Recent developments on the Kakeya and restriction problems

Notes for IPAM Tutorial, Mar 19-23
Oscillatory integrals and Dispersive Equations

Terence Tao (UCLA)

`www.math.ucla.edu/~tao/preprints` (in preparation)

Abstract: We survey the development of ideas and some recent results on the interconnected Kakeya and restriction problems. The first lecture shall focus mostly on Kakeya, the second on restriction. (These conjectures are related to many other problems too, but we shall not attempt to survey them all).

What is the Kakeya conjecture?

- Let $n \geq 2$. A *Besicovitch set* is defined as a (compact) subset of $\mathbf{R}^n$ which contains a unit line segment in every direction. These sets can have measure zero (Besicovitch, 1918).

- Originally, Besicovitch was led to constructing such sets from the *Kakeya needle problem*: what is the smallest amount of area needed to rotate a unit line segment (a needle) around in the plane? Based on the above construction it is easy to show that one can rotate a needle completely using arbitrarily small area.

- Besicovitch's example in the plane, despite having Lebesgue measure 0, still had Minkowski and Hausdorff dimension 2. The *Kakeya conjecture* asserts that in fact all Besicovitch sets have dimension n.

- Recall that the *(upper) Minkowski dimension* $\dim_M(E)$ of a set $E \subset \mathbf{R}^n$ is defined as the infimum of all exponents d such that for any $0 < \delta \ll 1$, the set E can be covered by $O(\delta^{-d})$ balls of radius δ.

- The *Hausdorff dimension* $\dim_H(E)$ is defined as the infimum of all exponents d such that for any $0 < \delta \ll 1$, the set E can be covered by balls $B(x_i, r_i)$ of radius $r_i \leq \delta$ such that $\sum_i r_i^d \lesssim 1$. Clearly $\dim_H(E) \leq$

$\dim_M(E)$, so that the Minkowski form of the Kakeya conjecture is easier than the Hausdorff.

- By themselves, the Minkowski and Hausdorff forms of the Kakeya conjecture are only mathematical curiosities. However, there is a more quantitative version of the conjecture which is much better suited for applications.

- Fix $0 < \delta \ll 1$. For any function f on $\mathbf{R}^n$ and direction $\omega \in \S^{n-1}$, define the *Kakeya maximal function*

$$f^*(\omega) = \sup_{T:T//\omega} \frac{1}{|T|} \int_T |f|,$$

where T ranges over all $1 \times \delta$ tubes which are oriented in the direction ω. This operator naturally arises in harmonic analysis in connection with directional differentiation and the x-ray transform.

- Clearly $\|f^*\|_\infty \lesssim \|f\|_\infty$. One can ask whether the maximal function is also bounded on L^p uniformly in δ. If one applies the Kakeya maximal function to a characteristic function of a δ-neighborhood of Besicovitch's construction, one can show that the L^p bound must grow at least logarithmically in δ. Also by testing f to be a characteristic function of a δ-ball, we see that the L^p norm of the maximal function must be at least $\delta^{n/p-1}$ for $1 \leq p \leq n$. This leads to the *Kakeya maximal function conjecture* that

$$\|f^*\|_{L^p(S^{n-1})} \lessapprox \delta^{n/p-1} \|f\|_p$$

for all $1 \leq p \leq n$, where we use $A \lessapprox B$ to denote the inequality $A \leq C_\varepsilon \delta^{-\varepsilon} B$ for all $\varepsilon > 0$.

- This inequality is trivial for $p = 1$; the objective is to make p as large as possible. This conjecture was first formulated explicitly in this form by Bourgain in 1991, although variants of it were considered earlier by Cordoba, Christ, Rubio de Francia, Duoandikoetxea, and others. One can also consider the (L^p, L^q) mapping problem, but this is not too different the (L^p, L^p) problem.

Relation between the Minkowski, maximal, and Hausdorff conjectures.

- It is clear that the maximal conjecture is related to the other two problems - indeed, any small-dimensional Besicovitch set could be used to construct a counterexample to L^p boundedness of the Kakeya maximal function. To make the connection more precise we need some notation.

- Let $\mathbf{T}$ be a family of about δ^{1-n} tubes, such that the directions of the tubes in $\mathbf{T}$ are δ-separated. Fix $0 < \lambda \leq 1$, and for each $T \in \mathbf{T}$, let $Y(T)$ be a subset of T of density λ:

$$|Y(T)| = \lambda |T|.$$

- The Kakeya maximal function estimate at exponent p turns out to be equivalent to the estimate

$$|\bigcup_{T \in \mathbf{T}} Y(T)| \gtrapprox \lambda^p \delta^{n-p}. \tag{0.1}$$

  One direction is easy: apply the estimate to the characteristic function of $\bigcup_{T \in \mathbf{T}} Y(T)$. For the other direction, one needs to pass to a discretized restricted weak-type formulation, and also use "factorization theory" exploiting the rotation invariance. (See e.g. Bourgain, 1991). Thus the maximal conjecture, like the other two conjectures, limits the extent to which lines or tubes in distinct directions can be compressed together.

- Suppose we only knew (0.1) when $\lambda = 1$, so that $Y(T) = T$. Then (0.1) implies Besicovitch sets having Minkowski dimension at least p. (Actually, it is equivalent to these sets having *lower* Minkowski dimension at least p. To prove the bound for the upper dimension it suffices to prove (0.1) for infinitely many dyadic δ, but not necessarily all δ).

- Suppose we only knew (0.1) when $\lambda \approx 1$. Then (0.1) implies (but is not quite equivalent to) Besicovitch sets having Hausdorff dimension at least p. (The idea is to pigeonhole the balls used to cover the set into dyadic scales. By the pigeonhole principle there must exist scale δ which covers $1/\log(1/\delta)^2$ (say) of the set). In fact one can take $\lambda = 1/\log\log(1/\delta)^{1+\varepsilon}$ by concatenating some dyadic scales together.

- Thus the maximal conjecture implies the Hausdorff conjecture, which in turn implies (and is very close to) the Minkowski conjecture.

- Another equivalent (dual) formulation of (0.1) is

$$\| \sum_{T \in \mathbf{T}} \chi_T \|_{p'} \lessapprox \delta^{\frac{n}{p}-1}.$$

This form is more directly useful for applications to restriction theorems and similar problems, as we will discuss next lecture.

