Didier Henrion, Alessandro Rudi

Solving moment and polynomial optimization problems on Sobolev spaces

IPAM UCLA May 2025

arXiv:2401.07734

Outline

1. Optimal transport and moment problems
2. Moments in finite dimension
3. Towards infinity with Sobolev and Fourier
4. Approximation results
5. Examples

Outline

1. **Optimal transport and moment problems**

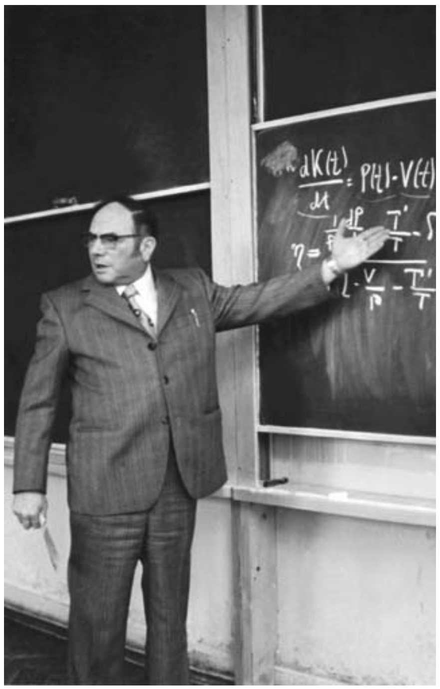
2. Moments in finite dimension

3. Towards infinity with Sobolev and Fourier

4. Approximation results

5. Examples

My background is in control and convex optimization
so I see optimal transport as a **linear optimization** problem



One of the first documented instances of linear programming [L. V. Kantorovich. 1942-48] is a **convex relaxation** of a difficult non-convex optimization problem [G. Monge. 1781]

Given probability measures μ, ν and cost function c, solve

$$\begin{aligned} \inf_{\pi} & \int c(x,y)d\pi(x,y) \\ \text{s.t.} & \int p(x)d\pi(x,y) = \int p(x)d\mu(x),\ \forall p \\ & \int q(y)d\pi(x,y) = \int q(y)d\nu(y),\ \forall q \end{aligned}$$

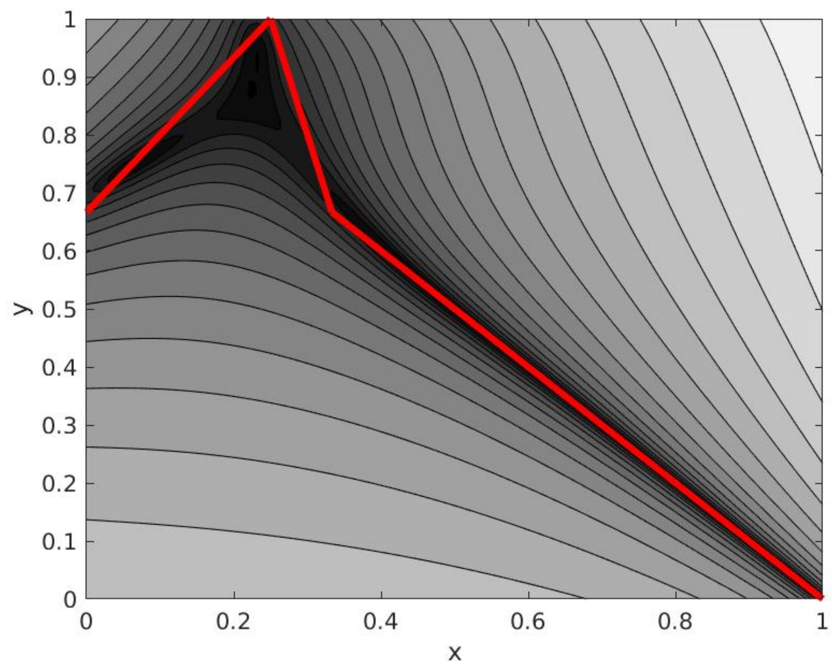
where the unknown is a probability measure π

If c, p, q are polynomial and the support of π is semialgebraic this is a particular instance of a **generalized problem of moments**

