s > \frac{n}{2}$. We state the precise result below:

Theorem 2.18. *Assume that the initial data $\phi[0] \in H^s(\mathbb{R}^n)$ for some $s \geq s_0 > \frac{n}{2}$. There exists a $T > 0$, depending only on the size of $\|\phi[0]\|_{H^{s_0}}$, and a unique solution ϕ of the system (31) ,(32) defined on the slab $[0,T] \times \mathbb{R}^n$ verifying,*

$$\|\phi[t]\|_{H^s} \leq C\|\phi[0]\|_{H^s}$$

for all $t \in [0,T]$ and C a constant depending only on T and s, $s_0 - \frac{n}{2}$ and n.

Proof We use the Littlewood-Paley notation of [70]. Thus, for a function $\phi(t,x)$ we denote the projections $P_k\phi(t,x) = \int e^{ix\cdot\xi}\chi(2^{-k}\xi)\hat{\phi}(t,\xi)d\xi$ where $\hat{\phi}(t,\xi)$ is the space Fourier transform of ϕ and $\chi(\xi) = \eta(\xi) - \eta(2\xi)$ with η a non-negative smooth bump function supported on $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. Therefore $\chi(\xi)$ is supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and $\sum_{k\in\mathbf{Z}} \chi(2^{-k}\xi) = 1$ for all $\xi \neq 0$. We also define $P_{\leq k} = \sum_{l \leq k} P_l$. Also for any interval $I \subset \mathbf{Z}$ we define P_I in an obvious fashion.

Notation: *We shall frequently use the notation $A \lesssim B$ to denote $A \leq cB$ for some constant $c > 0$ which does not depend on any of the important parameters used in our estimates.*

Following [70] we introduce the notation

$$\|\Phi\|_{S_k} = \sup_{q,r\in\mathcal{A}} 2^{k(\frac{1}{q}+\frac{n}{r}-1)}\left(\|\Phi\|_{L_t^q L_x^r} + 2^{-k}\|\partial_t\Phi\|_{L_t^q L_x^r}\right) \tag{37}$$

[10] These were used to take advantage of the presence of the special null quadratic form $Q_0(u,v) = m^{\alpha\beta}\partial_\alpha u\partial_b v$ in the special expression of wave maps to the standard sphere used in [70], [71].

[11] This is due to the fact that one needs to treat other other types of null quadratic forms than Q_0. Also, the fact that we use a wave equation, see (34), in Φ corresponding to the first derivatives of the map rather than the map itself, adds additional complications.

where $\mathcal{A} = \{(q,r)/2 \le q, r \le \infty,\ \frac{1}{q} + \frac{n-1}{2r} \le \frac{n-1}{4}\}$ is the set of admissible Strichartz exponents. Recall that,

Proposition 2.19 (Strichartz type estimates)**.** *For any fixed integer k and $\phi(t,x)$ a function on $\mathbb{R}\times\mathbb{R}^n$ such that the support of $\hat{\phi}(t,\xi)$ is included in the dyadic region $2^{k-1} \le |\xi| \le 2^{k+1}$ we have the estimate,*

$$\|\phi\|_{S_k} \lesssim \|\phi(0)\|_{H^{\frac{n-2}{2}}} + \|\partial_t\phi(0)\|_{H^{\frac{n-4}{2}}} + 2^{k\frac{n-4}{2}}\|\Box\phi\|_{L^1_t L^2_x}.$$

In what follows we recall the definition of frequency envelope given in [70].

Definition 2.20. A frequency envelope is an l^2 sequence $c = (c_k)_{k\in\mathbf{Z}}$ verifying

$$c_k \lesssim 2^{\sigma|k-k'|}c_{k'} \tag{38}$$

for all $k, k' \in \mathbf{Z}$. Here σ is a fixed positive constant; as in [70] we take $0 < \sigma < \frac{1}{2}$. In addition we need $0 < \sigma < \frac{n-4}{4}$ and $0 < \sigma < \frac{n-3}{4(n-1)}$.

We say that the $\dot{H}^s$ norm of a function f on $\mathbb{R}^n$ lies underneath an envelope c if, for all $k \in \mathbf{Z}$, $\|P_k f\|_{\dot{H}^s} \le c_k$. We shall write $f <<_s c$ or simply $f << c$ when there is no danger of confusion. It is easy to check that if $\|f\|_{\dot{H}^s} \le \varepsilon$ then there exists an envelope $c \in l^2$ such that $\|c\|_{l^2} \lesssim \varepsilon$ and $f <<_s c$. Indeed we can simply take, $c_k = \sum_{k'\in\mathbf{Z}} 2^{-\sigma|k-k'|}\|P_k\|_{\dot{H}^s}$.

Definition 2.21. Fix $0 < \sigma < \min(\frac{1}{2}, \frac{n-4}{4}, \frac{n-3}{4(n-1)})$ and c a frequency envelope. We say that the initial data $\phi[0] = \Big(\phi(0) = \varphi, \partial_t\phi(0) = \psi = \psi_0^a e_a\Big)$ lies underneath c if, relative to our frame e_a, we have for all $k \in \mathbf{Z}$,

$$\|P_k\phi[0]\|_{\dot{H}^{\frac{n}{2}}} \le c_k.$$

We shall use the short hand notation $\phi[0] << c$.

The proof of our main theorem can be easily reduced to the following:

Proposition 2.22. *(Main Proposition) Let c be a frequency envelope*[12] *with $\|c\|_{l^2} \le \varepsilon$, $0 < T < \infty$ and $\Phi = (\phi^a_\alpha)$ verify the equations* (31), (32), *and therefore also* (34). *Assume that, according to Definition 2.21, the initial data verifies the smallness condition $\phi[0] << c$. Assume also the bootstrap assumption,*

$$\|P_k\Phi\|_{S_k([0,T]\times\mathbb{R}^n)} \le 2Cc_k \tag{39}$$

for all $k \in \mathbf{Z}$. Then in fact, for sufficiently small ε, and all $k \in \mathbf{Z}$,

$$\|P_k\Phi\|_{S_k([0,T]\times\mathbb{R}^n)} \le Cc_k. \tag{40}$$

Remark 2.23. In view of the scale invariance of both our equations and the smallness condition $\phi[0] << c$ it suffices to prove (40) for $k = 0$. Let $\Psi = P_0\Phi$. We need to prove that,

$$\|\Psi\|_{S_0([0,T]\times\mathbb{R}^n)} \le Cc_0 \tag{41}$$

[12] verifying (38) with $\sigma < \min(\frac{1}{2}, \frac{n-4}{4}, \frac{n-3}{4(n-1)})$.

Remark 2.24. To prove (41) we would like to apply Theorem 2.19 to the equation obtained by applying the projection P_0. to (34) i.e.,

$$\Box\Psi = P_0(R_\mu \cdot \partial^\mu \Phi + E).$$

A straightforward application of the Strichartz inequalities will not work however. Indeed according to Theorem 2.19

$$\|\Psi\|_{S_0} \le c_0 + \|P_0(R_\mu \cdot \partial^\mu \Phi + E)\|_{L^1_t L^2_x}$$

The cubic term E presents no difficulty, the problem comes up when we try to estimate $P_0(R_\mu \cdot \partial^\mu \Phi)$ more precisely the part of it which corresponds to the interaction between low frequencies of R and frequencies of Φ comparable to those of Ψ. More precisely the most dangerous terms are of the form $\tilde{R} \cdot \partial\Psi$ with $\tilde{R} = P_{\le -10}R$. To estimate $\|\tilde{R} \cdot \partial\Psi\|_{L^1_t L^2_x}$ relative to the available Strichartz norms we are forced to take Ψ in the energy norm $L^\infty_t L^2_x$. This leaves us with $\tilde{R}$ in the norm $L^1_t L^\infty_x$ for which we don't have any Strichartz estimates.

Definition 2.25. We shall use, in what follows, Tao's convention to call an acceptable error any function, or matrix valued function, F on $[0,T] \times \mathbb{R}^n$ such that

$$\|F\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^n)} \le C^3 \varepsilon c_0 \tag{42}$$

Proposition 2.26. *The matrix valued function $P_0\Phi = \Psi$ verifies the equation*

$$\Box\Psi = -2\tilde{R}_\mu \cdot \partial^\mu \Psi + \textit{error} \tag{43}$$

where $\tilde{R}_\mu = P_{\le -10}R_\mu = \Gamma \cdot \tilde{\Phi}_\mu$ and $\tilde{\Phi}_\alpha = P_{\le -10}\Phi_\alpha$. Here "error" refers to an acceptable error term in the sense of (42).

Proposition 2.27. *The matrix valued function Ψ verifies an equation of the form,*

$$\Box\Psi = -2\partial_\mu\tilde{\Delta} \cdot \partial^\mu \Psi + \textit{error} \tag{44}$$

where the potential $\tilde{\Delta}$ verifies the following properties:

i.) The $N \times N$ matrix $\tilde{\Delta}$ is antisymmetric i.e. $\tilde{\Delta}^t = -\tilde{\Delta}$. The space Fourier transform of each component of $\tilde{\Delta}$ is supported in $|\xi| \le 2^{-10}$.

ii.) The following estimates hold for any $\tilde{\Delta}_k = P_k\tilde{\Delta}$:

$$\|\tilde{\Delta}_k\|_{S_k} \lesssim 2^{-k} C c_k \tag{45}$$

$$\|\partial\tilde{\Delta}_k\|_{S_k} \lesssim C c_k \tag{46}$$

Also,

$$\|\Box\tilde{\Delta}_k\|_{S_k} \lesssim 2^k C c_k \tag{47}$$

iii.) Set $\bar{R}_\mu = \tilde{R}_\mu - \partial_\mu\tilde{\Delta}$. The following estimates hold for all $P_k\bar{R}$,

$$\|P_k\bar{R}\|_{L^1_t L^\infty_x} \lesssim C^2 c_k^2 \tag{48}$$

$$\|P_k\bar{R}\|_{L^\infty_t L^\infty_x} \lesssim 2^k C^2 c_k^2 \tag{49}$$

Let M be a large integer, depending on T, which will be chosen below. Define the real $N \times N$ matrix valued function U to be

$$U = I + \sum_{-M<k\le -10} U_k$$

with the U_k defined inductively as follows,

$$\begin{aligned} U_k &= 0 \qquad \text{for all} \quad k < -M \\ U_{-M} &= I \\ U_k &= \tilde{\Delta}_k \cdot U_{<k} \qquad \text{for all} \quad -M < k \le -10 \end{aligned} \tag{50}$$

with $U_{<k} = \sum_{k'<k} U_k$. Due to the fact that the matrices $\tilde{\Delta}_k = P_k \tilde{\Delta}$ are antisymmetric we find the identity

$$U_k^t \cdot U_{<k} + U_{<k}^t \cdot U_k = 0$$

whence,

$$U_{<k}^t \cdot U_{<k} - I = \sum_{k'<k} U_{k'}^t \cdot U_{k'} \tag{51}$$

Using this identity we can prove inductively that

$$\begin{aligned} \|U_{<k}\|_{L_t^\infty L_x^\infty} &\le 2 \\ \|U_k\|_{L_t^\infty L_x^\infty} &\lesssim Cc_k \qquad \text{for} \quad k > -M \end{aligned} \tag{52}$$

as well as

$$\|U_k\|_{L_t^2 L_x^\infty} \lesssim C 2^{-k/2} c_k \qquad \text{for} \quad k > -M. \tag{53}$$

Also,

$$\begin{aligned} \|\partial U_{<k}\|_{L_t^\infty L_x^\infty} &\lesssim 2^k C^2 c_k \\ \|\partial U_k\|_{L_t^\infty L_x^\infty} &\lesssim 2^k C^2 c_k \end{aligned} \tag{54}$$

and

$$\begin{aligned} \|\partial U_{<k}\|_{L_t^2 L_x^\infty} &\lesssim 2^{k/2} C c_k \\ \|\partial U_k\|_{L_t^2 L_x^\infty} &\lesssim 2^{k/2} C c_k \end{aligned} \tag{55}$$

as well as,

$$\begin{aligned} \|\Box U_{<k}\|_{L_t^2 L_x^{n-1}} &\lesssim 2^{k(\frac{3}{2} - \frac{n}{n-1})} C c_k \\ \|\Box U_k\|_{L_t^2 L_x^{n-1}} &\lesssim 2^{k(\frac{3}{2} - \frac{n}{n-1})} C c_k \end{aligned} \tag{56}$$

Proposition 2.28. *Assume that ε is sufficiently small depending on C and M sufficiently large depending on T, C, ε. Then the matrices U verify the following properties:*

i.) Approximate orthogonality:

$$\|U^t U - I\|_{L_t^\infty L_x^\infty}, \ \|\partial(U^t U - I)\|_{L_t^\infty L_x^\infty} \lesssim C^2 \varepsilon \tag{57}$$

In particular, for small ε, U is invertible and we have,

$$\|U\|_{L_t^\infty L_x^\infty}, \ \|U^{-1}\|_{L_t^\infty L_x^\infty} \lesssim 1 \tag{58}$$

ii.) *Approximate gauge condition:*

$$\|\partial_\mu U - \partial_\mu \tilde{\Delta} \cdot U\|_{L^1_t L^\infty_x} \lesssim C^2 \varepsilon \tag{59}$$

iii.) *We also have,*

$$\|\partial U\|_{L^\infty_t L^\infty_x}, \ \|\partial U\|_{L^1_t L^\infty_x} \lesssim C^2 \varepsilon \tag{60}$$

$$\|\Box U\|_{L^2_t L^{n-1}_x} \lesssim C^2 \varepsilon \tag{61}$$

Following [70] we are now ready to perform the gauge transformation

$$\Psi = U \cdot W \tag{62}$$

W verifies the equation

$$\begin{aligned} \Box W &= -2U^{-1}(\partial_\mu U - \partial_\mu \tilde{\Delta} \cdot U)\partial^\mu W \\ &- 2U^{-1}\partial_\mu \tilde{\Delta} \cdot (\partial^\mu U) U^{-1}\Psi - U^{-1}(\Box U) U^{-1}\Psi + \text{error} \end{aligned} \tag{63}$$

In view of Proposition 2.28 we derive,

Proposition 2.29. *The matrix valued function W verifies an equation of the form*

$$\Box W = \textit{error}.$$

Therefore, if ε is sufficiently small,

$$\|\Psi\|_{S_0} \lesssim \|W\|_{S_0} \le \|\Psi[0]\|_{H^{\frac{n-2}{2}}} + CC^3\varepsilon c_0 \le Cc_0.$$

This is precisely (41) which ends the proof of the Main Proposition.