The geometric method

- There have been two broad approaches to obtaining progress on these Kakeya problems. The first is geometric, and relies mostly on combinatorics and basic facts in incidence geometry. The ideas here seem to extend to many other contexts. A more recent approach is arithmetic, and will be discussed later; it works very well in high dimensions for the Kakeya problem but there are few extensions of this approach to other problems as yet.

- The first argument of this type is due mainly to [Córdoba 1977], with related work by Davies and Fefferman. The key geometric ingredient is that any two lines which are not parallel can only intersect in at most one point. More quantitatively, two δ-tubes at an angle ~ 1 can only intersect in a δ-ball. (and two tubes at angle $\sim \theta$ intersect in a union of $\sim 1/\theta$ δ-balls). This is already enough to estimate

$$\| \sum_{T \in \mathbf{T}} \chi_T \|_2$$

quite accurately, allowing one to prove all three Kakeya conjectures when $p = 2$ and $n \geq 2$. From a modern viewpoint, Córdoba's argument shows that tubes in a plane which have distinct directions are essentially disjoint (up to a multiplicity of about ≈ 1).

- Córdoba's argument is not as effective in higher dimensions. This is because one expects most pairs of tubes T, T' to not intersect at all (so that estimating the intersection by a δ-ball becomes increasingly inefficient).

- Another method was developed by Christ, Rubio de Francia, Duoandikoetxea, and Drury $\sim$ 1986, and the geometric ingredient now is that any two points which are not equal have only one line between them. More quantitatively, any two δ-balls at a distance ~ 1 can only have $O(1)$ δ-tubes connecting them (and those at distance $\sim r$ can have $O(r^{1-n})$ tubes connecting them). This eventually leads (after a little work) to all three Kakeya conjectures holding true at the exponent $(n+1)/2$.

- (Actually, the above authors were not working directly on Besicovitch sets, but on the maximal function operator and the closely related problem of x-ray estimates.)

- Very roughly, the numerology leading to $(n+1)/2$ is as follows. Suppose that a Besicovitch set has dimension d. Then each line segment has codimension $d-1$ in the set. Since there are an $n-1$-dimensional family of line segments, we expect each point x_0 in the Besicovitch set to be contained in a $(n-1)-(d-1)$-dimensional family of lines. Form the union of all the lines that go through x_0 (this union is usually called a "bush"). Since any two lines intersect in at most one point, these lines are distinct away from x_0, and so this set has dimension $(n-1)-(d-1)+1$. But this bush is inside the Besicovitch set, hence

  $$(n-1)-(d-1)+1 \leq d, \text{ or } d \geq (n+1)/2.$$

  (This "bush" argument is first explicit in a 1991 paper of Bourgain).

- This argument is also a little inefficient, because not every pair of points in the Besicovitch set are joined by a line segment in the set. (For instance, Córdoba's argument prevents this from happening in two dimensions). By pursuing this idea, Bourgain was able in 1991 to improve the $(n+1)/2$ exponent slightly in every dimension. In 1995 Wolff unified Córdoba's argument with the bush argument to obtain an $(n+2)/2$ bound. The main new geometric idea was to consider not only a bush, but a larger object usually called a "brush", which is the union of all the line segments which intersect a "stem" line segment T_0 (as opposed to a single point x_0). The point is that just as in a bush, the lines in a brush are all essentially disjoint due to Córdoba's argument. (One can foliate the brush into 2-dimensional planes containing T_0, and then one applies Córdoba's argument to each plane separately). Since a brush has one higher dimension than a bush, we thus should obtain

$$(n-1) - (d-1) + 2 \leq d, \text{ or } d \geq (n+2)/2.$$

- Wolff was in fact able to obtain this bound for the maximal function as well as the dimension estimates, as well as a more technical x-ray bound. There were two technical issues that had to be dealt with for the maximal problem. One was that of small angles - the bristles brush might be almost parallel to the stem, making it smaller measure (but then one could locate several mostly disjoint brushes). The other was that many of sets $Y(T)$ might only clump on one end of T as opposed to being spread out throughout T (which creates difficulties when obtaining lower bounds for the size of the brush).

- The latter difficulty was overcome by Wolff's *two-ends reduction*. This reduction allows one to assume that the sets $Y(T)$ obey the estimate

$$|Y(T) \cap B(x,r)| \lessapprox r^\sigma \lambda |T|$$

  for some $\sigma > 0$ and all $B(x,r)$ - in other words, the $Y(T)$ must spread out over more than one end of the tube, and the average separation between two points in $Y(T)$ is ≈ 1. For if too many of the sets $Y(T)$ disobeyed this estimate, then they would cluster in an $r \times \delta$ tube rather than a $1 \times \delta$ tube. One could then rescale in a favorable manner. (This trick has since been used to good effect in other problems such as L^p estimates for averages along curves. The idea is also related to the powerful *induction on scales* strategy developed by Wolff, Bourgain, and others to handle oscillatory integral operators).

- A little while later, it was realized that the small angle issue could be almost completely eliminated. Just as the two ends reduction mostly eliminates "small distance" issues, the so-called *bilinear reduction* eliminates small angle issues. The idea is to replace the estimate

$$\|\sum_{T \in \mathbf{T}} \chi_T\|_{p'} \lessapprox \delta^{\frac{n}{p}-1}$$

by the variant

$$\|(\sum_{T\in\mathbf{T}}\chi_T)(\sum_{T'\in\mathbf{T}'}\chi_{T'})\|_{p'/2}\lesssim\delta^{2(\frac{n}{p}-1)}$$

where $\mathbf{T},\mathbf{T}'$ are two families of tubes such that the directions of $\mathbf{T}$ make an angle of ~ 1 with the directions of $\mathbf{T}'$. The former estimate implies the latter by Hölder's inequality; the converse is also true (one squares the first inequality, uses a dyadic decomposition of the angle, and uses linear transformations to rescale small angle interactions to large angle interactions); see T., Vargas, Vega, 1998.

- Because of the two reductions, we can morally assume (as a first approximation, at least) that any two tubes of interest intersect at angle ~ 1, and any two points of interest are separated by ~ 1, although in the rigorous argument things are not quite this simple.