It can be solved with the **moment-SOS hierarchy**, a family of numerically tractable finite-dimensional convex relaxations, typically **semdefinite optimization** problems

[J. B. Lasserre. A semidefinite programming approach to the generalized problem of moments. Math. Prog. 112:65–92, 2008]

The optimal transport map can be recovered from the moments with the Christoffel-Darboux kernel, see [S. Marx et al. Semi-algebraic approximation using Christoffel-Darboux kernel. Constr. Approx. 54(3):391-429, 2021] and [O. Mula, A. Nouy. Moment-SoS methods for optimal transport problems. Numer. Math. 156:1541–1578, 2024]



Moment-SOS techniques were developed originally for finite-dimensional polynomials, and then used for optimal transport and generalized problems of moments on finite-dimensional spaces

Ongoing extensions to non-commutative and infinite-dimensional polynomials

This talk focuses on the simplest possible **infinite-dimensional** extension of moment-SOS techniques

Outline

1. Optimal transport and moment problems

2. **Moments in finite dimension**

3. Towards infinity with Sobolev and Fourier

4. Approximation results

5. Examples

Polynomial optimization problem **(POP)**

$$p^* = \min_{x \in X} p(x)$$

where

$$p = \sum_{a \in \mathbb{N}^n_d} p_a x^a \in \mathbb{R}[x]_d$$

is a given multivariate polynomial of degree d expressed e.g. in the monomial basis

$$x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

indexed by $a \in \mathbb{N}^n_d := \{a \in \mathbb{N}^n : |a| := a_1 + a_2 + \cdots + a_n \leq d\}$, and

$$X := \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, 2, \ldots, m\}$$

is a compact basic semialgebraic set described by given multivariate polynomials $g_i \in \mathbb{R}[x]$

Solving POP

$$p^* = \min_{x \in X} \sum_{a \in \mathbb{N}^n_d} p_a x^a$$

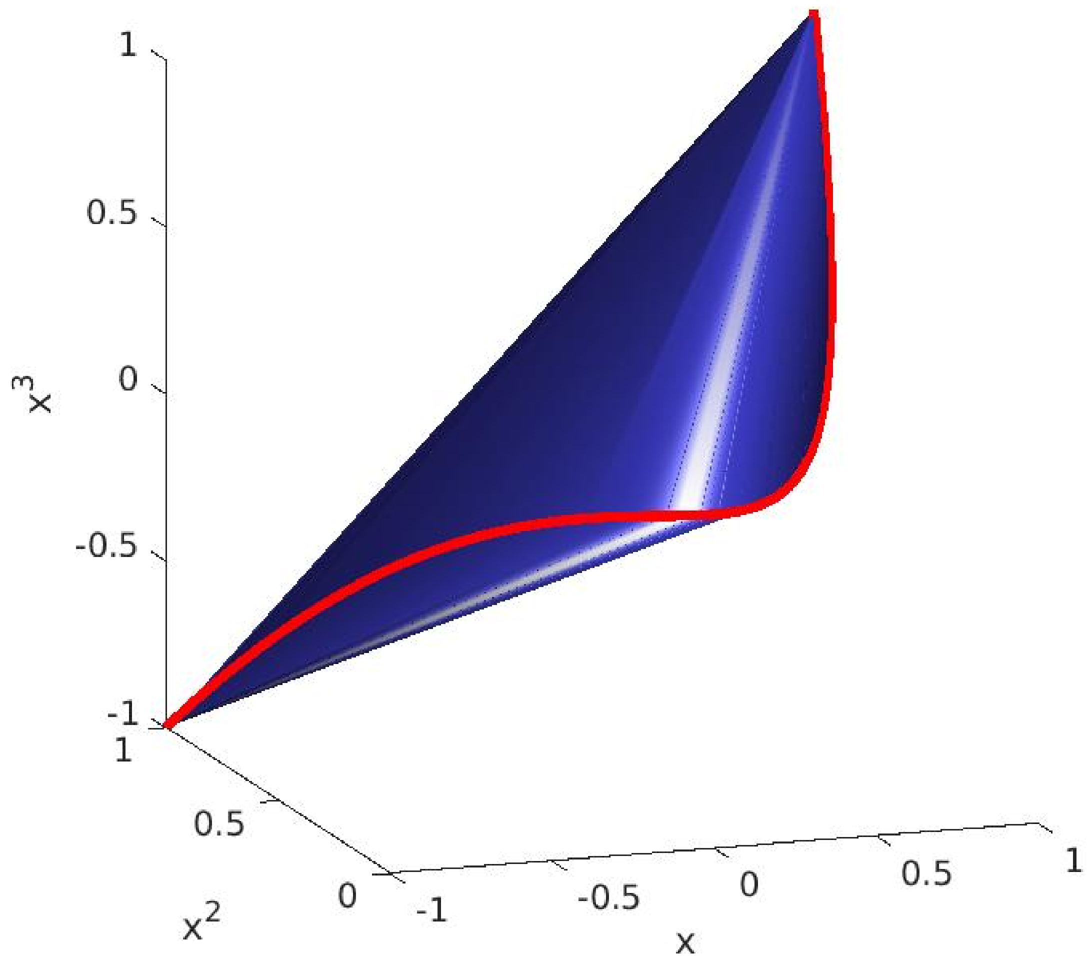
is equivalent to solving the linear problem