3. Lecture III: General Relativity and quasilinear wave equations

We start with our primary motivation to study the issue of optimal well-posedness for quasilinear wave equations.

3.1. Cauchy problem in General Relativity.

The primary object of Einstein's general relativity is the space-time. This can be defined as a class of equivalence of differentiable, oriented four dimensional Lorentz manifolds $(\mathcal{M}, g)$. We say that two Lorentz manifolds $(\mathcal{M}, g)$, $(\mathcal{M}', g')$ are equivalent if there exists a diffeomophism $\mathcal{M} \longrightarrow \mathcal{M}'$ such that $g' = \Phi_* g$. A space-time is simply a class of equivalence of such Lorentz manifolds. The space-time metric g has to satisfy the *Einstein Field Equations*,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}$$

with $R_{\alpha\beta}$ the Ricci curvature, R the scalar curvature of the metric and $T_{\alpha\beta}$ the energy-momentum tensor of some matterfield defined on $(\mathcal{M}, g)$. We restrict ourselves to the particular case of vacuum i.e. $T \equiv 0$ in which case the equations take the form,

$$R_{\alpha\beta} = 0.$$

The evolution character of the Einstein equations is much more subtle than for other equations as there is no intrinsic definition of a time variable t. It may seem, therefore, equally ambiguous to talk about the initial value problem. This can be defined, however, as follows:

Definition 3.2. An initial data set is a triplet $(\Sigma, \bar{g}, \bar{k})$ consisting of a three dimensional complete Riemannian manifold $(\Sigma, \bar{g})$ and a 2-covariant symmetric tensor k on Σ verifying the constraint equations:

$$\bar{\nabla}^j \bar{k}_{ij} - \bar{\nabla}_i \,\mathrm{tr}\bar{k} = 0$$
$$\bar{R} - |\bar{k}|^2 + (\mathrm{tr}\bar{k})^2 = 0$$

where $\bar{\nabla}$ is the covariant derivative, $\bar{R}$ the scalar curvature of $(\Sigma, \bar{g})$. The scalar $\mathrm{tr}\,\bar{k}$ is the trace of $\bar{k}$ with respect to $\bar{g}$.

Definition 3.3. An initial data set is said to be flat, or trivial, if it corresponds to a complete space-like hypersurface in Minkowski space with its induced metric and second fundamental form. An initial data set is said to be assymptotically flat(AF) if there exists a system of coordinates (x^1, x^2, x^3) defined in a neighborhood of infinity[13] on Σ relative to which the metric $\bar{g}$ approaches the euclidean metric and $\bar{k}$ approaches zero[14] as

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x_3)^2} \to \infty.$$

Definition 3.4. A Cauchy development of an initial data set $(\Sigma, \bar{g}, \bar{k})$ is a spacetime manifold $(\mathcal{M}, g)$ verifying the Einstein equations together with an embedding[15] $i : \Sigma \longrightarrow \mathcal{M}$ such that $\bar{g}$, $\bar{k}$ are the induced first and second fundamental forms of Σ in $\mathcal{M}$. A development is required to be also globally hyperbolic[16] in order to assure the unique dependence of solutions on the data. A future development of $(\Sigma, \bar{g}, \bar{k})$ consists of a globally hyperbolic manifold $(\mathcal{M}, g)$ with boundary, verifying the Einstein equations, and embedding i as before which identifies Σ to the boundary of $\mathcal{M}$.

The most primitive question asked about the initial value problem, solved satisfactory for very large classes of evolution equations, is that of local existence and uniqueness of solutions. For the Einstein equations this type of result was first established by Y.Choquet-Bruhat [7] with the help of wave coordinates which allowed her to cast the Einstein equations in the form of a system of nonlinear wave equations, to which one can apply the standard local existence theory based only on energy estimates and the $L^\infty \subset H^s$ Sobolev inequality.

[13] We assume, for simplicity, that Σ is diffeomorphic to $\mathbb{R}^n$. A neighborhood of infinity means the complement of a sufficiently large compact set on Σ.

[14] Because of the constraint equations the assymptotic behavior cannot be arbitrarily prescribed. A precise definition of asymptotic flatness has to involve the ADM mass of (Σ, g). Taking the mass into account we write $g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + o(r^{-1})$. According to the positive mass theorem $M \geq 0$ and $M = 0$ implies that the initial data set is flat.

[15] The constraint equations correspond to the contracted Codazzi and twice contracted Gauss equations of the embedding.

[16] Each causal curve in $\mathcal{M}$ intersects Σ at precisely one point.

Definition 3.5. A system of coordinates x^α are called *wave coordinates* if $\Box_g x^\alpha = 0$. Given such a system the Einstein vacuum equations take the following form,

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = N_{\mu\nu}(g_{\alpha\beta}, \partial g_{\alpha\beta}) \tag{64}$$

In the case of nonlinear systems of differential equations the local existence and uniquenes result leads, through a straigtforward extension argument, to a global result. The formulation of the same type of result for the Einstein equations is a little more subtle; it was done by Y. Choque-Bruhat and R.P. Geroch in [8]. The result asserts that *for each initial data set there exists a unique maximal future development.* Thus any construction, obtained by an evolutionary approach from a specific initial data set, must be necessarily contained in its maximal development. This may be said to solve the problem of global[17] existence and uniqueness in General Relativity; all further questions may be said to concern the qualitative properties of the maximal development. The central issue is that of existence and character of singularities. First we can define a regular maximal development as one which is complete in the sense that all future time-like and null geodesics are complete. A well known theorem of Penrose asserts that, under certain quite extreme conditions[18] on an initial data set the corresponding future maximal development is necessarily incomplete. In addition, with the exception of the flat Minkowski space itself, all explicit AF solutions of the Einstein Vacuum equations, such as the Kerr family, have turned out to be singular. At the opposite end of Penrose's theorem we have the following completeness result, see [12].

Theorem 3.6 (Global Stability of Minkowski)**.** *Any assymptotically flat initial data set which is sufficiently close*[19] *to the trivial one has a complete maximal future development* $(\mathcal{M}, g)$*. Moreover the spacetime* $\mathcal{M}$ *is close to the Minkowski space in the sense that its curvature tensor tends to zero along any timelike or null geodesics. Its causal structure, however, is asymptotically nontrivial.*

Problem *(Strong stability of the Minkowski space)* *Does there exists a scale invariant smallness condition*[20] *such that all developments, whose initial data sets* (Σ, g, k) *verify it, have complete maximal future developments.*

Remark 3.7. The relationship between the old stability result of theorem 3.6 and strong stability may be compared with the situation for Wave Maps in the wake of Tao's result [71]. Indeed an older result due to Sideris[63], provides global existence for sufficiently small data in a weighted Sobolev norm containing a large number of derivatives. Tao's new result requires only the smallnes of a scale invariant norm containing the minimum number of derivatives of the initial map.

Remark 3.8. The strong stability of the Minkowski spacetime should be viewed itself as an intermediate step towards the even loftier goal of proving the well known

[17] This is of course misleading, for equations defined in a fixed background global is a solution which exists for all time. In general relativity however we have no such background as the spacetime itself is the unknown. The connection with the classical meaning of a global solution requires a special discussion concerning the proper time of timelike geodesics.

[18] If $(\Sigma, \bar{g})$ is a noncomapct Riemannian manifold containing a closed trapped surface.

[19] The precise condition requires weighted H^s- Sobolev norms involving 4 derivatives of the metric $\bar{g}$ and 3 derivatives of the second fundamental form $\bar{k}$.

[20] involving, locally, the L^2 norm of $\frac{3}{2}$ derivatives of g and $\frac{1}{2}$ derivatives of k.

Penrose's Cosmic Censorship conjectures. We state these in the Complements to Lecture III.

A proof of the strong stability of the Minkowski space is a long term goal. We present below a far simpler, yet still very challenging, intermediate, conjecture, which motivates the subject matter of the next section.

Conjecture *(Finite L^2 - Curvature Conjecture) The Bruhat-Geroch result can be extended to initial data sets (Σ, g, k) with,*

$$R(g) \in L^2_{(loc)}(\Sigma) \qquad \text{and} \qquad k \in H^1_{(loc)}(\Sigma).$$

The L^2- *Boundedness Conjecture* is the most reasonable goal we can aspire to at this moment in connection with the strong stability of the Minkowski space conjecture. One might compare it with the proof of local(and global in this case !) well posedness for the YM equations in $\mathbb{R}^{1+3}$, see [34]. It is however going to be a lot more complicated to resolve. We might first want to settle for a simpler goal; to prove well posedness for the system (64) in $H^{2+\epsilon}(\mathbb{R}^3)$. To simplify a little bit consider wave equations of the form

$$G^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\phi = N(\phi, \partial\phi), \tag{65}$$

subject to initial conditions $\phi[0] \in H^s$. Here $G^{\alpha\beta}(\phi)$ is a family of Lorentz metrics in $\mathbb{R}^{1+3}$ depending smoothly on ϕ in a small neighborhood of the origin and N is quadratic in $\partial\phi$. We want to show that the IVP is well posed for $s > 2 + \epsilon$. The result analogous to this for semilinear wave equations of the form

$$\Box\phi = N(\phi, \partial\phi)$$

is very easy to prove[21], it depends only on the Strichartz type estimates of the form:

$$\|\partial\psi\|_{L^2_t L^\infty_x([0,T]\times\mathbb{R}^n)} \lesssim \|\phi[0]\|_{H^s(\mathbb{R}^n)} \tag{66}$$

for any $s > \frac{n}{2} + \frac{1}{2}$, $n = 3$. In particular it does not require that the nonlinear terms satisfy any special structure such as the null condition.

3.9. **Optimal well posedness for quasilinear wave equations.** I plan to report here on my recent results obtained in collaboration with I. Rodnianski[42] concerning the issue of optimal well posedness for quasilinear wave equations. Motivated by quasilinear wave equations of the form (64) and (67), we consider the simplified model equation in $\mathbb{R}^{n+1}$, $n \geq 3$ of the form,

$$\partial_t^2\phi - g^{ij}(\phi)\partial_i\partial_j\phi = N(\phi, \partial\phi) \tag{67}$$

with N a smooth function quadratic in $\partial\phi$. The classical local existence and uniqueness results, based on H^s- energy estimates and the Sobolev embedding theorem, show that (67) is locally well posed in the Sobolev space H^s for any $s > \frac{n}{2} + 1$. The H^s energy estimates, obtained by standard integration by parts and commutator lemmas, take the form

$$\|\phi(t)\|_{H^s} \leq M(\|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}})$$

[21] This result turns out to be optimal, in general, according to the well-known counterexample of H. Lindblad, [49]. His results shows that local well posedness fails for H^s data, $s \leq 2$, in dimension $n = 3$.

with M depending continuously on the integral $\int_0^t \|\partial\phi\|_{L^\infty_x}$. The Sobolev estimate $\|\partial\phi\|_{L^\infty} \le c\|\phi\|_{H^s}$ for $s > \frac{n}{2}+1$ allows one to conclude that for sufficiently small time t we can bound the right hand side of the above inequality purely in terms of $\|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}}$. These local apriori estimates, for $s > \frac{n}{2}+1$, can easily be turned into a local existence and uniqueness result.

The crucial quantity $\int_0^t \|\partial\phi\|_{L^\infty_x}$ could be better controlled with the aid of Strichartz estimates (66). This leads to the gain of $\frac{1}{2}$ derivatives mentioned above.

In the quasilinear case the metric g^{ij} depends on the solution ϕ and can therefore have only as much regularity as ϕ has. Thus one needs to address the problem of proving the Strichartz estimates for linear wave equations $\Box_g\phi = \partial_t^2 - g^{ij}\partial_i\partial_j\phi = 0$ with rough coefficients. The conditions needed for the coefficients g^{ij} of the linearized wave operator $\Box_g$ have to be consistent with the dependence on ϕ of the nonlinear coefficients $g^{ij}(\phi)$ in (67). In view of the expected boundedness of the Sobolev norms $\|\phi\|_{H^s}$ we may assume[22] that the norms $\|g\|_{H^s}$ are bounded. Moreover, as the H^s norm of a solution of the quasilinear problem (67) at time t is controlled by the H^s norm of the initial data as long as $\int_0^t \|\partial\phi\|_{L^\infty_x}$ is bounded, we can inductively assume that the metric $g^{ij} = g^{ij}(\phi)$ satisfies also the condition $\int_0^t \|\partial g\|_{L^\infty_x} \le B_0$. In view of our experience with Strichartz estimates in flat space we do not expect have direct access to the norm of $\|\partial\phi\|_{L^1_t L^\infty_x}$ but, locally in time this can be controlled by the $\|\partial\phi\|_{L^2_t L^\infty_x}$. Thus, to close the argument and derive an improved local existence and uniqueness result for (67), we need to prove the boundedness of H^s norms as well as a local $L^2_t L^\infty_x$ - Strichartz estimate for solutions ϕ to the linearized wave equation $\Box_g\phi = 0$.

Historical remarks: The first important work concerning Strichartz type estimates for solutions to $\Box_g\phi = 0$ with rough [23] coefficients is due to H. Smith [60]. He showed that the standard Strichartz estimates hold for equations with $C^{1,1}$ coefficients. However the $C^{1,1}$ condition is too strong for applications to quasilinear equations. Moreover some important counterexamples of H. Smith and C. Sogge[61] showed that the standard Strichartz estimates fail if the coefficients are rougher than $C^{1,1}$.

This was the situation before the important breakthrough of H. Bahouri and J.-Y. Chemin. In [1] they showed that some weaker form of the Strichartz estimates still survive for equations with coefficients rougher than $C^{1,1}$, compatible with applications to quasilinear equations. Namely, they were able to establish a Strichartz estimate with a loss

$$\|\partial\psi\|_{L^2_t L^\infty_x} \lesssim \|\psi[0]\|_{H^{\frac{n}{2}+\frac{1}{2}+\sigma}}, \tag{68}$$

for solutions to the variable coefficient wave equation $\Box_g\psi = 0$, with coefficients verifying assumptions of the type discussed above. As long as the loss $\sigma \le \frac{1}{2}$ such an estimate can be applied to prove a local well posedness result for the quasilinear problem considered here, stronger than the classical one. The results of H. Bahouri

[22] These are typical bootstrap assumptions.