- Other geometric arguments have been developed, for instance by Schlag, based on other incidence geometry facts. In dimensions four and higher one can improve Wolff's $(n+2)/2$ bound by a small amount, to $(n+2)/2+10^{-10}$, for the Minkowski dimension only (Laba, T., 2000); one has to study the scale $\sqrt{\delta}$ as well as δ and obtain a dichotomy, in that at least one of the scales has a slightly larger value of $|\bigcup_T Y(T)|$ than Wolff's argument allows. It is quite likely that one can do better, especially in very high dimension, although as we shall see the competing arithmetic method is already quite superior in this regime.

- However, there appears to be a limit with what one can do with thee "incidence geometry + combinatorics" paradigm, especially in three dimensions. One limitation is that these arguments do not use the structure of $\mathbf{R}$ much, and in fact work for very general fields (finite fields, $\mathbf{C}$, p-adics, etc). Also, these arguments do not really exploit the fact that all the tubes T point in different directions. If one passes to fields other than $\mathbf{R}$ and slightly relaxes the requirement that the tubes T point in different directions, then one can obtain examples which show that the $(n+2)/2$ bound is sharp in three dimensions (for instance, for the complex field one can take the Heisenberg group $\{(z_1,z_2,z_3): Im(z_3)=Im(z_1\overline{z_2})\}$).

The arithmetic method

- It is possible to view the Kakeya problem in a more arithmetic way. Let E be a d-dimensional Besicovitch set, which contains a line segment in every direction. Suppose for the sake of argument that the line segments join points $(x, 0)$ in the hyperplane $\{x_n = 0\}$ to points $(y, 1)$ in the hyperplane $\{x_n = 1\}$. Let $G \subset \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ denote the set of all pairs (x, y) which can be obtained in this manner. Since two points determine exactly one line, every line segment determines a different element of G, so (heuristically at least) G is $n-1$-dimensional. In fact, if we let $\pi_{-1} : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ be the subtraction map

$$\pi_{-1} : (x, y) \mapsto x - y$$

then $\pi_{-1}(G)$ is $n-1$-dimensional. On the other hand, since E is d-dimensional, we expect most slices $\{x \in E : x_n = t\}$ to be $d-1$ dimensional. But these slices contain the sets $\{(1-t)x+ty : (x, y) \in G\}$. Thus if we let $\pi_r(G)$ be the maps

$$\pi_r : (x, y) \mapsto x + ry$$

and $\pi_\infty : (x, y) \mapsto y$, then we expect $\pi_r(G)$ to be $d-1$-dimensional for a very large set of slopes r (but not -1).

- Thus a low-dimensional Kakeya set produces some conflict between a large range of the subtraction map π_{-1} on one hand, and a small range of other projections on the other. For instance, since π_0 and π_∞ both have a $d-1$-dimensional range, we see that G is at most $2(d-1)$-dimensional, which again proves the $d \geq (n+1)/2 = \frac{1}{2}(n-1) + 1$ bound.

- These observations are quite old (perhaps even dating to a 1969 paper of Kahane) but they could not be used to make effective progress on the Kakeya problem until 1999, in which Bourgain connected these ideas with the work of Gowers on the Balog-Szemerédi theorem in additive combinatorics. We will not go into detail here, but suffice to say that these results allow one to control the size of one projection $\pi_{-1}(G)$ non-trivially in terms of several other projections. For instance, for finite sets G Bourgain showed the bound

$$\#\pi_{-1}(G) \leq \min(\#\pi_0(G), \#\pi_1(G), \#\pi_\infty(G))^{2-\sigma}$$

for $\sigma = 1/13$; this was later improved to $\sigma = 1/6$ (Katz, T. 2000). This gives the bounds $d \geq \frac{13}{25}(n-1)+1$ and $n \geq \frac{6}{11}(n-1)+1$ respectively for the Minkowski dimension. (The Hausdorff bound is also true, but requires additional results on locating arithmetic progressions of length three in a set of density $\lambda \sim 1/\log\log(1/\delta)^{1+\varepsilon}$. One can also push these arguments to the maximal problem but at great cost in exponents). The arguments are purely combinatorial and are based on such arithmetic identities as

$$(a-b) = (a-b') - (a'-b') + (a'-b)$$

and

$$a+b = c+d \iff a-d = c-b.$$

- One could prove Kakeya if $\sigma = 1$. Unfortunately, we have the bound $\sigma \leq 0.36907\ldots$ (Katz, T., 2000), recently improved to $\sigma \leq 0.20824\ldots$ (Ruzsa 2001). However, we can circumvent this limitation by adding more projections. For instance, we have

$$\#\pi_{-1}(G) \leq \min(\#\pi_0(G), \#\pi_1(G), \#\pi_2(G), \#\pi_\infty(G))^{2-\sigma}$$

  with $\sigma = 1/4$ (Katz, T. 2000; the current upper bound is $\sigma \leq 1/2$). The best result we have of this type is with $\sigma = .32486\ldots$ (Katz, T. 2001), but this requires a huge number of projections (to get within ε of $0.32486\ldots$ requires $\exp(\exp(C\log(1/\varepsilon)^2))$ projections with our current argument). The dimension bounds thus obtained improve upon Wolff's bound in 7 and higher dimensions, but are inferior in lower dimensions. It should however be possible to pursue this method further, although it seems quite ambitious to get σ all the way down to 1. Perhaps some new combinatorial ideas are needed to make significantly more progress, though.

Hybrid techniques

- The geometric techniques seem most effective in low dimensions, while the arithmetic techniques are most effective in high dimensions. This seems to be because the geometric methods rely on the two-dimensional case, in which everything is OK by Córdoba's argument, whereas the arithmetic methods rely on slices of the Besicovitch set instead of the full Besicovitch set, thus potentially losing a dimension or so.

- A few recent results have combined both techniques to obtain some new results. For instance, in the Minkowski case Wolff's bound of $5/2$ in three dimensions can be improved, with some substantial effort, to $5/2 + 10^{-10}$ by pushing the geometric methods heavily to obtain very strong structural properties on the Besicovitch set (e.g. almost all the line segments through a point need to lie in a plane), and only then apply the arithmetic approach. Also, by running the arithmetic arguments without taking slices one can save about half a dimension in some of the above results (for instance, the $\sigma = 1/4$ argument, which would ordinarily correspond to $d \geq (4n+3)/7$, can be improved to $d \geq (4n+5)/7$ by exploiting Wolff's observation that the bristles of a brush are essentially disjoint).

