# Quadratic Wasserstein distance between quantum dynamical systems

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# Classical transport plans

A transport plan $\omega$ between probability spaces $(X, \mu)$ and $(Y, \nu)$ is a probability measure on $X \times Y$ such that

$$\omega(U \times Y) = \mu(U) \quad \text{and} \quad \omega(X \times V) = \nu(V).$$

A standard measure theoretic result from probability theory says that

$$\int_{X \times Y} 1_U \otimes 1_V \, d\omega = \omega(U \times V) = \int_Y E_\omega(1_U) 1_V \, d\nu$$

for a uniquely determined Markov operator or channel $E_\omega$. This extends to

$$\int_{X \times Y} a \otimes b \, d\omega = \int_{X \times Y} a(x) b(y) \, d\omega(x, y) = \int_Y E_\omega(a) b \, d\nu$$

also written more compactly as

$$\boxed{\omega(a \otimes b) = \nu(E_\omega(a) b)}$$

where $a$ and $b$ are complex-valued functions on $X$ and $Y$ respectively.

# Classical transport plans: channels

As mentioned, in

$$\omega(a \otimes b) = \nu(E_\omega(a)b)$$

$E_\omega$ is a channel between appropriate $*$-algebras (observable algebras) of complex-valued measurable functions on $X$ and $Y$ respectively, say

$$E_\omega : A \to B.$$

In particular,

$$\int_Y E_\omega(a) d\nu = \int_X a \, d\mu, \qquad \text{i.e.,} \qquad \nu\left(E_\omega(a)\right) = \mu(a)$$

in the same compact notation for integrals.

Conversely, any such channel defines a transport through the formula above. That is:

We can represent transport plans as these channels.

# Classical transport plans: an interpretation

Think of the transport plan $\omega$ as dynamics, or a process, taking an observable $a$ of the first system, to an observable $E_\omega(a)$ of the second system,

$$a \;\mapsto\; E_\omega(a)$$

(a Heisenberg picture).

Then one could think of

$$\nu(E_\omega(a)b)$$

as a measure of correlation between $a$ after transport, and an observable $b$ of the second system.

That is, we can think of the transport plan $\omega$ itself in these terms, since

$$\omega(a \otimes b) = \nu(E_\omega(a)b).$$

This interpretation will be employed heuristically when we discuss cost later on.

# Classical transport plans: the diagonal or identity plan

A transport plan $\omega$ can also be written as

$$\omega(a \otimes b) = \nu(E_\omega(a)b) = \int_Y E_\omega(a)b \, d\nu = \int_{Y \times Y} E_\omega(a) \otimes b \, d\delta_\nu$$

where the diagonal measure $\delta_\nu$ associated to $\nu$ is defined by

$$\delta_\nu(V_1 \times V_2) = \nu(V_1 \cap V_2)$$

for all $V_1, V_2 \subset Y$. It is clearly a transport plan between $(Y, \nu)$ and itself, the diagonal plan, which we can view as the identity transport plan,

$$E_{\delta_\nu} = \mathrm{id}_B \, .$$

A transport plan $\omega$ can thus always be written in terms of $\delta_\nu$ and the representing channel $E_\omega$. In our compact notation:

$$\boxed{\omega(a \otimes b) = \delta_\nu(E_\omega(a) \otimes b).}$$

Hence $\delta_\nu$ is a basic transport plan from which all others between $(X, \mu)$ and $(Y, \nu)$ can be built using $E_\omega$.

# Classical transport plans: abstract diagonal plans

In integral form, the definition $\delta_\nu(V_1 \times V_2) = \nu(V_1 \cap V_2)$ of $\delta_\nu$, reads

$$\int_{Y\times Y} b_1 \otimes b_2 \, d\delta_\nu = \int_Y b_1 b_2 \, d\nu \qquad \text{i.e.,} \qquad \delta_\nu(b_1 \otimes b_2) = \nu(b_1 b_2)$$

where $(b_1 \otimes b_2)(y_1, y_2) = b_1(y_1)b_2(y_2)$ and $(b_1 b_2)(y) = b_1(y)b_2(y)$.