[23] For C^∞ metrics the Strichartz estimates were proved by Kapitanski[25] and by Mockenhaupt-Seeger-Sogge [52].

and J.-Y. Chemin are based on Strichartz estimates with a loss of $\sigma > \frac{1}{4}$ for the linearized problem $\square_g \psi = 0$ where the metric g satisfying conditions of the type discussed above. Later, D. Tataru[72] showed that the loss[24] in the Strichartz estimate can be improved $\sigma > \frac{1}{6}$. In fact [72] provides the precise relationship between the smoothness of the metric, below $C^{1,1}$, and the corresponding loss in the Strichartz estimates. Recently H. Smith and D. Tataru[62] have shown that these results are also sharp and therefore there is no hope of improving Tataru's results based purely on linear theory.

Both the results of Bahouri-Chemin and those of Tataru are based on two major ingredients.

a) *Paradifferential Calculus:* Consider the dyadic projection operator P_λ, corresponding to a cut-off relative to the spatial frequencies $\frac{\lambda}{2} \le |\xi| \le 2\lambda$, $\lambda \in 2^{\mathbf{N}}$. Let $\phi^\lambda = \phi_\lambda = P_\lambda \phi$. We have $\phi = \sum_\lambda \phi_\lambda$. If ϕ satisfies the quasilinear wave equation (67) then ϕ^λ verifies the equation

$$\square_{g_{\le \lambda^a}} \phi^\lambda = -\partial_t^2 \phi^\lambda + g^{ij}_{\le \lambda^a} \partial_i \partial_j \phi^\lambda = R^a_\lambda, \tag{69}$$

with $g_{\le \lambda^a} = \sum_{\mu \le \lambda^a} P_\mu g$, $0 < a < 1$ and R_λ an easy to control error term. The original idea of Bahouri-Chemin, later refined by Tataru, was to prove a Strichartz estimate, without losses, for (69) in a frequency dependent time interval I_λ of size $|I_\lambda| \lesssim \lambda^{1-a}$. It is this loss in the control of the size of the time interval I_λ which is responsible for the loss of differentiability of (71).

b) *Parametrix based proof of the Dispersive inequality:* As we know the major *hard* ingredients in the proof of Strichartz inequalities are the energy and dispersive inequalities. Both Bahouri-Chemin and Tataru use a parametrix construction[25] for solutions to the approximate linearized equation (69). They derive the dispersive inequality for the explicit form of the approximate solutions, provided by the parametrix construction, using the method of stationary phase.

c) *Eikonal equation:* A major ingredient in both methods is the construction of solutions to the eikonal equation,

$$u_t^2 - g^{ij}_{\le \lambda^a} \partial_i u \partial_j u = 0, \qquad u(0, x, \xi) = x \cdot \xi. \tag{70}$$

as well as associated transport equations.

In [30] I have developed a different, more geometric approach, to the proof of the dispersive inequality for the equations (69) based on a modification of the vectorfields method discussed in Lecture I. Recall indeed, see proposition (1.9), that we were able to prove the dispersive inequality without any explicit representation

[24] The immediate consequence of these results is local well posedness for the quasilinear problem (67) in the space H^s with $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$, if $n \ge 3$, and $s > \frac{n}{2} + \frac{5}{6}$ for $n = 2$.

[25] Bahouri-Chemin use a standard progressing wave approximation while Tataru uses a parametrix based on the FBI transform.

of solutions, just the commutation properties of the standard wave equation with the scaling, boosts and angular momentum vectorfields. To implement such a strategy for quasilinear equations of the type (69) we would like a similar family of Killing and conformal Killing vectorfields for the metric $g_{\le\lambda^a}$ This is, of course, not possible; a general metric does not possess any symmetries. The best we can do is to construct vectorfields whose deformation tensors are as small as possible. In [30] I have outlined how to make such a construction based on the same eikonal equation (79) but with different initial conditions, corresponding not to a family of null hyperplanes as before but on a family of outgoing characteristic cones with vertices on the time axis. This direct geometric approach had allowed me to recover the Tataru's results for quasilinear wave equations in dimension three.

By using a similar geometric approach as in [30], recently Rodnianski and I, [42], were able to improve the well posedness results of Bahouri-Chemin and Tataru by taking into account the nonlinear character of the equation (67). We do not simply prove a Strichartz estimates for solutions of $\Box_g\phi = 0$ with bounds on $\|g\|_{H^s}$ and $\|\partial g\|_{L^1_t L^\infty_x}$, we make use in an essential way of the fact that the coefficients $g^{ij}(\phi)$ of the equation (67) verify themselves an equation of the type $\Box_g g_{ij} = N$ with N depending only on ϕ and $\partial\phi$.

Our main result is included in the following theorem.

Theorem 3.10. *The quasilinear initial value problem* (67) *in* $\mathbb{R}^{3+1}$ *is locally well posed in* H^{s_*} *for* $s_* > s_0 = 2 + \frac{2-\sqrt{3}}{2}$. *Namely, for any initial data* $\phi[0] \in H^{s_*}$ *there exists a sufficiently small interval* $[0,T]$ *with* $T = T(\|\phi[0]\|_{H^{s_*}})$ *such that problem* (67) *has a unique solution* $\phi \in C([0,T], H^{s_*}) \cap C^1([0,T], H^{s_*-1})$. *In addition,* ϕ *satisfies a Strichartz type estimate*

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le cT^{s_*-s_0}\|\phi[0]\|_{H^{s_*}}. \tag{71}$$

Moreover we believe that, with a relatively small modification of the proofs in [42], we now have all the tools needed to show that the Einstein equations[26] are locally well posed in $H^{2+\epsilon}$.

3.11. **Sketch of the proof of Theorem 3.10.** After a series of reductions, which are now canonical(see the complements to this lecture), we can reduce the proof of the Theorem to the following situation:

Assume given a family[27]the of Riemannian metrics $h^\lambda_{ij}(t,x)dx^i dx^j$ defined on a time slab $I_\lambda \times \mathbb{R}^3$ with $I_\lambda = [0, t_*]$, $t_* \le \lambda^a$ verifying the following conditions

[26]in a specific gauge such as wave coordinates.

[27]The metrics h_λ are in fact the rescales of $g_{\le\lambda^a}$ appearing in (**??**), more precisely we have $h_\lambda(t,x) = g_{\le\lambda^a}(\lambda^{-1}t, \lambda^{-1}x)$.

$$\|\partial^{1+m} h_\lambda\|_{L^1_{I_\lambda} L^\infty_x} \lesssim \lambda^{-(1-a)(m+1)}, \tag{72}$$

$$\|\partial^{1+m} h_\lambda\|_{L^2_{I_\lambda} L^\infty_x} \lesssim \lambda^{-\frac{2-a}{2}-(1-a)m}, \tag{73}$$

$$\|\partial^{1+m} h_\lambda\|_{L^\infty_{I_\lambda} L^\infty_x} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m}, \tag{74}$$

$$\|\partial_x^{\frac12+m}(\partial^2 h_\lambda)\|_{L^\infty_{I_\lambda} L^2_x} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m}, \tag{75}$$

$$\|\partial^m \bar{\Box}_{h_\lambda} h_\lambda\|_{L^1_{I_\lambda} L^\infty_x} \lesssim \lambda^{-(2-a)-(1-a)m}, \tag{76}$$

Here $\bar{\Box}_{h_\lambda} = \partial_t^2 - h_\lambda^{ij}\partial_i\partial_j$ We need to prove the following decay estimates for solutions to the covariant wave equation

$$\Box_{h_\lambda}\psi = 0 \tag{77}$$

where $\Box_{h_\lambda}\psi - \frac{1}{\sqrt{\det h_\lambda}}\partial_t\sqrt{\det h_\lambda}\,\partial_t\psi + \frac{1}{\sqrt{\det h_\lambda}}\partial_i(h_\lambda^{ij}\sqrt{\det h_\lambda}\,\partial_j\psi)$

Theorem 3.12. *Assume ψ is a solution of (77) in $[0,t_*]\times\mathbb{R}^n$, $t_*\lambda^a$ with initial data $\psi[0]$ supported in a ball of radius $\frac12$ centered at the origin in physical space. Assume that the metrics h_λ verify (138)–(142) with $a < -1+\sqrt{3}$. Then, for all $t\in[0,t_*]$,*

$$\|\partial\psi(t)\|_{L^\infty} \le (1+t)^{-1+\epsilon}\|\psi[0]\|_{H^k}. \tag{78}$$

The proof is based on a curved spacetime analogue of Proposition 4.11 discussed in the Complements to Lecture I. Before stating it we need to make some definitions:

Definition 3.13. A null pair consists of two vectorfields $L, \underline{L}$, verifying

$$H(L\,,\,L) = H(\underline{L}\,,\,\underline{L}) = 0, \quad H(L\,,\,\underline{L}) = -2.$$

A null frame, associated to the null pair $L, \underline{L}$, consists of 4 linearly independent vectors $e_1, e_2, e_3 = \underline{L}, e_4 = L$ verifying

$$H(e_A, L) = H(e_A, \underline{L}) = 0, \quad H(e_A, e_B) = \delta_{AB}, \quad \text{for } A = 1,2$$

Definition 3.14. The optical function[28] u is defined to be the unique forward solution to the eikonal equation

$$H^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0, \qquad \text{or} \quad \partial_t u = |\nabla u|_h \tag{79}$$

with the boundary condition $u = t$ on the time axis. The 2-surfaces of intersection between the level hypersurfaces Σ_t of the time function t and the level hypersurfaces C_u of u are denoted by $S_{t,u}$.

Definition 3.15. Given such u we define a canonical null pair as follows:

$$L = T + N \qquad \underline{L} = T - N \tag{80}$$

[28] In what follows we drop the labels λ from h_λ. We denote by H the spacetime metric $-dt^2 + h_{ij}dx^i dx^j$.

where,

$$T = \partial_t, \qquad N = -\frac{h^{ij}\partial_j u}{|\nabla u|_h}\partial_i.$$

Observe that $T + N$ are the generating normals of the null cones C_u. We can complement $L, \underline{L}$ to create a null frame spanning the whole tangent space. This can be done by choosing an arbitrary orthonormal frame e_A on $S_{t,u} = C_u \cap \Sigma_t$. We rewrite the full null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$.

We also introduce the following:

i) *Energy-momentum tensor:* $Q_{\alpha\beta} = \partial_\alpha\psi\partial_\beta\psi - \frac{1}{2}H_{\alpha\beta}(H^{\mu\nu}\partial_\mu\psi\partial_\nu\psi)$

ii) *Modified energy-momentum tensor:* $\bar{Q}_{\alpha\beta} = Q_{\alpha\beta} + (n-1)t\psi\partial_\beta\psi - \frac{n-1}{2}\psi^2$

iii) *Modified Morawetz vectorfield:* We set $K = \frac{1}{2}(\underline{u}^2 L + u^2 L)$ with $\underline{u} = 2t - u$.

iv) *Modified deformation tensor:* $\bar{\pi} = {}^{(K)}\pi - 4tH$

v) *Conformal Energy:* $\mathcal{Q}_0[\psi](t) = \int_{\Sigma_t} \bar{Q}(K, \partial_t)[\psi]$.

vi) *Full Conformal Energy:* $\mathcal{Q}[\psi](t) = \mathcal{Q}_0[\psi](t) + \mathcal{Q}_0[\partial_t\psi](t) + \mathcal{Q}_0[\partial_t^2\psi](t)$

vii) *Null components of Q:* The components of Q relative to a null frame are given by:

$$Q_{\mathbf{44}} = (D_{\mathbf{4}}\psi)^2, \qquad Q_{\mathbf{34}} = |\nabla\!\!\!/\psi|^2, \tag{81}$$

$$Q_{\mathbf{33}} = (D_{\mathbf{3}}\psi)^2, \qquad Q_{\mathbf{3}A} = D_{\mathbf{3}}\psi\nabla\!\!\!/_A\psi, \tag{82}$$

$$Q_{\mathbf{4}A} = D_{\mathbf{4}}\psi\nabla\!\!\!/_A\psi, \qquad Q_{AB} = \nabla\!\!\!/_A\psi\nabla\!\!\!/_B\psi + \frac{1}{2}(D_{\mathbf{3}}\psi D_{\mathbf{4}}\psi - |\nabla\!\!\!/\psi|^2)\delta_{AB}. \tag{83}$$

viii) *Conformal energy density* Using vii) we easily calculate:

$$Q(K, \partial_t)[\psi] = \frac{1}{4}\big(\underline{u}^2(D_{\mathbf{4}}\psi)^2 + u^2(D_{\mathbf{3}}\psi)^2 + (\underline{u}^2 + u^2)|\nabla\!\!\!/\psi|^2\big)$$

The proof of theorem 3.12 can be reduced to the following:

Theorem 3.16 (Boundedness Theorem). *Under the same assumptions as in theorem 3.12 we have,*

$$\mathcal{Q}(t) \lesssim \mathcal{Q}(0).$$

We sketch below some of the main ideas in the proof of the boundedness theorem. We restrict ourselves to a discussion of the boundedness of $\mathcal{Q}_0$.

Step 1. Introduce the quantity

$$\mathcal{E}_0[\psi](t) = \int_{\Sigma_t} (\underline{u}^2 (D_{\mathbf{4}}\psi)^2 + u^2 (D_{\mathbf{3}}\psi)^2 + \underline{u}^2 |\nabla\!\!\!/ \psi|^2 + \psi^2) \lesssim \mathcal{Q}_0[\psi](t). \tag{84}$$

By a simple comparison theorem, similar to part iii) of proposition 4.11, we show that

$$\mathcal{E}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t) \tag{85}$$

Step 2. The key ingredient in the proof theorem 3.16 is the following curved spacetime version of the generalized conformal identity of proposition 4.11 discussed in the complement to Lecture I.