- One can also use these techniques to push some of the above Minkowski and Hausdorff results to maximal function results, and perhaps even to other statements such as x-ray estimates.

Related estimates

- The ideas and techniques used for the Kakeya problem have also been applied to other problems of a similar flavor. We briefly discuss some of these.

- Kolasa, Wolff, Schlag, and Sogge have studied Kakeya-type problems in which lines are replaced by circles in the plane, building upon earlier work of Marstrand, Besicovitch, Rado, and Kinney. Several new issues arise, notably the fact that circles can be tangent to each other and thus have a quite large intersection in the δ-discretized model; this cannot be so easily eliminated by the bilinear reduction. A key combinatorial tool (already developed for discrete analogues of these problems by Szemeredi, Trotter, Clarkson, et al.) is the *cell decomposition*, in which one selects r circles at random for some relatively small r, uses them to divide the plane into about r^2 pieces or "cells", works on each piece separately, and then adds up. The reason this can be advantageous is that each circle will usually only visit about r of these cells, although there are additional difficulties coming from the fact that circles can be tangent to the walls of the cells. Thanks to this tool, the circular maximal functions in the plane have been understood quite satisfactorily (similar to how Kakeya maximal functions are satisfactorily understood in two dimensions).

- Wolff, Erdogan, and Christ have studied Kakeya problems in which the line segments are restricted to be light rays, or have slopes restricted to some generic curve in S^{n-1}. Such operators are related to some general classes of Fourier integral operators (as studied by Greenleaf, Seeger, Oberlin, and others), but are completely non-oscillatory in nature and so one expects to be able to prove good estimates for them directly without appeal to FIO theory. The techniques here are partly based on such geometric arguments as the brush argument, but also use some related developments of Christ, Wright, and myself on averaging operators along curves (in which some Kakeya techniques are combined with "iterated TT^*" methods).

- There are natural analogues of the Kakeya problem in which line segments are replaced with higher-dimensional spaces such as k-planes. Early work in this direction is by Christ in 1984; there were some improvements by Bourgain in 1991 but this is still largely an undeveloped area, and it is likely that many of the arguments above will transfer to the k-plane case.

- One can study variants of the Kakeya problem in which line segments are replaced by β-dimensional subsets of line segments for some $0 < \beta \leq 1$. Such sets arose (for unrelated reasons) in work of Furstenburg so one may term these *Furstenburg sets*. In 1996 Wolff posed the question of bounding the dimension of these Furstenburg sets; even in two dimensions this is not completely settled. It appears that a resolution of this two-dimensional problem is essential in order to progress much further on the Kakeya problem, especially in three dimensions.

- Another problem in this area is the *Falconer distance set conjecture*, which asserts that if a set E has Hausdorff dimension 1 in the plane, then its distance set $\{|x-y| : x, y \in E\}$ has dimension 1 in the real line. This conjecture has both Kakeya-type aspects and oscillatory integral aspects, and is closely related to the Furstenburg set problem as well as the *Erdös ring problem* - does there exist sub-rings of $\mathbf{R}$ of dimension

1/2? This last statement is relevant to the Kakeya problem as the existence of such a ring would allow one to construct a "Heisenberg group for $\mathbf{R}^3$". These problems have attracted interest by Falconer, Mattila, Schlag, Bourgain, Wolff, Katz, and others; some interconnections can be found in [Katz, T., 2001].

- One can also try to replace lines by other curves, such as geodesics on a Riemannian manifold. Counter-examples by Bourgain (for curves) and Sogge and Minicozzi (for geodesics) show that some of the above results do not transfer to the general curve case, but it is unclear how generic this phenomenon is, and what can be salvaged from this.

The restriction conjecture

- We now discuss the *restriction problem*, which is a typical example of an oscillatory integral problem which is related to Kakeya-type questions. (There are many other related problems - Bochner-Riesz, convergence of Schrödinger solutions, Strichartz type estimates - which we will not discuss due to lack of time.)

- Let f be an element of $L^p(\mathbf{R}^n)$ for some $1 < p \leq 2$. It is well known that the Fourier transform $\hat{f}$ need not be continuous despite being in $L^{p'}$ (by Hausdorff-Young), and hence it does not necessarily make sense to evaluate $\hat{f}$ on a point. Indeed, if f is merely in L^2, then by Plancherel $\hat{f}$ is just an arbitrary L^2 function, and cannot be evaluated meaningfully on any set of zero measure. If f is in L^p, then f cannot be evaluated on, say, a hyperplane.

- However, in 1967 Stein observed that for certain p strictly between 1 and 2, the Fourier transform of $\hat{f}$ could be meaningfully restricted to curved surfaces such as the sphere or paraboloid. The exact range of p for which this could be done was uncertain, and we quickly summarize the progress as follows.

- Work by Fefferman, Sjölin, and finally Stein and Tomas showed this was possible for $1 \leq p \leq 2(n+1)/(n+3)$, while the conjecture failed for $p \geq 2(n+1)/(n-1)$. Thus one could conjecture the restriction phenomenon to hold for $1 \leq p < 2(n+1)/(n-1)$. This was proven in two dimensions by Fefferman and Córdoba (and generalized to other oscillatory integrals by Carleson and Sjölin).

- In $\sim$ 1977 Córdoba gave a proof of the restriction conjecture in two dimensions which relied, in a large part, on the successful resolution of the Kakeya conjecture in two dimensions. Conversely, it was realized that the restriction conjecture would imply the Kakeya conjecture (this dates back to [Beckner, Carbery, Semmes, Soria, 1989] and perhaps earlier). In 1991 Bourgain showed how this method could be generalized to arbitrary dimension, so that any non-trivial progress on Kakeya would give a non-trivial restriction result. However, even if we had a complete solution to Kakeya in higher dimension, the best technology at our disposal would only give us a partial solution to restriction.