$$p^* = \min_{y \in M_d(X)} \sum_{a \in \mathbb{N}^n_d} p_a y_a$$

in the finite-dimensional convex **moment body**

$$\begin{aligned} M_d(X) &:= \{(y_a)_{a \in \mathbb{N}^n_d} : y_a = \int_X x^a d\mu(x) \text{ for some } \mu \in P(X)\} \\ &= \text{conv}\{(x^a)_{a \in \mathbb{N}^n_d} : x \in X\} \end{aligned}$$

where $P(X)$ denotes the set of probability measures on X



Moment body $M_3([-1,1])$

The truncated moment problem consists of determining whether a given vector belongs to the moment body

Can be approximated with semidefinite representable sets built from polynomial sums of squares (SOS) of increasing degrees

POP can be solved with the **moment-SOS hierarchy**

[D. Henrion, M. Korda, J. B. Lasserre. The moment-SOS hierarchy. 2020]

[J. Nie. Moment and polynomial optimization. 2023]

[T. Theobald. Real algebraic geometry and optimization. 2024]

Outline

1. Optimal transport and moment problems
2. Moments in finite dimension
3. **Towards infinity with Sobolev and Fourier**
4. Approximation results
5. Examples

What can be said in **infinite dimension** ?

How can we make sense of a POP

$$p^* = \min_{x \in X} p(x)$$

if X is a compact set of a Hilbert space $\mathscr{H}$?

Can we solve the moment problem and hence the POP in $\mathscr{H}$ with an infinite-dimensional moment-SOS hierarchy ?

Motivation: optimization and control of polynomial differential equations (ordinary, stochastic, partial) - not covered today

Let $\mathscr{H}$ be a separable Hilbert space with a scalar product $\langle .,. \rangle$ and a complete orthonormal system $(e_k)_{k=1,2,\dots}$ with $e_1 = 1$

The **monomial** of degree $a \in c_0(\mathbb{Z}^n)$ is the finite product

$$x^a := \prod_{k=1,2,\dots} \langle x, e_k \rangle^{a_k}$$

and a **polynomial** in H is a linear combination of monomials

$$p(x) = \sum_{a \in \operatorname{spt} p} p_a x^a$$

with scalar coefficients p_a indexed in the support $\operatorname{spt} p \subset c_0(\mathbb{Z}^n)$

The **algebraic degree** of $p(.)$ is $d := \max_{a \in \operatorname{spt} p} \sum_k a_k$ and
the **harmonic degree** of $p(.)$ is $n := \max_{a \in \operatorname{spt} p} \{k \in \mathbb{N} : a_k \neq 0\}$