Proposition 3.17 (Generalized conformal identity)**.** *Let ψ be a solution to $\Box_h \psi = 0$. Then,*

$$\mathcal{Q}_0[\psi](t) = \mathcal{Q}_0[\psi](t_0) - \frac{1}{2} \int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta} \bar{\pi}_{\alpha\beta} + \frac{n-1}{2} \int_{[t_0,t]\times\mathbb{R}^3} \psi^2 \Box_h(t). \tag{86}$$

Step 3. To prove the boundedness theorem we need to control $\int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta}\bar{\pi}_{\alpha\beta}$. Expanding relative to a null frame, we have

$$\begin{aligned} Q^{\alpha\beta}\bar{\pi}_{\alpha\beta} &= \frac{1}{4} Q_{44}\bar{\pi}_{33} + \frac{1}{4} Q_{33}\bar{\pi}_{44} + \frac{1}{2} Q_{34}\bar{\pi}_{34} \\ &- Q_{3A}\bar{\pi}_{4A} - Q_{4A}\bar{\pi}_{3A} + Q_{AB}\bar{\pi}_{AB} \end{aligned}$$

We need an estimate of the form,

$$\begin{aligned} \int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta}\bar{\pi}_{\alpha\beta} &\lesssim \lambda^{-\epsilon} \sup_{[t_0,t]} \int_{\Sigma_\tau} (\underline{u}^2 (D_{\mathbf{4}}\psi)^2 + u^2 (D_{\mathbf{3}}\psi)^2 + \underline{u}^2 |\nabla\!\!\!/ \psi|^2 + \psi^2) \\ &= \lambda^{-\epsilon} \sup_{[t_0,t]} \mathcal{E}_0[\psi](\tau) \lesssim \lambda^{-\epsilon} \sup_{[t_0,t]} \mathcal{Q}_0[\psi](\tau) \end{aligned}$$

with an arbitrary $\epsilon > 0$. Consider in particular the term,

$$\int_{[t_0,t]\times\mathbb{R}^3} Q_{33}\bar{\pi}_{44} = \int_{[t_0,t]\times\mathbb{R}^3} |D_3\psi|^2 \bar{\pi}_{44}$$

To obtain the desired estimate we need,

$$|\bar{\pi}_{44}(\tau,\cdot)| \lesssim u^2 \tau^{-1} \lambda^{-\epsilon}.$$

In other words $\bar{\pi}_{44}$ must be very small in the wave zone $|u| << t$. In fact, to estimate most of the terms in $\int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta}\bar{\pi}_{\alpha\beta}$ we need the following estimates for the modified deformation tensor $\bar{\pi}$ of K:

Proposition 3.18. *The null components of the modified deformation tensor of K verifies the estimates:*

$$\begin{aligned} |\bar{\pi}_{\mathbf{44}}| &\le u^2 \lambda^{-\epsilon} \tau^{-1} & |\bar{\pi}_{\mathbf{34}}| &\le \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, \\ |\bar{\pi}_{\mathbf{33}}| &\le \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, & |\bar{\pi}_{\mathbf{3}A}| &\le \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, \\ |\bar{\pi}_{\mathbf{4}A}| &\le u\,\underline{u} \lambda^{-\epsilon} \tau^{-1}, & |\bar{\pi}_{AB}| &\le \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}. \end{aligned}$$

Step 4. The estimates above would take care of all but the term

$$\frac{1}{2}\int_{[t_0,t]\times\mathbb{R}^3} \bar{\pi} D_{\mathbf{3}}\psi\, D_{\mathbf{4}}\psi, \quad \text{with} \qquad \bar{\pi} = \delta^{AB}\bar{\pi}_{AB}$$

To estimate this term we have to use an integration by parts argument which would bring in $D_{\mathbf{4}}\bar{\pi}$ and $\nabla\!\!\!/\bar{\pi}$ for which we need similar estimates as those in proposition 3.18.

Step 5. We need to prove the asymptotic estimates of proposition 3.18, as well as similar ones for $D_{\mathbf{4}}\bar{\pi}$ and $\nabla\!\!\!/\bar{\pi}$. This relies on tke analysis of the eikonal equation (79). Indeed it is easy to see, by a straightforward calculation that $\bar{\pi}$ depends on the second fundamental form k of the time foliation Σ_t and, more importantly, on the Hessian D^2u. Since $k_{ij} = -\frac{1}{2}\partial_t h_{ij}$ we have a very good control on k. The hessian of u, on the other hand, verifies a Ricatti type equation of the form,

$$D_L(D^2u) + D^2u \cdot D^2u = R(L, L)$$

with $R(L, L)$ represents the components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ twice contracted with L. At a first glance it looks as if D^2u has the same differentiability properties as those of the curvature tensor R. Moreover to estimate D^2u at some time t we need to integrate $R(L, L)$ on a geodesic generator of the null cones C_u. Thus, in view of the assumption (138), since R depends on two derivatives of h, it seems that the best estimate we can hope to get is

$$\begin{aligned} \|D^2u(t)\|_{L^\infty} &\lesssim \int_0^t |\partial^2 h(\tau,\cdot)| d\tau \\ &\lesssim \lambda^{-2(1-a)} \end{aligned}$$

Arguments along this line would only give the optimal result obtained by Tataru. To do better we have to make use of the nonlinear assumption (142). The ability to make use of that assumption is the crucial advantage of our geometric method.

To do this we need to decompose D^2u relative to our canonical null frame. This can be done by introducing the quantities:

$$\chi_{AB} = < D_A e_4, e_B >, \quad \eta_A = \frac{1}{2} < D_{\mathbf{3}} e_4, e_A > \tag{87}$$

Also, set,

$$b^{-1} = \partial_t u = |\nabla u|_h$$

We also split χ_{AB} into $\mathrm{tr}\chi = \delta^{AB}\chi_{AB}$ and traceless part $\hat{\chi}_{AB}$. They satisfy the following equations:

$$L(b) = -b\, k_{NN}, \tag{88}$$

$$L(\mathrm{tr}\chi) + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - k_{NN}\mathrm{tr}\chi - R_{44}, \tag{89}$$

$$D\!\!\!\!/_{\mathbf{4}}\hat{\chi}_{AB} + \frac{1}{2}(\mathrm{tr}\chi)\hat{\chi}_{AB} = -k_{NN}\hat{\chi}_{AB} - \hat{\alpha}_{AB}, \tag{90}$$

$$D\!\!\!\!/_{\mathbf{4}}\eta_A + \frac{1}{2}(\mathrm{tr}\chi)\eta_A = -(k_{BN} + \eta_B)\hat{\chi}_{AB} - \frac{1}{2}(\mathrm{tr}\chi)k_{AN} - \frac{1}{2}\beta_A, . \tag{91}$$

where $\alpha_{AB} = R_{A4B4}$, $\hat{\alpha} = \delta^{AB}\alpha_{AB}$ and $\beta_A = R_{A43B}$. If we try to estimate $\mathrm{tr}\chi$, $\hat{\chi}$, η from their corresponding transport equations (89), (90) and respectively (91) we run into precisely the same difficutlies mentioned above. We circumvent these difficulties using the following two ideas:

Idea 1 Observe that the curvature term R_{44} appearing in the transport equation for trχ has a special structure. In fact we have,

Lemma 3.19 (Remarkable decomposition)**.** *The $R_{44} = e_4^\mu e_4^\nu R_{\mu\nu}$ component of the Ricci curvature admits the following decomposition:*

$$R_{44} = L(z) - \frac{1}{2} e_4^\mu \, e_4^\nu \Box_h H_{\mu\nu} + Error, \tag{92}$$

with the functions z and $Error$ obeying pointwise estimates $|z| \lesssim |\partial H|$ and $|Error| \lesssim (\partial H)^2$.

In view of the Lemma we can rewrite the transport equation for trχ as follows:

Proposition 3.20. *Let $y = tr\chi - \frac{2}{s}$ with $s = t - u$ and the functions z and error as in the Lemma above. Then,*

$$\begin{aligned} L(y+z) + tr\chi(y+z) &= \frac{1}{2}(y+z)^2 + \frac{2}{s} z - \frac{1}{2} z^2 - |\hat{\chi}|^2 \\ &- k_{NN} \, tr\chi + \frac{1}{2} e_4^\nu e_4^\mu \Box_h H_{\mu\nu} - Error, \end{aligned}$$

Idea 2: The equation for $y + z$ seems substantially better then the one for trχ. Unfortunately one can not do the same trick for $\hat{\chi}$ and η. Moreover the equation for $y + z$ is itself coupled with the transport equation for $\hat{\chi}$. To solve this problem we appeal to an entirely different geometric equation, namely the Codazzi equations for χ. This has the form,

$$(\mathrm{d}\!\!\!/\mathrm{iv}\, \hat{\chi})_A + \hat{\chi}_{AB} k_{BN} = \frac{1}{2}(\nabla\!\!\!\!/_A \mathrm{tr}\chi + k_{AN} \mathrm{tr}\chi) - R_{B4AB} \tag{93}$$

Observe that the equation $\mathrm{d}\!\!\!/\mathrm{iv}\, \hat{\chi}_A = F_A$ is an elliptic Hodge system on the surfaces $S_{t,u}$. Using the coupled system between the above transport equation for $y + z$ and (93) we derive estimates of the type:

$$\begin{aligned} |\mathrm{tr}\chi - \frac{2}{s}| &\lesssim \lambda^{-\bar{a}} \\ |\hat{\chi}| &\lesssim \lambda^{-\bar{a}} \end{aligned}$$

with

$$\bar{a} = 1 - \frac{a^2}{2}.$$

This is a lot better then the estimates by $\lambda^{-2(1-a)}$ obtained before.

Remark 3.21. We have a similar approach to treat η. The idea is to obtain a divergence -curl system for η coupled with a transport equation which is similar to the one for trχ.

Remark 3.22. The idea of using coupled elliptic Hodge systems with favorable transport equations appears, in a very different context in [12], in connection with the global stabilty of the Minkowski space.

4. Complements to Lecture I

The important nonlinear wave equations such as Yang -Mills, Wave Maps and the Einstein equations have fundamental symmetries intimately connected to their geometric character. An in depth knowledge of Differential Geometry, in particular Lorentzian Geometry, is as at least as important in the study of these equations as Harmonic Analysis. In what follows we shall investigate the relationship between the conformal structure of Minkowski spacetime and estimates for the wave equation (1). In particular we shall use the generalized energy estimates discussed below in Lecture III. I assume some familiarity with basic notion of differential geometry, such as Lie and covariant differentiation as well as the curvature tensor.

4.1. **Conformal structure of Minkowski spacetime.**

Definition 4.2. Consider a spacetime $(\mathcal{M}, g)$, i.e.a Lorentzian manifolds with g a Lorentz metric of signature $(-1, 1 \dots, 1)$.

A diffeomorphism $\Phi : \mathcal{U} \subset \mathcal{M} \to \mathcal{M}$ is said to be a conformal isometry if, at every point p, $\Phi_* g = \Lambda^2 g$, i.e.,

$$(\Phi^* g)(X, Y)|_p = g(\Phi_* X, \Phi_* Y)|_{\Phi(p)} = \Lambda^2 g(X, Y)|_p$$

with $\Lambda \neq 0$. If $\Lambda = 1$, Φ is called an isometry of $\mathcal{M}$.

Definition 4.3. A vector field K which generates a one parameter group of isometries, respectively, conformal isometries is called a Killing, respectively, conformal Killing vector field.

Let K be such a vector field and Φ_t the corresponding 1-parameter group. Since the $(\Phi_t)_*$ are conformal isometries, we infer that $\mathcal{L}_K g$ must be proportional to the metric g. Moreover $\mathcal{L}_K g = 0$ if K is a Killing vector field.

Observe that C has the important property of being conformal invariant with respect to a conformal isometry.

Definition 4.4. Given an arbitrary vector field X we denote ${}^{(X)}\pi$ the deformation tensor of X defined by the formula

$${}^{(X)}\pi_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha \ .$$

The tensor ${}^{(X)}\pi$ measures, in a precise sense, how much the diffeomorphism generated by X differs from an isometry or a conformal isometry.

Proposition 4.5. *The vector field X is Killing if and only if ${}^{(X)}\pi = 0$. It is conformal Killing if and only if ${}^{(X)}\pi$ is proportional to g.*

To prove the "if" part we can choose local coordinates $x^0, x^1, ..., x^n$ such that $X = \frac{\partial}{\partial x^0}$. It then immediately follows that, relative to these coordinates, the metric g is independent of x^0.

Proposition 4.6. *On any spacetime $\mathcal{M}$ there can be no more than $\frac{1}{2}(n+1)(n+2)$ linearly independent Killing vector fields.*

The spacetime which possesses the maximum number of Killing and conformal Killing vector fields is the Minkowski spacetime. Let us review its associated isometries and conformal isometries.

Let x^μ be an inertial coordinate system, positively oriented, we have:

1. Translations: for any given vector $a = (a^0, a^1, ..., a^n) \in \mathbf{M}$,

$$x^\mu \to x^\mu + a^\mu$$

2. Lorentz rotations: Given any $\Lambda = \Lambda^\rho_\sigma \in \mathbf{O}(1, n)$,

$$x^\mu \to \Lambda^\mu_\nu x^\nu$$

3. Scalings: Given any real number $\lambda \neq 0$,

$$x^\mu \to \lambda x^\mu$$

4. Inversion: Consider the transformation $x^\mu \to I(x^\mu)$, where

$$I(x^\mu) = \frac{x^\mu}{(x, x)}$$

defined for all points $x \in \mathbf{M}$ such that $(x, x) \neq 0$.

The first two transformations are isometries of $\mathbf{M}$, the group generated by them is called the Poincarè group. The last two type of transformations are conformal isometries. the group generated by all the above transformations is called the Conformal group. Let us list the Killing and conformal Killing vector fields which generate the above transformations.

i. The generators of translations in the x^μ directions, $\mu = 0, 1, ..., n$:

$$T_\mu = \frac{\partial}{\partial x^\mu}$$

ii. The generators of the Lorentz rotations in the (μ, ν) plane:

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

iii. The generators of the scaling transformations:

$$S = x^\mu \partial_\mu$$

iv. The generators of the inverted translations:

$$K_\mu = 2x_\mu (x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho) \frac{\partial}{\partial x^\mu}$$

Theorem 4.7. *Any conformal Killing vector field in $\mathbf{M}^{n+1}$, for $n > 1$ is a linear combination, with real constant coefficients, of the vector fields T, L, S and K. If $n = 1$ all conformal vector fields are given by the formula*

$$f(t+x)(\partial_t + \partial_x) + g(t-x)(\partial_t - \partial_x)$$

where f, g are arbitrary smooth, functions of one variable.

4.8. **Generalized energy estimates.** Consider the wave equation $\square\phi = 0$ in the flat Minkowski space $\mathbb{R}^{n+1}$ and its associated energy meomentum tensor,

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m_{\alpha\beta}(m^{\mu\nu}\partial_\mu\phi\partial_\nu\phi).$$

Proposition 4.9. *The energy momentum tensor is symmetric and divergenceless i.e.,*

$$\partial^\beta Q_{\alpha\beta} = 0.$$

Also, for any future, timelike, vectorfields X, Y we have $Q(X,Y) \geq 0$.