- As some idea of the current state of affairs we briefly summarize progress on the three-dimensional case for restriction to the sphere. In this case we expect the restriction phenomenon when $p' > 3$. Tomas, Stein, and Sjölin proved $p' \geq 4$. This was improved to $p' > 4 - \frac{2}{15}$ by Bourgain in 1991, then to $p' > 4 - \frac{2}{11}$ by Wolff in 1995. Work by Moyua, Vargas, Vega, and then by T., Vargas, Vega in 1998 improved this to $p' \geq 4 - \frac{2}{9}$. The current best result is $p' \geq 4 - \frac{2}{7}$ [Vargas, T. 2000]. Further progress has been made by Bourgain, Vargas, T., and most notably Wolff on the cone restriction problem, which we will discuss later. As in the Kakeya problem, progress has been aided by two useful reductions, the *bilinear reduction* and the *induction on scales* argument (the analogue of the two ends reduction for oscillatory integrals).

Duality

- To obtain a restriction phenomenon at exponent p for a surface S, we seek an estimate of the form

$$\|\hat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbf{R}^n)}$$

for all test functions f and some exponent q (the choice of exponent q is not as important as the choice of p, and q only plays a minor role). By duality this restriction estimate is equivalent to the *extension* or *adjoint restriction* estimate

$$\|\widehat{g d\sigma}\|_{L^{p'}(\mathbf{R}^n)} \lesssim \|g\|_{L^{q'}(S)}$$

for all g on the surface S, where $d\sigma$ is surface measure on S.

- Suppose we let S be the sphere $S = S^{n-1}$. Standard computations using stationary phase (or Bessel functions) give

$$|\widehat{d\sigma}(x)| \sim e^{\pm 2\pi i |x|} |x|^{-(n-1)/2} \text{ as } |x| \to \infty.$$

  This function is in $L^{p'}$ only when $p < 2n/(n+1)$, so this is therefore a necessary condition for the restriction phenomenon to hold. The *restriction conjecture* asserts that this is the only condition needed.

- In this dualized formulation we see that this problem is also connected to PDE. Functions of the form $\widehat{gd\sigma}$, when S is the sphere, are solutions to the Helmholtz equation. Similarly when S is a paraboloid, we obtain solutions to the Schrodinger equation; when S is a cone, we get solutions to the wave equation, etc.

Local restriction estimates

- The extension estimate stated above is a global estimate, requiring one to bound $\widehat{gd\sigma}$ on all of $\mathbf{R}^n$. The modern viewpoint on how to obtain these global estimates is to first prove *local* estimates of the form

$$\|\widehat{gd\sigma}\|_{L^{p'}(B(x,R))} \lesssim R^{\alpha} \|g\|_{L^{q'}(S)}$$

  for all balls $B(x, R)$ with $R \gg 1$ and some exponent $\alpha \geq 0$. A typical such estimate is the estimate (first observed by Agmon and Hörmander)

$$\|\widehat{gd\sigma}\|_{L^2(B(x,R))} \lesssim R^{1/2} \|g\|_{L^2(S)};$$

  nowadays we would view this as Plancherel's theorem combined with the uncertainty principle (if space is localized to scale R, then one can blur out the sphere at scale $1/R$, which costs $R^{1/2}$ in L^2 norm).

- Of course if $\alpha = 0$ then one could take limits and obtain a global estimate. However, even when $\alpha > 0$ it is possible to convert these local restriction estimates to global ones, albeit at a cost in the p and q indices. The idea is to use the decay of $\widehat{d\sigma}$, which implies that the operators $g \mapsto \widehat{gd\sigma}$ and $f \mapsto \hat{f}|_S$ are somewhat localized.

- One way to make this heuristic rigorous is via the TT^* principle (since the composition of the restriction operator with its adjoint is the convolution operator $f \mapsto f * \widehat{d\sigma}$). In the early arguments of Fefferman, Stein and Tomas this was achieved in the special case $q = 2$; if one begins with the Agmon-Hörmander estimate and applies this argument one eventually ends up with a restriction estimate for $1 \leq p \leq 2(n+1)/(n+3)$. (This result is known as the Tomas-Stein restriction theorem).

- In 1991 Bourgain observed that this technique of passing from local to global restriction estimates could be extended to general exponents q (but at the cost of some efficiency of exponents). This potentially allows local restriction estimates other than the Agmon-Hörmander estimate to be used as input to the Tomas-Stein argument. Some other variants of the Tomas-Stein method, with the same goal of passing from local to global, were then developed by Bourgain, Moyua, Vargas, Vega, and the author; however our arguments are still not completely satisfactory (unless α is extremely close to 0).

Wave packets

- Henceforth we shall localize to a ball $B(0, R)$ for some $R \gg 1$.

- The connection between restriction and Kakeya comes by breaking up $\widehat{gd\sigma}$ into *wave packets* - objects which are localized on $R \times \sqrt{R}$ tubes and have a fixed frequency.

- The wave packets for the restriction problem were discovered by Knapp. Suppose for instance that g is a bump function adapted to a spherical cap of width $R^{-1/2}$ centered at ω. Then a simple calculation shows that $\widehat{gd\sigma}(x) \approx \psi_T(x)e^{2\pi i\omega\cdot x}$, where ψ_T is a function concentrated on the $R \times \sqrt{R}$ tube T oriented at ω and centered at the origin. (Knapp used this example to place the necessary condition $q \leq \frac{n-1}{n+1}p'$ for a global restriction estimate to hold). As a first approximation one should think of ψ_T as a smoothed out version of χ_T.

- Slightly more generally, if g is the same bump function but modulated by a plane wave $\exp(-2\pi i\xi \cdot x_0)$, then the Fourier transform $\widehat{gd\sigma}$ is similar except that the tube T is centered at x_0 rather than 0.

- More generally still, if g is an arbitrary L^2 function on the same cap as previously, then $\widehat{gd\sigma}$ is an L^2 combination of Knapp examples supported on disjoint tubes, all oriented in the direction ω_0.

- For general g, we thus see that

$$\widehat{gd\sigma}(x) \approx \sum_{\omega} \sum_{T \| \omega} c_{T,\omega} \psi_T(x) e^{2\pi i \omega \cdot x}$$

  where ω ranges over a $R^{-1/2}$ separated set of directions, T ranges over a separated set of $R \times \sqrt{R}$ tubes oriented in the direction ω, and the size of the co-efficients $c_{T,\omega}$ are controlled in an appropriate sense by some $L^{q'}$ norm of g. Thus we have decomposed $\widehat{gd\sigma}$ into a linear combination of Knapp examples. Our task is then to estimate the $L^{p'}$ norm of the above oscillatory sum.