Our Hilbert space $\mathscr{H}$ is the **Sobolev space** of complex functions on the n-dimensional unit torus T^m whose derivatives up to order m are square integrable:

$$H^m(T^n) := \{x \in L^2 : T^n \to \mathbb{C} : \|x\|^2_{H^m(T^n)} < \infty\}$$

where

$$\|x\|^2_{H^m(T^n)} := \sum_{a \in \mathbb{N}^n_m} \int_{T^n} \left\| \frac{\partial^{|a|} x(\theta)}{\partial\theta_1^{a_1} \ldots \partial\theta_n^{a_n}} \right\|_2^2 d\theta$$

Let μ be a Radon measure on the unit ball B of $\mathscr{H}$

The **moment** of μ of index $a \in c_0(\mathbb{Z}^n)$ is

$$y_a := \int_B x^a d\mu(x), \quad x^a := \prod_{k=1,2,\ldots} \langle x, e_k \rangle_{L^2}^{a_k}$$

for the scalar product

$$\langle x, e_k \rangle_{L^2} := \int_{T^n} x(\theta) e_a(\theta) d\theta, \quad e_a(\theta) := e^{-2\pi \mathbf{i} \langle a, \theta \rangle_{\mathbb{R}^n}}$$

Given a set $A \subset c_0(\mathbb{Z}^n)^N$, let us define the Sobolev moment cone

$$C(A) := \{(y_a)_{a \in A} : y_a = \int_B x^a d\mu(x) \text{ for some } \mu\}$$

The **Sobolev truncated moment problem** consists of asking whether a given vector $y \in \mathbb{C}^N$ belongs to $C(A)$

Define the Fourier transform

$$F : L^2(T^n) \to \ell_2(\mathbb{Z}^n),\ \ x \mapsto c$$

with Fourier coefficients

$$c := (c_a)_{a\in\mathbb{Z}^n},\ \ c_a := \langle x, e_a\rangle_{L^2}$$

Its adjoint is the inverse Fourier transform

$$F^* : \ell_2(\mathbb{Z}^n) \to L^2(T^n),\ \ c \mapsto x$$

with

$$x = \langle c, e\rangle_{\ell^2} := \sum_{a\in\mathbb{Z}^n} c_a e_{-a},\ \ e := (e_a)_{a\in\mathbb{Z}^n}.$$

Parseval's identity:

$$\|x\|^2_{H^m(T^n)} = \|c\|^2_W := \sum_{a\in\mathbb{Z}^n} w_a |c_a|^2,\ \ w_a := (1 + \langle a, a\rangle_{\mathbb{Z}^n})^m$$

The Sobolev unit ball

$$B := \{x \in H^m(T^n) : \|x\|_{H^m(T^n)} \leq 1\}$$

becomes a **compact ellipsoid** in the Fourier coefficients

$$E := \{Fx : x \in B\} = FB = \{c \in \ell_2(\mathbb{Z}^n) : \sum_{a \in \mathbb{Z}^n} w_a |c_a|^2 \leq 1\}$$

which is included in a Hilbert cube

$$E \subset \{c \in \ell_2(\mathbb{Z}^n) : |c_a|^2 \leq \frac{1}{w_a} = \frac{1}{(1 + \langle a, a \rangle_{\mathbb{Z}^n})^m}, a \in \mathbb{Z}^n\}$$

Let $\nu := F_{\#}\mu$ be the image measure of μ through F, i.e.

$$\nu(A) = \mu(\{x \in B : Fx \in A\})$$

for all Borel sets $A \subset E$

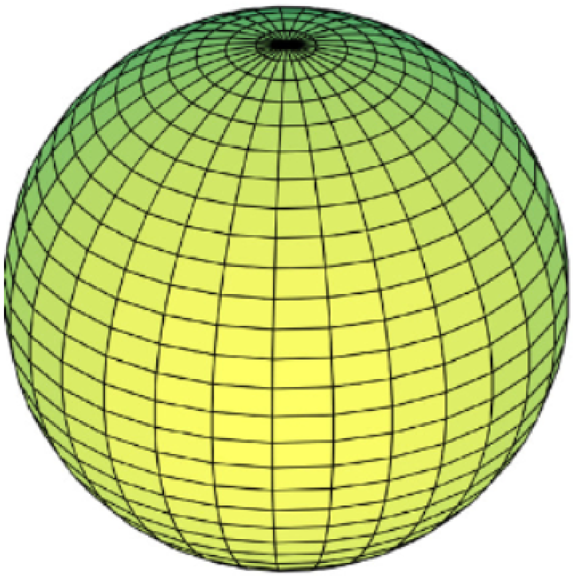
Geometrically, Sobolev moments on the ball B become Fourier moments on the ellipsoid E:

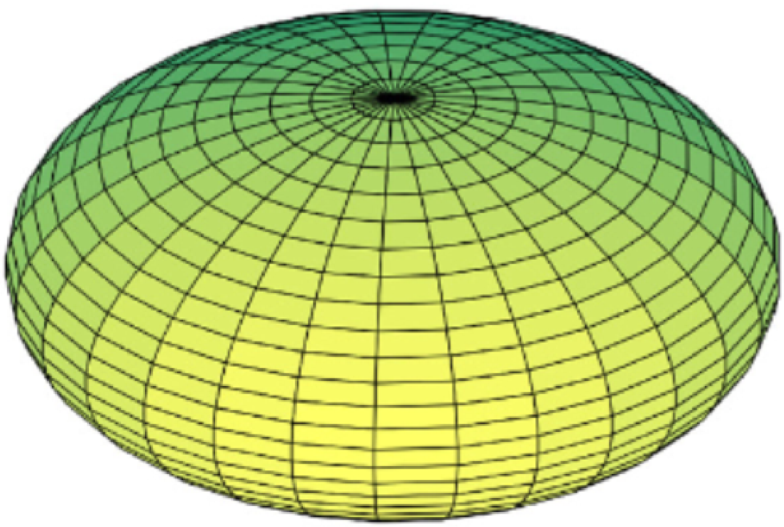
$$y_a = \int_B x^a \, d\mu(x) = \int_E c^a \, d\nu(c)$$

Our Sobolev moment problem is a Fourier moment problem in

$$C(A) := \{(y_a)_{a \in A} : y_a = \int_E c^a d\nu(c) \text{ for some } \nu\}$$

in a compact set E defined on $\mathbb{C}[c]$, the ring of complex polynomials with **countably infinitely** many variables $c = (c_a)_{a \in \mathbb{Z}^n}$ i.e. **moment polynomials**