Proof The first two statements are obvious. To check the third consider, at any point, a null pair[29] $L, \underline{L}$ spaning a plane containing X and Y. Thus X, Y are linear combinations of $L, \underline{L}$ with positive coefficents. The positivity is then an immediate consequence of the following simple but important calculation,

$$\begin{aligned} Q(L,L) &= L(\phi)^2 && (94)\\ Q(L,\underline{L}) &= |\nabla\!\!\!/\phi|^2 && (95)\\ Q(\underline{L},\underline{L}) &= L(\phi)^2 && (96) \end{aligned}$$

where $|\nabla\!\!\!/\phi| = \sum_A |e_A(\phi)|^2$ with $(e_A)_{A=1,\dots,n-1}$ an orthonormal frame spanning the orthogonal complement of $L, \underline{L}$. ■

Let X be an arbitrary vectorfield and consider the spacetime momentum vectorfield associated to it $P_\alpha = Q_{\alpha\beta}X^\beta$. Using the symmetry and the divergenceless properties of Q along with the definition of the deformation tensor $\pi = {}^{(X)}\pi$ we infer that,

$$\partial_\alpha P^\alpha = \frac{1}{2}Q^{\alpha\beta}\pi_{\alpha\beta}.$$

Thus, if X is a Killing vectorfield, we have $\partial_\alpha P^\alpha = 0$ which leads by integration to the familiar conservation laws. Thus, in particular, if we integrate in the spacetime slab $[t_0, t] \times \mathbb{R}^n$ we derive,

$$\int_{\Sigma_t} Q(X,\partial_t)dx = \int_{\Sigma_{t_0}} Q(X,\partial_t)dx \tag{97}$$

with ∂_t playing here the role of future oriented unit normal to the spacelike hypersurfaces Σ_t. Taking X to be also the Killing vectorfield ∂_t we derive the familiar

[29]two future directed linearly independent null vectors $L, \underline{L}$ verifying $< L,\ L >=< \underline{L},\ \underline{L} >= 0$ and the normalization condition $< L, \underline{L} >= -2$.

law of conservation of energy

$$\int_{\Sigma_t} Q(\partial_t, \partial_t) dx = \int_{\Sigma_{t_0}} Q(\partial_t, \partial_t) dx$$

with $Q(\partial_t, \partial_t) = \frac{1}{2}\Big(|\partial_t \phi|^2 + \sum_{i=1}^n |\partial_i \phi|^2\Big)$ the usual energy density. The laws of conservation of linear and angular momentum are also included in (97) by choosing $X = \partial_i$ or $X = L_{ij}$.

If instead of integrating the identity $\partial^\alpha P_\alpha$ over the slab $[0, T] \times \mathbb{R}^n$ we integrate over the future domain[30],

$$\mathcal{D} = \{(t, x)/|x - y| \le T - t,\ t_0 \le t \le t_1 \le T\},$$

we derive the conservation law,

$$\int_{\Sigma_{t_1}} Q(X, \partial_t) dx + \int_{\underline{C}(t_0, t_1)} Q(X, \underline{L}) = \int_{\Sigma_{t_0}} Q(X, \partial_t) dx \tag{98}$$

where $\underline{C}(t_0, t_1)$ is the exterior boundary of $\mathcal{D}$, $\underline{L} = \partial_t - \sum_i \frac{(x^i - x_0^i)}{|x - x_0|} \partial_i$ the null vectorfield normal[31] to the incoming null hypersurface $\underline{C}$ and the flux $\int_{\underline{C}(t_0,t_1)} Q(X, \underline{L}) = \int_{t_0}^{t_1} dt \int_{S_{\underline{u},t}} Q(X, \underline{L})$, with $S_{\underline{u},t}$ the sphere of intersection between $\underline{C}$, and Σ_t. In the particular case of $X = \partial_t$ both the energy integrals on Σ_t and the flux are positive which leads to a simple proof of the domain of dependence properties of the wave equation.

The procedure outlined above, leading to (97), can be extended to conformal Killing vectorfields.

Proposition 4.10. *Let $Q_{\alpha\beta} = Q_{\alpha\beta}[\phi]$ the corresponding energy momentum tensor to a solution of $\Box\phi = 0$. Let X be a conformal Killing vectorfield, i.e. $\pi = {}^{(X)}\pi = \mathcal{L}_X m = \Omega m$, and $tr\pi = m^{\alpha\beta}\pi_{\alpha\beta}$. It is easy to check that $\Box\Omega = 0$; in fact, in the particular case of $X = K_0$, $\Omega = 4(n+1)t$. Let*

$$\bar{P}_\alpha = Q_{\alpha\beta} X^\beta + \frac{n-1}{4(n+1)} tr\pi \phi \partial_\alpha \phi - \frac{n-1}{8(n+1)} \partial_\alpha (tr\pi) \phi^2.$$

We have,

$$\partial^\alpha \bar{P}_\alpha = 0.$$

Applying the proposition to the conformal Killing vectorfield $X = K := K_0$ and integrating the corresponding divergence free equation on a time slab $[t_0, t] \times \mathbb{R}^n$ we infer the following[32]:

---

[30] or domain of dependence

[31] in the sense of the spacetime Minkowski metric

[32] Part i and ii of the proposition are due to C. Morawetz [**?**]. For part iii see [**?**], pages 310–313.

Proposition 4.11. *Let* $\overline{Q}(K_0, T_0) = Q(K_0, T_0) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2$, *with* $T_0 = \partial_t$ *the unit normal to* Σ_t *and* ϕ *a solution to* $\square\phi = 0$.

i.) *The following conformal conservation law holds true,*

$$\int_{\Sigma_t} \overline{Q}(K_0, T_0) = \int_{\Sigma_{t_0}} \overline{Q}(K_0, T_0) \tag{99}$$

ii.) *Moreover we have,*

$$\int_{\Sigma_t} \overline{Q}(K_0, T_0) = \frac{1}{4}\Big(\int_{\Sigma_t} \underline{u}^2 (L'\phi)^2 + \int_{\Sigma_t} 2(t^2 + r^2)|\nabla\!\!\!/\phi|^2 + \int_{\Sigma_t} u^2 (\underline{L}'\phi)^2 \Big) \tag{100}$$

where $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, $u = t - r$, $\underline{u} = t + r$ *and* $\underline{u}L'(\phi) = \underline{u}L(\phi) + (n-1)\phi$, $u\underline{L}'(\phi) = u\underline{L}(\phi) + (n-1)\phi$.

iii.) *Also, if* $n \geq 3$, *there exists a constant* $c > 0$ *such that,*

$$\int_{\Sigma_t} \overline{Q}(K_0, T_0) \geq c\Big(\int_{\Sigma_t} \underline{u}^2 (L\phi)^2 + \int_{\Sigma_t} 2(t^2 + r^2)|\nabla\!\!\!/\phi|^2 + \int_{\Sigma_t} u^2 (\underline{L}\phi)^2 + \phi^2 \Big) \tag{101}$$

Definition 4.12. We introduce the following norms:

$$\begin{aligned} E[\phi](t) &= \int_{\Sigma_t} |\phi(t)|^2 \\ \mathcal{Q}[\phi](t) &= \int_{\Sigma_t} \overline{Q}[\phi](K_0, T_0) \\ \mathcal{E}[\phi](t) &= \int_{\Sigma_t} \Big(\underline{u}^2 |L\phi|^2 + 2(t^2 + r^2)|\nabla\!\!\!/\phi|^2 + u^2|\underline{L}\phi|^2 + |\phi|^2 \Big) \end{aligned}$$

We introduce also the higher order norms:

$$\begin{aligned} E_s[\phi](t) &= \sum_{j=0}^{s} \sum_{\Gamma_{i_1}, .., \Gamma_{i_j}} E[\Gamma_{i_1}, .., \Gamma_{i_j}\phi](t) \\ \mathcal{Q}_s[\phi](t) &= \sum_{j=0}^{s} \sum_{\Gamma_{i_1}, .., \Gamma_{i_j}} \mathcal{Q}[\Gamma_{i_1}, .., \Gamma_{i_j}\phi](t) \end{aligned}$$

with Γ any of the vectorfields

$$S = t\partial_t + \sum_i x_i\partial_i, \quad L_i = x_i\partial_t + t\partial_i, \quad \Omega_{ij} = x_i\partial_j - x_j\partial_i.$$

Remark 4.13. Observe that

$$E_{s+1}(t) \lesssim \mathcal{Q}_s(t) = \mathcal{Q}_s(0).$$

4.14. **Decay Estimates.** It is well known that solutions of the free wave equation $\Box\phi = 0$, in $\mathbb{R}^{n+1}$, decay in the uniform norm like $t^{-\frac{n-1}{2}}$ as $t \to \infty$. Moreover the decay is faster away from the outgoing characteristic directions. This type of information can be derived from explicit representation of solutions in physical or Fourier space. The goal of this section is to show how such information can be obtained without using explicit representation, just the generalized energy estimates discussed in the previous section.

Consider the canonical null pair $E_\pm = \partial_t \pm \partial_r$, as well as the angular vectorfields, $A_i = \partial_i - \frac{x_i}{r}\partial_r$. Clearly,

$$\sum_i |A_i\phi| \lesssim |\nabla\!\!\!/\phi| \lesssim \sum_i |A_i\phi|$$

. Also,

$$|\partial_r\phi| + \sum_i |A_i\phi| \lesssim |D\phi| \lesssim |\partial_r\phi| + \sum_i |A_i\phi|$$

We can also easily check the following simple algebraic identities,

$$\begin{aligned}
\frac{1}{2}\big((t+r)E_+ + (t-r)E_-\big) &= S \\
\frac{1}{2}\big((t+r)E_+ - (t-r)E_-\big) &= \sum_i \frac{x_i}{|x|}L_i \\
tA_i &= L_i - \frac{x_i}{|x|}\sum_j \frac{x_j}{|x|}L_j \\
t\Omega_{ij} &= x_iL_j - x_jL_i
\end{aligned}$$

From the first two identities we easily derive,

$$\begin{aligned}
|E_+\phi(t,x)| &\lesssim \frac{1}{t}|\Gamma\phi(t,x)| \\
|E_-\phi(t,x)| &\lesssim \frac{1}{\big|t-|x|\big|}|\Gamma\phi(t,x)|
\end{aligned} \tag{102}$$

with $|\Gamma\phi| = |S\phi| + |L\phi|$.

$$|\nabla\!\!\!/\phi(t,x)| \lesssim \frac{1}{t}|\Gamma\phi(t,x)|. \tag{103}$$

Clearly, we also have,

$$|\partial\phi(t,x)| \lesssim \frac{1}{\big|t-|x|\big|}|\Gamma\phi(t,x)|$$

or, more generally,

$$|\partial^N\phi(t,x)| \lesssim \frac{1}{\big|t-|x|\big|^N}|\Gamma^N\phi(t,x)| \tag{104}$$

where $|\Gamma^N\phi| = \sum |\Gamma_1 \dots \Gamma_N\phi|$ with $\Gamma_1, \dots, \Gamma_N$ any of the vectorfields $S, L_1, \dots L_n$.

Combining the above inequalities with the definition of our norms we derive

$$\begin{aligned} t\|E_+\phi(t)\|_{L^2} &\lesssim E_1(\phi)^{\frac12}\\ t\|\nabla\!\!\!/\,\phi(t)\|_{L^2} &\lesssim E_1(\phi)^{\frac12}\\ \|uE_-\phi(t)\|_{L^2} &\lesssim E_1(\phi)^{\frac12}\end{aligned}$$

We are now ready to prove the following:

Proposition 4.15. *Let $\Box\phi = 0$ with initial data verifying the assumptions above. Then, for all $t \geq \Lambda_0$, $s > \frac{n}{2}$,*

$$\|\phi(t)\|_{L^\infty} \lesssim \big(\frac{1}{t}\big)^{\frac{n-1}{2}} E_s[\phi](t)^{\frac12} \tag{105}$$

$$\|(1+|u|)^k\partial^k\phi(t)\|_{L^\infty} \lesssim \big(\frac{1}{t}\big)^{\frac{n-1}{2}} E_{s+k}[\phi](t)^{\frac12} \tag{106}$$

Also,

$$\begin{aligned}\|E_+\phi(t)\|_{L^\infty} &\lesssim \big(\frac{1}{t}\big)^{\frac{n+1}{2}} E_{s+1}[\phi](t)^{\frac12}\\ \|\nabla\!\!\!/\,\phi(t)\|_{L^\infty} &\lesssim \big(\frac{1}{t}\big)^{\frac{n+1}{2}} E_{s+1}[\phi](t)^{\frac12}\\ \|(1+|u|)E_-\phi(t)\|_{L^\infty} &\lesssim \big(\frac{1}{t}\big)^{\frac{n-1}{2}} E_{s+1}[\phi](t)^{\frac12}\end{aligned}$$

The proof is based on the following Lemma

Lemma 4.16. *Let $u(x)$ be a smooth, compactly supported function on $\mathbb{R}^n$, $n \geq 2$. We have,*

$$|u(x)| \leq C\frac{1}{|x|^{n-1}}\bigg(\|\partial_r u\|_{L^1} + \|(r\nabla\!\!\!/)^{n-1}\partial_r u\|_{L^1}\bigg) \tag{107}$$

As an immediate corrolary to the Lemma we deduce the following:

$$|u(x)| \leq C\frac{1}{|x|^{n-1}}\sum_{k=0}^{n-1}\|\partial_r\Omega^k u\|_{L^1} \tag{108}$$

with $\Omega^k u$ representing all derivatives of the form $\Omega_1\ldots\Omega_k u$, with Ω_i the angular momentum vectorfields $\Omega_i = \in_{ijk}\, x_j\partial_k$. We now apply the inequality (108) to the function $u(x) = \phi^2(t,x)$, for some fixed t.

$$\begin{aligned}|\phi(t,x)|^2 &\lesssim \frac{1}{|x|^{n-1}}\sum_{k=0}^{n-1}\sum_{i+j=k}\|\Omega^i\phi(t)\|_{L^2}\|\partial\Omega^j\phi(t)\|_{L^2}\\ &\lesssim \frac{1}{|x|^{n-1}}E_n[\phi]\end{aligned}$$

from where we infer that, for $|x| \geq \frac{t}{2}$,

$$|\phi(t,x)| \lesssim \left(\frac{1}{t}\right)^{\frac{n-1}{2}} E_n^{\frac{1}{2}}[\phi] \tag{109}$$

Consider now the region $|x| \leq \frac{t}{2}$. Let χ be a test function supported in the region $|x| \leq 1$ with $\chi(x) = 1$ if $|x| \leq \frac{1}{2}$. Apply to the function $u(x) = \chi(x/t)\phi^2(t,x)$ the Sobolev inequality,

$$|u(x)| \lesssim \|D^n u\|_{L^1}.$$

Thus, for $|x| \leq \frac{t}{2}$, taking into account (104) and then (31),

$$\begin{aligned} |\phi(t,x)|^2 &\lesssim \sum_{i+j+k=n} \|D^i\chi(x/t)\|_{L_\infty} \cdot \|D^j\phi(t)\|_{L^2(|x|\leq\frac{t}{2})} \cdot \|D^k\phi(t)\|_{L^2(|x|\leq\frac{t}{2})} \\ &\lesssim \sum_{i+j+k=n} t^{-i} \cdot t^{-j}\|\Gamma^j\phi(t)\|_{L^2} \cdot t^{-k}\|\Gamma^k\phi(t)\|_{L^2} \\ &\lesssim t^{-n}E_n[\phi] \end{aligned}$$

Therefore,

$$\sup_{|x|\leq\frac{t}{2}} |\phi(t,x)| \lesssim \left(\frac{1}{t}\right)^{\frac{n}{2}} E_n^{\frac{1}{2}}[\phi].$$

which together with (109) proves that

$$\|\phi(t)\|_{L^\infty} \lesssim \left(\frac{1}{t}\right)^{\frac{n-1}{2}} E^{\frac{1}{2}}[\phi]$$

as desired.