- As a model case we can assume that there is only one tube T_ω contributing from each direction ω, and that the co-efficients are all equal, so we reduce to estimating sums such as

$$\| \sum_{\omega} \psi_{T_\omega} e^{2\pi i \omega \cdot x} \|_{p'}.$$

- The above approach was first pioneered by Córdoba in the special case $p' = 4$. From a modern viewpoint, the idea is as follows. Experience with other oscillatory sums leads us to hope one can estimate the above oscillatory expression by the non-oscillatory square function

$$\| (\sum_{\omega} |\psi_{T_\omega} e^{2\pi i \omega \cdot x}|^2)^{1/2} \|_{p'}.$$

  (For instance, if one puts random signs in front of the oscillations, then these two quantities are almost surely equivalent thanks to Khinchin's inequality). The square function is basically

$$\| (\sum_{\omega} \chi_{T_\omega} \|_{2p'}^2.$$

  But the problem of bounding this expression is almost precisely the problem of obtaining estimates for the Kakeya maximal function.

- In Córdoba's original argument, p' was equal to 4, and one could pass to the square function by an explicit computation based on using Plancherel's theorem to evaluate the L^4 norm. This gave an alternate proof of the restriction conjecture in two dimensions, using the Kakeya conjecture in two dimensions. However it was not so clear what to do for other values of p'.

- By testing the restriction conjecture using a randomized sum of Knapp examples and applying the above arguments one can show that the restriction conjecture in $\mathbf{R}^n$ implies the Kakeya conjecture in $\mathbf{R}^n$ (this dates back at least to 1989 by Beckner, Carbery, Soria, and Semmes). In 1991 Bourgain showed how to reverse this process and show how Kakeya estimates could be used to obtain partial progress on restriction. The idea was again to pass to the square function as with Córdoba's argument, but losing a power of R in the process. (This was achieved using a discretized form of the Tomas-Stein inequality, localized to $\sqrt{R}$-balls, to control the oscillatory sums).

- This general technique is still basically the best way we know of to obtain restriction theorems in higher dimensions, except for the later refinements of the bilinear reduction and induction on scales, which we discuss later. We would love to know how to pass to the square function more efficiently; if it were not for the loss in R then we could show that the Kakeya conjecture implied the restriction conjecture. One ray of hope comes from a very recent result of Wolff, in which a very efficient square function estimate for the cone is proven.

Bilinear estimates

- In the last eight or so years it has been realized that linear restriction estimates such as
$$\|\widehat{f d\sigma}\|_{p'} \lesssim \|f\|_{q'}$$
should be studied together with their bilinear counterparts
$$\|\widehat{f d\sigma_1}\widehat{g d\sigma_2}\|_{p'/2} \lesssim \|f\|_{q'}\|g\|_{q'}$$
where $d\sigma_1$, $d\sigma_2$ are surface measures on two surfaces S_1, S_2. Such bilinear estimates were implicitly considered in the L^4 theory of Córdoba,

Fefferman, and Sjölin (since one can re-interpret L^4 estimates as bilinear L^2 estimates). Considerations from non-linear PDE led Klainerman-Machedon, Bourgain, Kenig-Ponce-Vega and others to further develop the theory of bilinear L^2 restriction estimates. Meanwhile, work by Bourgain, Moyua, Vargas, Vega, T. showed how these bilinear estimates could then be used to efficiently obtain linear estimates (not just vanilla (L^p, L^q) restriction estimates, but also slightly stronger variants, which generally have the flavor of "if we do not have the Knapp example, then we can improve upon standard restriction estimates").

- In fact, bilinear estimates are often easier to work with than their linear counterparts, especially when S_1 and S_2 are sufficiently transverse. This is because the wave packets coming from S_1 make a large angle of intersection with the wave packets from S_2, and one can take advantage of bilinear Kakeya estimates.

- A fundamental example of a bilinear restriction estimate is the following: if S_1 and S_2 are smooth compact hypersurfaces with boundary such that the normals of S_1 always make an angle ~ 1 with the normals of S_2, then we have the easy bound

$$\|\widehat{fd\sigma_1}\widehat{gd\sigma_2}\|_{L^2} \lesssim \|f\|_2 \|g\|_2.$$

We shall give two proofs of this estimate shortly. This estimate can be used to imply linear L^4 estimates such as the Fefferman-Córdoba estimate

$$\|\widehat{fd\sigma}\|_{L^4(B(0,R))} \lesssim R^\varepsilon \|f\|_4$$

when $d\sigma$ is Lebesgue measure on the unit circle in $\mathbf{R}^2$, or the Tomas-Stein-Sjölin estimate

$$\|\widehat{fd\sigma}\|_{L^4(\mathbf{R}^3)} \lesssim \|f\|_2$$

when $d\sigma$ is Lebesgue measure of the sphere in $\mathbf{R}^3$. The idea is to square $\|\widehat{fd\sigma}\|_4$ as $\|\widehat{fd\sigma}\widehat{fd\sigma}\|_2$, and then split this bilinear expression dyadically (using a Whitney-type decomposition) into expressions of the type above. To deal with the "parallel interactions" (when S_1 and S_2 are close to parallel) we use linear transformations to rescale this back to the "transverse interaction" case. (This strategy is in fact quite general and does not use the special properties of the exponents L^4, L^2).

- The above bilinear L^2 estimate is also the prototype for many bilinear estimates involving the so-called $X^{s,b}$ spaces which are very useful in the low-regularity (but sub-critical) study of non-linear wave and dispersive equations, but we will not discuss this further here.

- There are two known ways to prove this bilinear estimate. The first is via the Fourier transform, rewriting the estimate as

$$\|fd\sigma_1 * gd\sigma_2\|_{L^2} \lesssim \|f\|_2\|g\|_2.$$

But this follows from interpolation between the trivial estimate

$$\|fd\sigma_1 * gd\sigma_2\|_{L^1} \lesssim \|f\|_1\|g\|_1$$

(from Young's inequality) and

$$\|fd\sigma_1 * gd\sigma_2\|_{L^\infty} \lesssim \|f\|_\infty\|g\|_\infty$$

(from the transversality of S_1 and S_2).