Outline

1. Optimal transport and moment problems

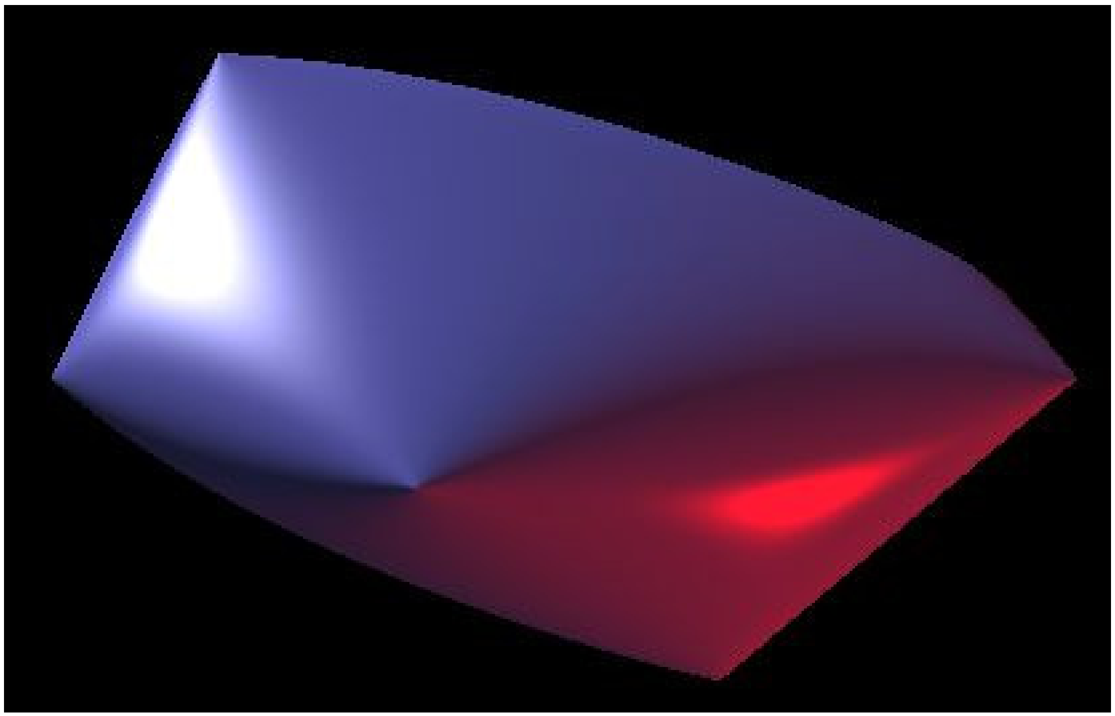
2. Moments in finite dimension

3. Towards infinity with Sobolev and Fourier

4. **Approximation results**

5. Examples

In complete analogy with the finite-dimensional case, cone $C(A)$ can be approximated by linear projections of linear sections of the semidefinite cone of increasing size



To construct these approximations and prove their convergence, we use SOS representations of positive polynomials in $\mathbb{C}[c]$

Jacobi's Archimedean Positivstellensatz for commutative unital algebras is Theorem 4 in [T. Jacobi. A representation theorem for certain partially ordered commutative rings. Math. Z. 237:259-273, 2001]

See also Theorem 2.1 in [M. Ghasemi, S. Kuhlmann, M. Marshall. Application of Jacobi's representation theorem to locally multiplicatively convex topological $\mathbb{R}$-algebras. J. Functional Analysis 266:1041-1049, 2014]

See also Theorem 3.9 in [M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski. An intrinsic characterization of moment functionals in the compact case. Int. Math. Res. Not. 2023(3):2281-2303, 2023]

Using Jacobi's Psatz we can approximate the moment cone

$$C(A) = \{(y_a)_{a \in A} : y_a = \int_E c^a \, d\nu(c) \text{ for some } \nu \text{ on } E \ \} \subset \mathbb{C}^N$$

for a given index set $A \in c_0(\mathbb{Z}^n)$, with a converging **hierarchy of semidefinite representable outer approximations** $C_{r,\rho}^{\text{out}}(A)$ indexed by the algebraic resp. harmonic degree r resp. ρ of the SOS certificates:

$$C_{r,\rho}^{\text{out}}(A) \supset C(A), \quad \overline{C_{\infty,\infty}^{\text{out}}}(A) = C(A)$$

For any $r \geq r_A$ and $\rho \geq \rho_A$, Hausdorff distance bound

$$d_H(C(A), C_{r,\rho}^{\text{out}}(A)) \ \leq \ 9\,(2\rho_A + 1)^n \, \frac{r_A^2}{r^2}$$

Using measures with SOS densities, we can design a converging hierarchy of semidefinite representable inner approx. $C_{r,\rho}^{\text{inn}}(A)$

We are now fully equipped to solve a Sobolev POP

$$p^* := \inf_{x \in B} p(x)$$

with finite harmonic degree $\delta(p)$ and finite algebraic degree $d(p)$

Equivalent to the Fourier POP

$$p^* = \min_{c \in E} p(c)$$

and the linear problem

$$p^* = \min_{\nu \in P(E)} \int_E p(c) d\nu(c) = \min_{\nu \in P(E)} \sum_{a \in A} \int_E c^a \nu(c)$$

on $P(E)$, the probability measures on the Fourier ellipsoid E, in turn equivalent to the linear moment problem

$$p^* = \min_{y \in C(A)} \sum_{a \in A} p_a y_a \text{ s.t. } y_0 = 1$$

Therefore we can design a moment-SOS hierarchy of lower bounds

$$p^{\mathsf{out}}_{r,\rho} := \min_{y \in C^{\mathsf{out}}_{r,\rho}(A)} \sum_{a \in A} p_a y_a$$