To prove Lemma 4.16 we write, in polar coordinates $x = r\xi$ with $\xi \in \mathbb{R}^{n-1}$,

$$u(r\xi) = -\int_r^\infty \partial_r u(\lambda\xi) d\lambda$$

Hence,

$$\int_{|\xi|=1} |u(r\xi)| d\sigma(\xi) \leq \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} |\partial_r u(y)| dy.$$

On the other hand, using the Sobolev inequality on the unit sphere S^{n-1},

$$|u(x)| \leq c_n \Big(\|u(r\,\cdot)\|_{L^1(S^{n-1})} + \|(r\nabla\!\!\!/)^{n-1}u(r\,\cdot)\|_{L^1(S^{n-1})} \Big)$$

which combined with the inequality above proves the desired result.

Definition 4.17. Let ϕ, ψ be two smooth functions on $\mathbb{R}^{n+1}$. We define,

$$\begin{aligned} Q_0(\phi,\psi) &= m^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi \\ Q_{\alpha\beta}(\phi,\psi) &= \partial_\alpha\phi\partial_\beta\psi - \partial_\alpha\psi\partial_\beta\phi \end{aligned}$$

In what follows we will estimate the null quadratic forms $Q_0(\phi,\psi)$ and $Q_{ij}(\phi,\psi)$ by using the decay estimates established before and appropriate decompositions of the null forms in the spirit of [28]. Assume that $\Box\phi = \Box\psi = 0$. Consider the null form $Q_0(\phi,\psi)$ and write it in the form,

$$Q_0(\phi,\psi) = -\frac{1}{2}(E_+\phi \cdot E_-\psi + E_+\phi \cdot E_-\psi) + \nabla\!\!\!/\phi \cdot \nabla\!\!\!/\psi$$

Now, in view of the results of Propositions 4.15,

$$\begin{aligned}\|Q_0(\phi,\psi)(t)\|_{L^2} &\le \frac{1}{2}\Big(\|E_+(\phi)(t)\|_{L^\infty}\|E_-(\psi)(t)\|_{L^2} \\ &+ \|E_-(\phi)(t)\|_{L^\infty}\|E_+(\psi)(t)\|_{L^2}\Big) + \|\nabla\!\!\!/\phi(t)\|_{L^\infty}\|\nabla\!\!\!/\psi(t)\|_{L^2} \\ &\lesssim \big(\frac{1}{t}\big)^{\frac{n+1}{2}} E_{n+1}[\phi]^{\frac{1}{2}} E_1[\psi]^{\frac{1}{2}}\end{aligned}$$

Similarly, to estimate $Q_{ij}(\phi,\psi)$ we write $\partial_i = A_i + \frac{x_i}{|x|}\partial_r$. Therefore,

$$Q_{ij}(\phi,\psi) = A_i\phi A_j\psi - A_j\phi A_i\psi + \partial_r\phi(\frac{x_i}{r}A_j\phi - \frac{x_j}{r}A_i)\phi$$

Thus taking into account the third formula in (32) as well as (31) we deduce the same type of estimate for Q_{ij}.

5. Complements to Lecture II

5.0.1. *Wave Maps.* A *wave map* from the Minkowski space-time into a Riemannian manifold (M,g) is a map $\phi : \mathbb{R}^{1+n} \to M$ which is a critical point with respect to compactly supported variations of the Lagrangian

$$\mathcal{L}[u] = \frac{1}{2}\int_{\mathbb{R}^{1+n}} < du, du > \, dt\, dx,$$

where $< du, du >= \sum_{\mu=0}^{n}\sum_{a,b} g_{ab}\partial_\mu u^a \partial^\mu u^b$ in local coordinates on M. The Euler-Lagrange equation for this variational problem is exactly of the form (WM), in local coordinates on M, with Γ^I_{JK} the Christoffel symbols of M in the local chart and $N = \dim M$ (see, e.g., Shatah-Struwe [58]).

5.0.2. *Maxwell-Klein-Gordon Equations.* In the following discussion, the summation convention is in effect. Greek indices are summed from 0 to n, roman indices from 1 to n. Recall that indices are raised and lowered relative to the Minkowski metric $m_{\mu\nu} = \text{diag}(-1,1,\dots,1)$. For example, $\Box = \partial^\mu\partial_\mu$ and $\Delta = \partial^j\partial_j$. We denote by i the imaginary unit.

The unknowns of the equations are a one-form $A_\mu dx^\mu$ (the gauge potential) and a scalar ϕ, both defined on the Minkowski space-time:

$$A_\mu : \mathbb{R}^{1+n} \to \mathbb{R},$$
$$\phi : \mathbb{R}^{1+n} \to \mathbf{C}.$$

The electromagnetic field is the two-form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The covariant derivative relative to the gauge potential is

$$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi.$$

We are looking for critical points of the Lagrangian

$$\mathcal{L}[A_\mu, \phi] = \int_{\mathbb{R}^{1+n}} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} \right) \, dt \, dx.$$

The corresponding Euler-Lagrange equations are

$$\partial^\mu F_{\mu\nu} = -\Im\big(\phi \overline{D_\nu \phi}\big), \tag{MKGa}$$

$$D^\mu D_\mu \phi = 0, \tag{MKGb}$$

where $\Im z$ denotes the imaginary part of z.

Let χ be a real-valued function on $\mathbb{R}^{1+n}$, and consider the transformation $(A_\mu, \phi) \to (\widetilde{A}_\mu, \widetilde{\phi})$ given by

$$\widetilde{A}_\mu = A_\mu - \partial_\mu \chi,$$
$$\widetilde{\phi} = e^{i\chi} \phi.$$

Clearly, the electromagnetic field is left unchanged by the gauge transformation $A_\mu \to \widetilde{A}_\mu$, and a simple calculation reveals that if (A_μ, ϕ) verifies (MKG), then so does $(\widetilde{A}_\mu, \widetilde{\phi})$ (keep in mind that D_μ depends on A_μ). This gives an equivalence relation on the set of pairs (A_μ, ϕ) verifying (MKG), and by a *solution* of the latter, we understand an equivalence class of such pairs.

Thus, we have gauge freedom; i.e., we are free to choose any representative of a given solution (equivalence class), and we may stipulate a condition that the gauge potential should satisfy. The traditional gauge conditions are:

- *Lorentz:* $\partial^\mu A_\mu = 0$,
- *Coulomb:* $\partial^j A_j = 0$,
- *Temporal:* $A_0 = 0$.

(MKG) in Lorentz gauge. Coupling the Lorentz condition with (MKG) yields the system

$$\Box A_\mu = -\Im\big(\phi \overline{\partial_\mu \phi}\big) + |\phi|^2 A_\mu, \tag{110a}$$

$$\Box \phi = -2i A^\mu \partial_\mu \phi + A^\mu A_\mu \phi, \tag{110b}$$

$$\partial^\mu A_\mu = 0. \tag{110c}$$

Now observe that if (A_μ, ϕ) satisfies (110a) and (110b) with initial data

$$A_\mu|_{t=0} = a_\mu, \qquad \partial_t A_\mu|_{t=0} = b_\mu, \tag{111a}$$

$$\phi|_{t=0} = \phi_0, \qquad \partial_t \phi|_{t=0} = \phi_1 \tag{111b}$$

satisfying the constraints

$$b_0 = \partial^j a_j, \qquad \Delta a_0 - |\phi_0|^2 a_0 = \partial^j b_j - \Im(\phi_0 \overline{\phi_1}), \tag{112}$$

then (110c) is automatically satisfied. For by (110a) and (110b), $u = \partial^\mu A_\mu$ solves

$$\Box u = |\phi|^2 u,$$

and by (111a) and (112), $u|_{t=0} = \partial_t u|_{t=0} = 0$. By uniqueness of solutions, $u = 0$.

Thus, (110c) is equivalent to the constraint (112) on the initial data, so we are left with (110a) and (110b). Therefore, (MKG) in Lorentz gauge is schematically of the form $\Box u = u\partial u + u^3$. Unfortunately, generic equations of this type do not have good local regularity properties, so the Lorentz gauge is not very useful for our purposes.

(MKG) in Coulomb gauge. Coupling the Coulomb condition with (MKG) gives

$$\Delta A_0 = -\Im\big(\phi\overline{\partial_t\phi}\big) + |\phi|^2 A_0, \tag{113a}$$

$$\Box A_j = -\Im\big(\phi\overline{\partial_j\phi}\big) + |\phi|^2 A_j - \partial_j\partial_t A_0, \tag{113b}$$

$$\Box\phi = -2iA^j\partial_j\phi + 2iA_0\partial_t\phi + i(\partial_t A_0)\phi + A^\mu A_\mu\phi, \tag{113c}$$

$$\partial^j A_j = 0. \tag{113d}$$

Here we have split the gauge potential into its time component A_0 and its spatial component $A = A_j dx^j$. We prescribe initial data at time $t = 0$:

$$A_j|_{t=0} = a_j, \qquad \partial_t A_j|_{t=0} = b_j, \tag{114a}$$

$$\phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1. \tag{114b}$$

No initial condition is imposed on A_0; if we set $a_0 = A_0|_{t=0}$, then by (113a), $\Delta a_0 - |\phi_0|^2 a_0 = -\Im(\phi_0\overline{\phi}_1)$.

Equation (113d) is automatically satisfied if the data are divergence-free:

$$\partial^j a_j = \partial^j b_j = 0. \tag{115}$$

For if (A_0, A, ϕ) satisfies (113a)–(113c), then $u = \partial^j A_j$ solves $\Box u = |\phi|^2 u$, and if (114) and (115) are satisfied, then $u|_{t=0} = \partial_t u|_{t=0} = 0$.

We are then left with the equations (113a)–(113c). The first of these, being an elliptic equation, is relatively easy to handle, so we leave it out of our model equations. The two remaining equations have terms of three types on the right hand side:

- "Elliptic terms" involving A_0; these are collectively denoted by $\mathcal{E}$.
- Cubic terms in A_j and ϕ; these are collectively denoted by $\mathcal{C}$.
- Quadratic terms with a null-form structure.

The terms falling into the latter category are $-\Im\big(\phi\overline{\partial_j\phi}\big)$ and $-2iA^j\partial_j\phi$. We now uncover the null-form structure inherent in these expressions (due to the Coulomb condition).

Split ϕ into its real and imaginary parts: $\phi = u + iv$. Then

$$-\Im\big(\phi\overline{\partial_j\phi}\big) = u\partial_j v - v\partial_j u,$$

so (113b) reads, as an equation of (time-dependent) one-forms on $\mathbb{R}^n$:

$$\square A = udv - vdu + \mathcal{C} - \lceil(\partial_\sqcup \mathcal{A}_\prime).$$

Apply d to both sides:

$$\square(dA) = 2du \wedge dv + d\mathcal{C}.$$

But

$$du \wedge dv = \frac{1}{2} Q_{jk}(u,v) dx^j \wedge dx^k,$$

whence

$$\square F_{jk} = Q_{jk}(u,v) + \partial \mathcal{C}.$$

The Coulomb gauge condition implies that $\partial^k F_{jk} = -\Delta A_j$, so we have

$$-\Delta \square A_j = \partial^k Q_{jk}(u,v) + \partial^2 \mathcal{C}.$$

Thus, modulo Riesz operators,

$$\square A = D^{-1} Q(\Re\phi, \Im\phi) + \mathcal{C}, \tag{116}$$

where Q is some linear combination of the null forms[33] Q_{jk}. Since the cubic term $\mathcal{C}$ is easier to estimate, we leave it out of our model problem.

Now consider equation (113c). Separating real and imaginary parts, we have

$$\square u = 2A \cdot \nabla v + \mathcal{C} + \mathcal{E},$$
$$\square v = -2A \cdot \nabla u + \mathcal{C} + \mathcal{E}.$$

(Here we consider A as a vector field by raising its indices; ∇ denotes the gradient in the space variables.) We claim that the terms $A \cdot \nabla u$ and $A \cdot \nabla v$ have a null-form structure, due to the fact that A is divergence-free (by the Coulomb condition). Let B_{jk} be the unique solution of

$$\Delta B_{jk} = \partial_j A_k - \partial_k A_j \tag{117}$$

(with appropriate regularity assumptions). By the Coulomb condition,

$$\Delta \partial^j B_{jk} = \Delta A_k, \quad \text{which implies} \quad \partial^j B_{jk} = A_k. \tag{118}$$

Thus,

$$A \cdot \nabla u = \partial^j B_{jk} \partial^k u = \frac{1}{2} Q_{jk}(u, B^{jk}).$$

The above equations for $u = \Re\phi$ and $v = \Im\phi$ can therefore be rewritten

$$\square \Re\phi = Q_{jk}(\Im\phi, B^{jk}) + \mathcal{C} + \mathcal{E},$$
$$\square \Im\phi = Q_{jk}(B^{jk}, \Re\phi) + \mathcal{C} + \mathcal{E}.$$

But in view of (117), B is of the form $D^{-1}A$ modulo Riesz operators. Combining this with (116) and discarding the terms $\mathcal{C}$ and $\mathcal{E}$ throughout, we obtain a system of the form ("MKG"), which is our model for (MKG).