- It is possible to refine this analysis by using the L^p theory of Radon-type transforms. For instance, for bounded transverse portions of the sphere or cone in $\mathbf{R}^3$ one has the improvement

$$\|fd\sigma_1 * gd\sigma_2\|_{L^2} \lesssim \|f\|_{12/7}\|g\|_{12/7}$$

(T., Vargas, Vega, 1998). These types of estimates have been useful for obtaining other L^p restriction estimates, though they have had limited application to non-linear PDE as yet.

- The other way to prove this estimate is via decomposition into wave packets. To simplify the exposition we shall only prove the slightly weaker estimate

$$\|\widehat{fd\sigma_1}\widehat{gd\sigma_2}\|_{L^2(B(0,R))} \lesssim R^\varepsilon\|f\|_2\|g\|_2$$

for $R \gg 1$ and $\varepsilon > 0$.

- We shall prove this estimate via Wolff's *induction on scales* strategy: assume the estimate true for $\sqrt{R}$, and use this together with the wave packet decomposition to obtain the estimate for R. (The estimate is easy when $R \sim 1$).

- From the wave packet decomposition, we may split $f = \sum_T f_T$, where T ranges over $\sqrt{R} \times R$ tubes oriented along directions normal to S_1 and $\widehat{f_T d\sigma_1}$ is essentially supported on T. Also from orthogonality considerations we have
$$\|f\|_2 \sim (\sum_T \|f_T\|_2^2)^{1/2}.$$
This decomposition can be accomplished by e.g. first decomposing f into caps of width $R^{-1/2}$, and then decomposing into tubes on the physical space side. Similarly we decompose $g = \sum_{T'} g_{T'}$. Finally, we decompose $B(0,R)$ into balls b of radius $\sqrt{R}$. We can thus write
$$\|\widehat{f d\sigma_1}\widehat{g d\sigma_2}\|_{L^2(B(0,R))}$$
as
$$(\sum_b \|(\sum_T \widehat{f_T d\sigma_1})(\sum_{T'} \widehat{g_{T'} d\sigma_2})\|_{L^2(b)}^2)^{1/2}.$$
Now for each b we may essentially restrict the tubes T to those tubes which contain b, and similarly for T':
$$(\sum_b \|(\sum_{T:b\subset T} \widehat{f_T d\sigma_1})(\sum_{T':b\subset T'} \widehat{g_{T'} d\sigma_2})\|_{L^2(b)}^2)^{1/2}.$$
We now apply the induction hypothesis at scale $\sqrt{R}$:
$$\sqrt{R}^{\varepsilon}(\sum_b \|\sum_{T:b\subset T} f_T\|_2^2 \|\sum_{T':b\subset T'} g_{T'}\|_2^2)^{1/2}.$$
Then we use orthogonality:
$$\sqrt{R}^{\varepsilon}(\sum_b (\sum_{T:b\subset T} \|f_T\|_2^2)(\sum_{T':b\subset T'} g_{T'}\|_2^2))^{1/2}.$$
For each T, T' there is basically only one ball b which is contained in both (because T, T' are transverse). Thus we may bound this by
$$\sqrt{R}^{\varepsilon}(\sum_T \sum_{T'} \|f_T\|_2^2 \|g_{T'}\|_2^2)^{1/2}$$

which by orthogonality again becomes

$$\sqrt{R}^{\varepsilon}\|f\|_2\|g\|_2$$

which is OK.

- This style of proof is much more involved than the short one based on Plancherel's theorem, but it appears to be more general - for instance, it has a good chance of working in curved space with quite rough metrics (where the quantity $\widehat{fd\sigma}$ is replaced by a solution to some PDE). Also, it can be adapted to prove bilinear L^p estimates for $p \neq 2$. One particularly striking instance of this is Wolff's bilinear restriction estimate for the cone, which gives the essentially optimal bilinear estimate when the two surfaces S_1, S_2 are transverse subsets of the cone, and the original functions f, g are assumed to be in L^2. This can be thought of as the bilinear version of the Tomas-Stein inequality, and is strictly stronger than it.

- Wolff's argument is a little different from the one presented above. Basically one breaks things up into wave packets as before, and then divides the spatial region $B(0,R)$ into "bad" balls $B(x,R^{1-\sigma})$, where many tubes are interacting with each other, and the remaining "good" region, where few tubes interact with each other. In the bad region one uses the induction hypothesis. In the good region one uses Kakeya estimates and square function estimates as in earlier work. The point is that one only needs Kakeya estimates outside of an exceptional set, and this is roughly analogous to proving Kakeya estimates assuming the "two-ends" condition. With this hypothesis one can obtain a wider class of Kakeya estimates, and this turns out to be crucial in order to obtain the optimal L^p estimates.

The future?

- Improving the arithmetic method for obtaining Kakeya estimates

- Understanding the geometric structure of "optimal" Besicovitch set examples (self-similarity? incidence properties? algebraic structure?)

- Combine geometric methods with the arithmetic method

- Related problems such as dimension bounds for Furstenburg sets
- Improving the Kakeya $\Rightarrow$ restriction machinery (may need the Furstenburg set stuff)
- Understanding the induction on scales technique better
- ?New techniques?

Acknowledgements: The author is a Clay Prize fellow and is supported by grants from the Packard and Sloan foundations. The writing of these notes was supported by IPAM.

References

[1] B. Barcelo, *On the restriction of the Fourier transform to a conical surface*, Trans. Amer. Math. Soc. **292** (1985), 321–333.

[2] W. Beckner, A. Carbery, S. Semmes, F. Soria, *A note on restriction of the Fourier transform to spheres.*, Bull. London Math. Soc. **21** (1989), no. 4, 394–398.

[3] J. Bourgain, *Besicovitch-type maximal operators and applications to Fourier analysis*, Geom. and Funct. Anal. **22** (1991), 147–187.

[4] J. Bourgain, *On the restriction and multiplier problem in* $\mathbf{R}^3$, Lecture notes in Mathematics, no. 1469. Springer Verlag, 1991.

[5] J. Bourgain, *A remark on Schrodinger operators*, Israel J. Math. 77 (1992), 1–16.

[6] J. Bourgain, *Estimates for cone multipliers*, Operator Theory: Advances and Applications, **77** (1995), 41–60.