as well as a moment-SOS hierarchy of upper bounds

$$p^{\mathsf{inn}}_{r,\rho} := \min_{y \in C^{\mathsf{inn}}_{r,\rho}(A)} \sum_{a \in A} p_a y_a$$

for increasing algebraic resp. harmonic relaxation degrees r, ρ

Theorem: For all $r \geq r' \geq d(p)$ and $\rho = \delta(p)$, it holds

$$p^{\mathsf{out}}_{r',\rho} \leq p^{\mathsf{out}}_{r,\rho} \leq p^{\mathsf{out}}_{\infty,\infty} = p^* = p^{\mathsf{inn}}_{\infty,\rho} \leq p^{\mathsf{inn}}_{r,\rho} \leq p^{\mathsf{inn}}_{r',\rho}$$

Outline

1. Optimal transport and moment problems

2. Moments in finite dimension

3. Towards infinity with Sobolev and Fourier

4. Approximation results

5. **Examples**

Consider the non-convex **harmonic Sobolev POP**

$$p^* = \min_{x \in B} \langle x, e_0 \rangle^4_{H^0(T)} + (\langle x, e_1 \rangle^2_{H^0(T)} - 1/4)^2$$

on $B \subset H^0(T)$, i.e. $n = 1$, $m = 0$ and harmonic degree $\rho = 1$

Equivalent to the harmonic Fourier POP

$$p^* = \min_{c_{-1}, c_0, c_1} c_0^4 + (c_1^2 - 1/4)^2 \text{ s.t. } c_{-1}^2 + c_0^2 + c_1^2 \leq 1.$$

With the outer moment-SOS hierarchy, at algebraic relaxation degree $r = 2$, we obtain the two global minimizers

$$c^*_{-1} = 0, c^*_0 = 0, c^*_1 = \pm 1/2$$

and the corresponding functions

$$x(\theta) = \pm e^{-2\pi \mathbf{i} \theta}/2$$

achieving the global minimum $p^* = p^{\text{out}}_{2,1} = 0$

Another class is the **algebraic** Sobolev POP

$$p^* = \inf_{x \in B} L(p(x, D^{a_1}x, \ldots, D^{a_l}x))$$

where p is a polynomial of $x \in B$ and its derivatives $D^{a_j}x$, $a_j \in \mathbb{N}^n$ and $L : L^\infty(T^n) \to \mathbb{R}$ is a given bounded linear functional

For example

$$L(p(x)) = \int_{T^n} (p_1(\theta)x(\theta) + p_2(\theta)\|Dx(\theta)\|_2^2) d\sigma(\theta)$$

where σ is a given probability measure on T^n and p_1, p_2 are given real polynomials of θ

The non-linearity hits the function value $x(\theta)$ and its derivatives, and hence this POP generally involves infinitely many harmonics

Consider e.g. the non-convex algebraic Sobolev POP

$$p^* = \inf_{x \in B} \int_T (x(\theta)^2 - 1/2)^2 d\sigma(\theta)$$

where σ is the Dirac measure at 0 on $B \subset H^0(T)$

Since $x(0) = \sum_{a \in \mathbb{Z}} c_a$, the problem is the algebraic Fourier POP

$$p^* = \inf_{c \in E} \frac{1}{4} - \sum_{a_1, a_2 \in \mathbb{Z}} c_{a_1} c_{a_2} + \sum_{a_1, a_2, a_3, a_4 \in \mathbb{Z}} c_{a_1} c_{a_2} c_{a_3} c_{a_4}$$

With the outer moment-SOS hierarchy, at algebraic relaxation degree $r = 2$ and harmonic relaxation degree $\rho = 0$, we obtain the two global minimizers $c_0^* = \pm\sqrt{2}/2$ and the corresponding functions

$$x^*(\theta) = \pm\sqrt{2}/2$$

achieving the global minimum $p^* = p_{2,0}^{\text{out}} = 0$

Didier Henrion, Alessandro Rudi

Solving moment and polynomial optimization problems on Sobolev spaces

IPAM UCLA May 2025

arXiv:2401.07734