[33] To be precise, the j-th component of Q is $\sum_k R_k Q_{jk}$, where $R_k = D^{-1}\partial^k$ is the k-th Riesz operator. Since we work with norms which only depend on the size of the Fourier transform, we ignore the Riesz operators.

5.0.3. *Yang-Mills Equations.* Let G be one of the classical, compact Lie groups of matrices (such as $SO(k,\mathbb{R})$ or $SU(k,\mathbf{C})$), and let $\mathfrak{g}$ be its Lie algebra. The unknown is a $\mathfrak{g}$-valued one-form $A_\mu dx^\mu$ on $\mathbb{R}^{1+n}$. The corresponding covariant derivative is

$$D_\mu H = \partial_\mu H + [A_\mu, H],$$

where H is any $\mathfrak{g}$-valued tensor field on $\mathbb{R}^{1+n}$ and $[\cdot,\cdot]$ is the matrix commutator.

The curvature is the $\mathfrak{g}$-valued two-form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Lagrangian is

$$\mathcal{L}[A_\mu] = -\frac{1}{4}\int < F_{\mu\nu}, F^{\mu\nu} > \,dt\,dx,$$

where $< \cdot,\cdot >$ is the inner product on $\mathfrak{g}$ inherited from the ambient space (e.g., $SO(k,\mathbb{R})$ embeds in $\mathbb{R}^{k^2}$, so its Lie algebra can be viewed as a subspace of the latter). The Euler-Lagrange equations are

$$D^\nu F_{\mu\nu} = 0. \tag{YM}$$

Let O be a G-valued function on $\mathbb{R}^{1+n}$. Consider the gauge transformation $A_\mu \to \widetilde{A}_\mu$, given by

$$\widetilde{A}_\mu = O A_\mu O^{-1} - \partial_\mu O O^{-1}.$$

A calculation shows that the curvature then transforms into

$$\widetilde{F}_{\mu\nu} = O F_{\mu\nu} O^{-1}.$$

Denoting by $\widetilde{D}_\mu$ the covariant derivative corresponding to $\widetilde{A}_\mu$, we then have

$$\widetilde{D}^\nu \widetilde{F}_{\mu\nu} = O D^\nu F_{\mu\nu} O^{-1},$$

so (119) is invariant under gauge transformations. We therefore have gauge freedom, and may impose a gauge condition on A_μ.

(YM) in Coulomb gauge. Relative to the Coulomb condition $\partial^j A_j = 0$, (119) takes the form (see [34])

$$\Delta A_0 = 2[\partial^j A_0, A_j] + [A^j, \partial_t A_j] + [A^j, [A_0, A_j]], \tag{119a}$$

$$\begin{aligned}\Box A_j + \partial_t \partial_j A_0 = &-2[A^k, \partial_k A_j] + [A^k, \partial_j A_k] + [\partial_t A_0, A_j] + 2[A_0, \partial_t A_j] \\ &- [A_0, \partial_j A_0] - [A^k, [A_k, A_j]] + [A_0, [A_0, A_j]],\end{aligned} \tag{119b}$$

$$\partial^j A_j = 0. \tag{119c}$$

Unfortunately, assuming the existence of a global Coulomb gauge forces a restrictive smallness assumption on the initial data. In [34] this difficulty was resolved by using local arguments. Following [47], we ignore this complication, and derive our model equation from the system (119).

As in the discussion of (MKG), (119c) reduces to a constraint on the initial data. The equation for A_0 is elliptic, so we ignore it. As for (119b), we only retain the first two terms on the right, since the other terms either involve A_0 (for which we expect to have better estimates than for A_j), or are cubic.

Now write (119b) as an equation of time-dependent, $\mathfrak{g}$-valued one-forms on $\mathbb{R}^n$ (ignoring all but the first two terms on the right):

$$\Box A + d(\partial_t A_0) = S + T,$$

where $A = A_j dx^j$, $S = -2[A^k, \partial_k A_j]dx^j$ and $T = [A^k, \partial_j A_k]dx^j$. Apply the exterior derivative d to both sides:

$$\Box dA = dS + dT.$$

Let B be the two-form (in this case $\mathfrak{g}$-valued) determined by equation (117). Thus

$$\Box \Delta B_{jk} = \partial_j S_k - \partial_k S_j + \partial_j T_k - \partial_k T_j.$$

By (118), it follows that

$$-\Box \Delta A_j = \partial^k \left(\partial_j S_k - \partial_k S_j + \partial_j T_k - \partial_k T_j\right),$$

so for the purposes of estimates in frequency space, we may replace (119b) by

$$\Box A = S + D^{-1} dT. \tag{120}$$

It remains to identify the null form structure hidden in S and dT. To begin with, we have

$$\begin{aligned} S_j = [\partial_k B^{kl}, \partial_l A_j] &= \frac{1}{2}[\partial_k B^{kl} - \partial_k B^{lk}, \partial_l A_j] \\ &= \frac{1}{2}\left([\partial_k B^{kl}, \partial_l A_j] - [\partial_l B^{kl}, \partial_k A_j]\right) \\ &= \frac{1}{2}\left(\partial_k B^{kl} \partial_l A_j - \partial_l B^{kl} \partial_k A_j + \partial_k A_j \partial_l B^{kl} - \partial_l A_j \partial_k B^{kl}\right), \end{aligned}$$

so each entry of the matrix S_j is a linear combination of terms of the form $Q_{kl}(A, B)$, where A and B stand for any two entries of A_j and B_{kl}. But by (118), we may replace B by $D^{-1}A$. Schematically,

$$S \sim Q(A, D^{-1}A). \tag{121}$$

Now consider the one-form T. We calculate:

$$\begin{aligned} (dT)_{jk} &= \partial_j [A^l, \partial_k A_l] - \partial_k [A^l, \partial_j A_l] \\ &= [\partial_j A^l, \partial_k A_l] - [\partial_k A^l, \partial_j A_l]. \end{aligned}$$

Thus, each entry of the matrix $(dT)_{jk}$ is a linear combination of terms of the form $Q_{jk}(A, A')$, where A and A' stand for any two entries of A_l, $1 \le l \le n$. Combining this with (121) and (120), we arrive at the model ("YM") for the Yang-Mills equations.

6. Complements to Lecture III

An important corollary of this result is the following[34]:

[34] This was recently given a direct proof in [40]

Theorem 6.1. *For any assymptotically flat initial data set $(\Sigma, \bar{g}, \bar{k})$ with maximal future development $(\mathcal{M}, g)$ one can find a suitable domain Ω_0 with compact closure in Σ such that the boundary $\mathcal{D}_I^+$ of its domain of influence*[35] *$\mathcal{C}^+(\Omega_0)$ in $\mathcal{M}$ has complete null geodesic generators*[36].

The above Corollary can be used to introduce the concept of complete future null infinity[37]:

Definition 6.2. We say that the maximal future development $(\mathcal{M}, g)$ of an AF initial data set $(\Sigma, \bar{g}, \bar{k})$ possesses as complete future null infinity if, for any positive real number A, we can find a domain Ω containing the set Ω_0 of Theorem 1.2 such that the boundary $\mathcal{D}^-$ of the domain of dependence of Ω in $\mathcal{M}$ has the property that each of its null geodesic generators has a total affine length, measured from $\mathcal{D}^- \cap \mathcal{D}_0^+$, of at least A.

The unavoidable presence of singularities, for sufficiently large initial data sets, has led Penrose to formulate two conjectures which go under the name of the weak and strong cosmic censorhip conjectures.

Conjecture 1 [*Weak Cosmic Censorship Conjecture* (WCC)] *Generic asymptotically flat initial data have maximal future developments possesing a complete future null infinity.*

The WCC conjecture does not preclude the possibility that singularities may be visible by local observers. This could lead to the paradoxical situation of lack of unique predictability of outcomes of observations made by such observers. The strong cosmic censorship was designed to forbid such undesirable feature of singularities.

Conjecture 2 [*Strong Cosmic Censorship*] *Generic initial data*[38] *sets have maximal future developments which are locally inextendible, in a continuous manner, as Lorentz manifolds.*

In more technical terms this means that, disregarding some possible exceptional initial conditions, the maximal future development of an initial data set is such that along any future, inextendible, timelike geodesic of finite length[39], the space-time curvature components, expressed relative to a parallel transported orthonormal frame along the geodesic, must become infinite as the value of the arclength tends to its limiting value.

[35] called also the causal future of Ω

[36] with respect to the corresponding affine parameter

[37] This concept is usually defined in the GR literature through the concept of a regular conformal compactification of a spacetime, by attaching a boundary at infinity. The definition given here, due to [10], avoids the technical issue of the precise degree of smoothness of the compctification.

[38] Not necessarily assymptotically flat. This conjecture has been often discussed in the context of comapct initial data sets.

[39] i.e. bounded proper time.

6.3. **Reduction of Theorem 3.10 to Theorem 3.12.** In what follows we describe the reduction of Theorem 3.10, to which we shall now refer as Theorem (**A**), to the proof of the decay estimate of Theorem 3.12 which will be refered as Theorem (**B**). The first five steps are now standard, see [2] and especially [72]. The reduction to the last step is typical to the geometric approach of [29].

6.4. **Step 1 Energy estimates.** As we have already noted above the energy estimates for our quasilinear wave equation imply that the $L^\infty_{[0,T]}H^s$ norm of a solution ϕ is controlled by the H^s norm of the initial data $\phi[0]$ provided that $\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x} \le 1$. The next proposition is a more precise version of this statement.

Proposition 6.5 (Energy estimate). *Let $\phi \in C([0,T],H^s)\cap C^1([0,T],H^{s-1})$ be a solution of (67) on the time interval $[0,T]$ for some $s \ge 1$ obeying the condition that $\|\phi\|_{L^\infty_{[0,T]}L^\infty_x} \le \Lambda_0$. Then ϕ verifies the following energy estimate.*

$$\|\phi\|_{L^\infty_{[0,T]}\dot H^s} \le C(\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x}, \Lambda_0)\|\phi[0]\|_{\dot H^s}. \tag{122}$$

Here cosntant $C(a,b)$ denotes the dependence on a, b and $\dot H^s$ are the usual homogeneous Sobolev spaces.

6.6. **Step 2 Reduction to the Strichartz type estimates.** By the Cauchy-Schwartz inequality, $\|\partial\phi\|_{L^1_{[0,T]}L^\infty_x} \le T^{\frac12}\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x}$. Thus the Strichartz inequality (71) and the smallness of the time interval $[0,T]$ can be used to close the energy estimates in the space H^{s_*}. This effectively yields the desired local well posedness of problem. Theorem (**A**) can be then reduced to the following bootstrap argument.

Theorem 6.7 (A1). *Let $\phi \in C([0,T],H^{s_*})\cap C^1([0,T],H^{s_*-1})$ be a solution of (67) on the time interval $[0,T]$, $T \le 1$. Assume that*

$$\|\phi\|_{L^\infty_{[0,T]}H^{s*}} + \|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le B_0, \tag{123}$$

with the constant $B_0 \le c_{s_*}^{-1}\Lambda_0$, where c_{s_*} is the Sobolev constant of the embedding $H^{s_*} \subset L^\infty$. Then ϕ satisfies the local in time Strichartz type estimate,

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le C(B_0)T^{s_*-s_0}\|\phi\|_{L^\infty_{[0,T]}H^{s*}}. \tag{124}$$

6.8. **Step 3 The dyadic version of the Strichartz type estimate and the paradifferential approximation.** For the purpose of proving the Strichartz type estimate (124) we may regard the quasilinear problem (67) as a linear wave equation for ϕ with rough coefficients. It is advantageous to mollify the coefficients and work with the family of linear wave equations with smooth coefficients dependent on a parameter. First we introduce functions ϕ^λ obtained by restricting ϕ to the dyadic piece of frequency $\sim \lambda$ in Fourier space. More precisely, let ζ be a smooth function with support in the shell $\{\xi : \frac12 \le |\xi| \le 2\}$. Here, ξ denotes the variable of the spatial Fourier transform. Let ζ also satisfy the condition $\sum_{k\in\mathbb{Z}}\zeta(2^k\xi) = 1, \quad \forall \xi \in \mathbb{R}^{\not\Vdash}/\{\not\vdash\}$. Let λ be a dyadic parameter $\lambda = 2^k$ with some $k \in \mathbb{Z}$ and denote by P_λ the "projector"

$$P_\lambda f(x) = f^\lambda(x) = \int e^{-ix\cdot\xi}\zeta(\lambda^{-1}\xi)\hat f(\xi)\,d\xi.$$

Define also

$$f_{\le\lambda} = S_\lambda f = \sum_{\mu\le\lambda} P_\mu f.$$

Theorem (**A1**) then follows from the following dyadic version of the Strichartz type estimates for $\phi^\lambda = P_\lambda\phi$.

Theorem 6.9 (**A2**). *Let ϕ be as in Theorem (**A1**). Fix a large parameter Λ. Then for each $\lambda\ge\Lambda$, the function ϕ^λ satisfies the Strichartz type estimate*

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \le C(B_0)\, c_\lambda T^{s_*-s_0}\|\phi\|_{L^\infty_{[0,T]}H^{s_*}}, \tag{125}$$

for constants c_λ such that $\sum_\lambda c_\lambda^2 \le 1$. A similar estimate also holds for $\phi_{\le\Lambda}$.

Remark (**A2**) *In the case of the low frequencies, the estimate (125) for $\phi_{\le\Lambda}$ follows trivially from the Sobolev inequality.*

$$\|\partial\phi_{\le\Lambda}\|_{L^2_{[0,T]}L^\infty_x} \le c\,T^{\frac12}\|\phi_{\le\Lambda}\|_{L^\infty_{[0,T]}H^{\frac52+\epsilon}} \le c\,\Lambda^{\frac52+\epsilon-s_*}T^{\frac12}\|\phi\|_{L^\infty_{[0,T]}H^{s_*}},$$

where c is the norm of the embedding $H^{\frac32+\epsilon}(\mathbb{R}^3)\subset L^\infty(\mathbb{R}^3)$. Since s_* is assumed to be sufficiently close to $s_0 = 2+\frac12(2-\sqrt3)$ and Λ is a fixed large parameter which could depend only upon B_0, we have the desired bound for the low frequency part of ϕ.

We restrict the attention to the large frequencies $\lambda\ge\Lambda$. In the next proposition we show that each ϕ^λ satisfies an inhomogeneous wave equation with the smooth metric $g^{ij}_{\lambda^a} = S_{\lambda^a}g^{ij}(S^a_\lambda\phi)$ for any fixed value of the parameter $a\in[0,1]$.