[7] J. Bourgain, *Some new estimates on oscillatory integrals*, Essays in Fourier Analysis in honor of E. M. Stein, Princeton University Press (1995), 83–112.

[8] J. Bourgain, *On the dimension of Kakeya sets and related maximal inequalities*, Geom. Funct. Anal. **9** (1999), no. 2, 256–282.

[9] J. Bourgain, *Harmonic analysis and combinatorics: How much may they contribute to each other?*, Mathematics: Frontiers and perspectives, IMU/Amer. Math. Society 2000, 13–32.

[10] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972): 287–299.

[11] M. Christ, J. Duoandikoetxea, and J. L. Rubio de Francia, *Maximal operators associated to the Radon transform and the Calderón-Zygmund method of rotations*, Duke Math. J. **53** (1986), 189-209.

[12] A. Córdoba, *Geometric Fourier analysis*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 215–226.

[13] A. Córdoba, *Maximal functions, covering lemmas and Fourier multipliers*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, pp. 29–50, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979.

[14] A. Córdoba, *The Kakeya maximal function and the spherical summation multipliers*, Amer. J. Math. 99 (1977), 1–22.

[15] R.O. Davies, *Some remarks on the Kakeya problem*, Proc. Cambridge Phil. Soc. **69** (1971), 417–421.

[16] S. Drury, *L^p estimates for the x-ray transform*, Ill. J. Math. **27** (1983), 125–129.

[17] M. Erdogan. *Mixed norm estimates for the X-ray transform restricted to a rigid well-curved line complex in R^4 and R^5*, preprint.

[18] C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. **94** (1971): 330–336.

[19] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44–52.

[20] T. Gowers, *A new proof of Szemerédi's theorem for arithmetic progressions of length four*, to appear in GAFA.

[21] J-P. Kahane, *Trois notes sur les ensembles parfaits linéaires*, Enseignment Math. 15 (1969), 185–192.

[22] N. Katz, I. Łaba, T. Tao, *An improved bound for the Minkowski dimension of Besicovitch sets in* $\mathbf{R}^3$, Annals of Math. 152 (2000), 383-446.

[23] N. Katz, T. Tao: *Bounds on arithmetic projections, and applications to the Kakeya conjecture*, Math Res. Letters **6** (1999), 625–630.

[24] N. Katz, T. Tao, *New bounds for the Kakeya problems*, in preparation.

[25] S. Klainerman, M. Machedon, *Space-time Estimates for Null Forms and the Local Existence Theorem*, Comm. Pure Appl. Math., **46** (1993), 1221–1268.

[26] I. Łaba, T. Tao: *An x-ray estimate in* $\mathbf{R}^n$, to appear, Revista Iberoamericana.

[27] I. Łaba, T. Tao: *An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension*, to appear.

[28] G. Mockenhaupt, A. Seeger, C.D. Sogge, *Wave front sets, local smoothing and Bourgain's circular maximal theorem.*, Ann. of Math. **136** (1992), 207–218.

[29] A. Moyua, A. Vargas, L. Vega, *Schrödinger Maximal Function and Restriction Properties of the Fourier transform*, International Math. Research Notices **16** (1996).

[30] A. Moyua, A. Vargas, L. Vega, *Restriction theorems and Maximal operators related to oscillatory integrals in* $\mathbf{R}^3$, to appear, Duke Math. J.

[31] W. P. Minicozzi II, C. D. Sogge, *Negative results for Nikodym maximal functions and related oscillatory integrals in curved space*, preprint.

[32] I. Ruzsa, *Sums of finite sets*, Number Theory: New York Seminar; Springer-Verlag (1996), D.V. Chudnovsky, G.V. Chudnovsky and M.B. Nathanson editors.

[33] W. Schlag, *A generalization of Bourgain's circular maximal theorem*, J. Amer. Math. Soc. **10** (1997), 103-122.

[34] W. Schlag, *A geometric proof of the circular maximal theorem*, Duke Math. J., to appear.

[35] W. Schlag, *A geometric inequality with applications to the Kakeya problem in three dimensions*, Geometric and Functional Analysis **8** (1998), 606–625.

[36] W. Schlag, C. D. Sogge, *Local smoothing estimates related to the circular maximal theorem*, Math. Res. Lett. **4** (1997): 1–15.

[37] C. D. Sogge, *Concerning Nikodym-type sets in 3-dimensional curved space*, preprint.

[38] E. M. Stein, *Some problems in harmonic analysis*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, pp. 3–20.

[39] E. M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[40] R. S. Strichartz, *Restriction of Fourier Transform to Quadratic Surfaces and Decay of Solutions of Wave Equations*, Duke Math. J., **44** (1977), 70 5–774.

[41] T. Tao, *Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates*, to appear, Math Z.

[42] T. Tao, *From rotating needles to stability of waves: emerging connections between combinatorics, analysis, and PDE*, Notices Amer. Math. Soc. 48 (2001), 294-303.

[43] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations*, to appear, Amer. J. Math.

[44] T. Tao, A. Vargas, L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc. **11** (1998), pp. 967–1000.

[45] T. Tao, A. Vargas, *A bilinear approach to cone multipliers I. Restriction Estimates*, GAFA 10 (2000), 185-215.

[46] T. Tao, A. Vargas, *A bilinear approach to cone multipliers II. Applications*, GAFA 10 (2000), 216-258.

[47] T. H. Wolff, *An improved bound for Kakeya type maximal functions*, Revista Mat. Iberoamericana. **11** (1995). 651–674.

[48] T. H. Wolff, *A mixed norm estimate for the x-ray transform*, Revista Mat. Iberoamericana. **14** (1998), 561-600.

[49] T. H. Wolff, *A Kakeya type problem for circles*, Amer. J. Math. **119** (1997), 985–1026.

[50] T. Wolff, *Recent work connected with the Kakeya problem*, Prospects in mathematics (Princeton, NJ, 1996), 129–162, Amer. Math. Soc., Providence, RI, 1999.

[51] T. Wolff, *Maximal averages and packing of one-dimensional sets*, Proceedings of the International Congress of Mathematics, Berlin 1998 Vol II, 755–764.

[52] T. H. Wolff, *A sharp bilinear cone restriction estimate*, to appear, Annals of Math.

[53] T. Wolff, *Local smoothing estimates in L^p for large p*, to appear.