Proposition 6.10. *Let ϕ be as in Theorem (**A1**). Fix the value of the parameter a, $a\in[0,1]$. Then for each $\lambda\ge\Lambda$, ϕ^λ verifies the equation*

$$\begin{aligned}\square_{g_{\le\lambda^a}}\phi^\lambda &= -\partial_t^2\phi^\lambda + g^{ij}_{\le\lambda^a}\partial_i\partial_j\phi^\lambda = R^a_\lambda,\\ \phi^\lambda|_{t=0} &= \phi^\lambda_0,\quad \partial_t\phi^\lambda|_{t=0} = \phi^\lambda_1.\end{aligned} \tag{126}$$

Furthermore, the Fourier support of the right-hand side R^a_λ is contained in the set $\{\xi:\ \lambda\le|\xi|\le 4\lambda\}$, and for all real $s>1$ and an arbitrary $t\in[0,T]$

$$\|R_\lambda(t)\|_{\dot H^s} \le c\,\lambda^s\|R_\lambda\|_{L^2_x} \lesssim C(B_0)\lambda^{1-a}c_\lambda\,\|\partial\phi\|_{L^\infty_x}\|\phi\|_{\dot H^s} \tag{127}$$

with the constants c_λ: $\sum_\lambda c_\lambda^2\le 1$.

6.11. **Step 4 Strichartz estimate on the frequency dependent intervals.** This step reduces the proof of Theorem (**A**) to the proof of the precise [40] Strichartz estimate for the linearized equation $\square_{g_{\lambda^a}}\psi = 0$ on small time interval I of size $\approx T\lambda^{-(1-a)}$. The loss of regularity in the final Strichartz estimate follows then as a result of summing these sharp Strichartz estimates over intervals I in $[0,T]$.

Theorem 6.12 (**A3**). *Let $\lambda\ge\Lambda$ and let ψ be a solution of the linear wave equation $\square_{g_{\lambda^a}}\psi = 0$ with initial data $\psi[0]$ such that the supp $\hat\psi[0]\in\{\frac12\lambda\le|\xi|\le 2\lambda\}$ and the metric g_{λ^a} is as defined in Proposition 6.10. Fix the value of the parameter a, $a = \frac{5-2s_*}{1-2s_*+2s_0} < -1+\sqrt3$. Then there exists a partition $\{\,I\,\}$ of the interval $[0,T]$*

[40] without losses as in the flat case

into subintervals I such that the size of each I, $|I| \le T\lambda^{-(1-a)}$, the number of the subintervals is approximately λ^{1-a}, and on each I, ψ satisfies a Strichartz estimate with a fixed, arbitrary small $\epsilon > 0$,

$$\|P_\lambda \partial\psi\|_{L_I^2 L_x^\infty} \le C(B_0)\,|I|^\epsilon \|\psi[0]\|_{\dot H^{2+\epsilon}} \tag{128}$$

Remark (A3) *According to the bootstrap assumption (123) the solution ϕ of the quasilinear problem verifies the estimate $\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \le B_0$. It easily follows that there exists a subpartition $\{\,I\,\}$ of the time interval $[0,T]$, with the total number of the subintervals I between λ^{1-a} and $2\lambda^{1-a}$ and the size of each I bounded by $T\lambda^{-(1-a)}$, such that on each I we have*

$$\|\partial\phi\|_{L_I^2 L_x^\infty} \le \lambda^{-\frac{1-a}{2}}\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \tag{129}$$

*This construction defines the subpartition $\{\,I\,\}$ mentioned in Theorem (**A3**).*

6.13. **Properties of the metric g_{λ^a}.** Inequality (128) is the Strichartz estimate for a solution of the wave equation with variable coefficients (metric) $\square_{g_{\le\lambda^a}}\psi = 0$ on the frequency dependent intervals I.

The metric $g_{\le\lambda^a} = S_{\lambda^a} g(S_{\lambda^a}\phi)$ depends upon the solution ϕ of the quasilinear problem. In the next proposition we state the properties of the family $g_{\le\lambda^a}$ which follow from the bootstrap condition (123) on ϕ and the construction of the partition $\{\,I\,\}$ described in the Remark (**A3**).

Proposition 6.14. *Let $\phi \in C([0,T], H^{s_*}) \cap C^1([0,T], H^{s_*-1})$ be a solution of (67) on the time interval $[0,T]$, $T \le 1$. Assume that ϕ verifies the assumption (123) of Theorem (**A1**). Consider the subpartition $\{\,I\,\}$ of the time interval $[0,T]$ as defined by Remark (**A3**). Then the family of metrics $g_{\le\lambda^a} = S_{\lambda^a} g(S_{\lambda^a}\phi)$ obeys the following conditions:*

For all subintervals I and all nonnegative integers m

$$\|\partial^{1+m} g_{\le\lambda^a}\|_{L_I^1 L_x^\infty} \le \lambda^{-(1-a)+am}\bar B_0, \tag{130}$$

$$\|\partial^{1+m} g_{\le\lambda^a}\|_{L_I^2 L_x^\infty} \le \lambda^{-\frac{1-a}{2}+am}\bar B_0, \tag{131}$$

$$\|\partial^{1+m} g_{\le\lambda^a}\|_{L_I^\infty L_x^\infty} \le \lambda^{\frac{a^2}{2}+am}\bar B_0, \tag{132}$$

$$\|\partial_x^{\frac12+m}(\partial^2 g_{\le\lambda^a})\|_{L_I^\infty L_x^2} \le \lambda^{\frac{a^2}{2}+am}\bar B_0, \tag{133}$$

$$\|\partial^m \square_{g_{\le\lambda^a}} g_{\le\lambda^a}\|_{L_I^1 L_x^\infty} \le \lambda^{-(1-a)+am}\bar B_0. \tag{134}$$

The constant $\bar B_0$ depend only on the constants M_0 and B_0.

Remark: *Observe that by the construction the frequencies of the metric $g_{\le\lambda^a}$ are truncated above λ^a only with respect to the Fourier variable dual to the spatial variable x. Therefore, each differentiation with respect to x introduces an additional* *factor of at most λ^a. However, using the fact that $g_{\le\lambda^a}$ depends on the solution of the wave equation, we can make the same conclusion for the time derivatives*

Theorem (**A3**) can be recast[41] as a result concerning local Strichartz estimates, on a fixed small subinterval I, for solutions to a linear wave equation with the background metric $g_{\le\lambda^a}$ satisfying the estimates of Proposition 6.14.

Theorem 6.15 (**A4**). *Let ψ be a solution of the linear wave equation*

$$\begin{aligned} &\Box_{g_{\le\lambda^a}}\psi = -\partial_t^2\psi + g^{ij}_{\le\lambda^a}\partial_i\partial_j\psi = 0,\\ &\psi|_{t=0} = \psi_0, \quad \partial_t\psi|_{t=0} = \psi_1 \end{aligned} \tag{135}$$

on the time interval I of length $|I| \le \lambda^{-(1-a)}$ with initial data $\psi[0]$ supported on the set $\{\xi : \frac{1}{2}\lambda \le |\xi| \le 2\lambda\}$ in Fourier space. Assume that the metric g_{λ^a} verifies (130)-(134) of Proposition 6.14 with the parameter a chosen such that $a < -1+\sqrt{3}$. Let P_λ be the projection on the set $\{\xi : \frac{1}{2}\lambda \le |\xi| \le 2\lambda\}$ in Fourier space. Then for a sufficiently large parameter Λ, all dyadic $\lambda \ge \Lambda$ and a fixed $\epsilon > 0$,

$$\|P_\lambda\,\partial\psi\|_{L^2_I L^\infty_x} \le C(\bar{B}_0)\,|I|^\epsilon \|\psi[0]\|_{\dot{H}^{2+\epsilon}} \tag{136}$$

with the constant $C(\bar{B}_0)$ independent of λ.

6.16. **Step 5 Rescaling.** It is convenient to replace the problem (135) by its rescaled version, so that the support of the initial data has frequencies $|\xi| \sim 1$ and the time interval I has length $\le \lambda^a$.

Translating the problem in time, if necessary, we can assume that the time interval I starts at $t = 0$. Introduce the family of the rescaled metrics h_λ

$$h_\lambda(t,x) = g_{\le\lambda^a}(\lambda^{-1}t, \lambda^{-1}x) \tag{137}$$

Proposition 6.14 implies that h_λ obeys the following estimates [42] on the time interval[43] $I = [0, t_*]$ with $t_* \le \lambda^a$:

$$\|\partial^{1+m}h_\lambda\|_{L^1_{I_\lambda}L^\infty_x} \lesssim \lambda^{-(1-a)(m+1)}, \tag{138}$$

$$\|\partial^{1+m}h_\lambda\|_{L^2_{I_\lambda}L^\infty_x} \lesssim \lambda^{-\frac{2-a}{2}-(1-a)m}, \tag{139}$$

$$\|\partial^{1+m}h_\lambda\|_{L^\infty_{I_\lambda}L^\infty_x} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m}, \tag{140}$$

$$\|\partial_x^{\frac{1}{2}+m}(\partial^2 h_\lambda)\|_{L^\infty_{I_\lambda}L^2_x} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m}, \tag{141}$$

$$\|\partial^m\bar{\Box}_{h_\lambda}h_\lambda\|_{L^1_{I_\lambda}L^\infty_x} \lesssim \lambda^{-(2-a)-(1-a)m}, \tag{142}$$

After rescaling Theorem (**A4**) transforms into

[41] We can therefore completely forget the origin of the metric $g_{\le\lambda^a}$, we only need to know (130)-(134).

[42] According to our convention $A \lesssim B$ means $A \le C\cdot B$ for some universal constant C. By the bootstrap assumption all the constants in our estimates may depend on the constant B_0. Thus we can treat B_0 as a universal constant and, in what follows, replace the dependence on it by $\lesssim$. In addition, the choice of the large frequency Λ, as in Theorem (**A3**), will be determined by the condition that $A \lesssim B$ may be replaced by $A \le \Lambda^\epsilon B$ with an arbitrary positive ϵ

[43] We keep the notation I for the rescaled time interval

Theorem 6.17 (**A5**). *Let ψ be a solution of the linear wave equation*

$$\begin{aligned} \Box_{h_\lambda}\psi &= -\partial_t^2\psi + h_\lambda^{ij}\partial_i\partial_j\psi = 0,\\ \psi|_{t=0} &= \psi_0, \quad \partial_t\psi|_{t=0} = \psi_1 \end{aligned} \tag{143}$$

on the time interval $[0, t_*]$ with $t_* \le \lambda^a$. Assume that the parameter $\lambda \ge \Lambda$ for a sufficiently large constant Λ and that the metric h_λ verifies (138)-(142) *with the parameter a such that $a < -1 + \sqrt{3}$. Let P be the operator of projection on the set $\{\xi :\ 1 \le |\xi| \le 2\}$ in Fourier space. Then*

$$\|P\,\partial\psi\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim |t_*|^\epsilon(\|\partial\psi_0\|_{L^2_x} + \|\psi_1\|_{L^2_x}) \tag{144}$$

with a constant independent of λ in the inequality $\lesssim$.

Remark: *Note that Theorem (***A5***) does not contain any assumptions on the Fourier support of the initial data $\psi[0]$.*

6.18. **Step 6 Decay estimates.** A variation of the standard TT^* type argument, see [29], allows us to reduce the Strichartz estimate (144) to a corresponding dispersive inequality, see (145). In the process we replace[44] the equation $\Box_{h_\lambda}\psi = 0$ by the *geometric* wave equation $\Box_{h_\lambda}\psi = -\frac{1}{\sqrt{\det h_\lambda}}\partial_t\sqrt{\det h_\lambda}\,\partial_t\psi + \frac{1}{\sqrt{\det h_\lambda}}\partial_i(h_\lambda^{ij}\sqrt{\det h_\lambda}\,\partial_j\psi) = 0$.

Theorem 6.19 (**A6**). *Let ψ be a solution of the linear wave equation*

$$\begin{aligned} &\Box_{h_\lambda}\psi = -\frac{1}{\sqrt{\det h_\lambda}}\partial_t\sqrt{\det h_\lambda}\,\partial_t\psi + \frac{1}{\sqrt{\det h_\lambda}}\partial_i(h_\lambda^{ij}\sqrt{\det h_\lambda}\,\partial_j\psi) = 0,\\ &\psi|_{t=0} = \psi_0, \quad \partial_t\psi|_{t=0} = \psi_1 \end{aligned} \tag{145}$$

on the time interval $[0, t_*]$ with $t_* \le \lambda^a$ and with initial data $\psi[0]$ supported in the set $\{\xi :\ \frac12 \le |\xi| \le 2\}$ in Fourier space. We consider only large values of the parameter $\lambda \ge \Lambda$. Assume that the metric h_λ verifies (138)-(142) *with the parameter a such that $a < -1 + \sqrt{3}$. Then for all $t \le t_*$ and a fixed arbitrary small $\epsilon > 0$*

$$\|P\,\partial\psi(t)\|_{L^\infty_x} \lesssim \frac{1}{(1+|t|)^{1-\epsilon}}\|\psi[0]\|_{L^1_x}. \tag{146}$$

We make the final reduction by decomposing the initial data $\psi[0]$ in the physical space into a sum of functions with essentially disjoint supports contained in balls of radius $\frac12$. Using the additivity of the L^1 norm and the standard Sobolev inequality we can reduce the dispersive inequality (146) to an $L^2 - L^\infty$ decay estimate.

Theorem 6.20 (**B**). *Let ψ be a solution of the linear wave equation* (145) *on the time interval $[0, t_*]$ with $t_* \le \lambda^a$ and with initial data $\psi[0]$ supported in the ball $B_{\frac12}(0)$ of radius $\frac12$ centered at the origin in the physical space. We fix a big constant Λ and consider only large values of the parameter $\lambda \ge \Lambda$. Assume that the metric*

[44] The two wave operators differ only by lower order terms in so far as the Strichartz estimates are concerned.

h_λ verifies (138)-(142) *with the parameter a such that $a < -1+\sqrt{3}$. Then for all $t \le t_*$, an arbitrary small $\epsilon > 0$, and a sufficiently large integer $m > 0$,*

$$\|P\,\partial\psi(t)\|_{L^\infty_x} \lesssim \frac{1}{(1+|t|)^{1-\epsilon}} \sum_{k=1}^{m} \|\partial^k\psi[0]\|_{L^2_x}. \tag{147}$$

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