This still works on an abstract abelian $*$-algebra $B$: Define

$$\delta_\nu(b_1 \otimes b_2) = \nu(b_1 b_2), \quad \text{for all } b_1, b_2 \in B,$$

with $\nu$ a positive linear functional on $B$, meaning $\nu(b^* b) \geq 0$.

I.e.,

$$\boxed{\delta_\nu = \nu \circ \varpi,}$$

where

$$\varpi : B \otimes B \to B$$

extends the product of $B$. Since $B$ is abelian, $\varpi$ is a $*$-homomorphism (it preserves all structure), ensuring positivity of $\delta_\nu$: For any $c \in B \otimes B$,

$$\delta_\nu(c^* c) = \nu(\varpi(c)^* \varpi(c)) \geq 0.$$

# Quantum transport plans

Consider possibly noncommutative observable algebras

$$A \quad \text{and} \quad B$$

and states (positive normalized linear functionals) $\mu$ and $\nu$ on them.

For example $$A = M_m \quad \text{and} \quad B = M_n\,,$$

then $$\mu(a) = \mathrm{Tr}(\rho_\mu a) \quad \text{and} \quad \nu(b) = \mathrm{Tr}(\rho_\nu b)$$

in terms of density matrices.

A transport plan $\omega$ between $\mu$ and $\nu$ is (naively) expected to be a state $\omega$ on $A \otimes B$ such that

$$\omega(a \otimes 1_B) = \mu(a) \quad \text{and} \quad \omega(1_A \otimes b) = \nu(b)$$

in accordance with the classical case. The positivity requirement

$$\omega(c^\dagger c) \geq 0$$

leads to a complication which we need to address.

# Quantum transport plans: the trouble with diagonal plans

Taking our cue from the basic role of diagonal plans in the classical case, we first attempt quantum diagonal plans to motivate certain structures to be employed throughout.

For a positive linear functional $\nu$ on an abstract $*$-algebra $B$, i.e., $\nu(b^*b) \geq 0$, the same construction as in the classical case,

$$\delta_\nu = \nu \circ \varpi \qquad \text{i.e.,} \qquad \delta_\nu(b_1 \otimes b_2) = \nu(b_1 b_2), \text{ for all } b_1, b_2 \in B,$$

is immediately problematic for non-abelian $B$:

The product map

$$\varpi : B \otimes B \to B$$

is no longer a $*$-homomorphism, for example,

$$\varpi((a_1 \otimes a_2)(b_1 \otimes b_2)) = a_1 b_1 a_2 b_2 \neq a_1 a_2 b_1 b_2 = \varpi(a_1 \otimes a_2)\varpi(b_1 \otimes b_2),$$

causing difficulty for the positivity of $\delta_\nu$.

# Quantum transport plans: a strategy for diagonal plans

We aim to sidestep the lack of commutativity of $B$ preventing the product map $\varpi$ from being a $*$-homomorphism.

Question: Can we represent the $*$-algebra $B$ on a Hilbert space $H_B$ such the algebra of linear operators on it contains both $B$ and a "copy" $B'$ of it such that they commute? I.e.,

$$B, B' \subset L(H_B) \quad \text{such that} \quad bb' = b'b \quad \text{for all } b \in B, b' \in B'.$$

If so, we can attempt to replace the abelian construction for $\varpi$ by

$$\varpi : B \otimes B' \to L(H_B)\,,$$

defined as extending

$$\varpi(b \otimes b') = bb'$$

to the whole tensor product $B \otimes B'$. This $\varpi$ is indeed a $*$-homomorphism.

Question: Will this allow a sensible and useful extension of diagonal plans to the quantum (noncommutative) case?

# Quantum transport plans: standard forms

The theory of standard forms of von Neumann algebras provide an ideal framework for this strategy. A von Neumann algebra $B$ is a $*$-algebra of bounded linear operators on a Hilbert space $H_B$ such that

$$B'' = B$$

where primes denote commutants in the algebra of bounded linear operator $L(H_B)$ on $H_B$. That is, $B'$ consists of all operators commuting with everything in $B$. Similarly for $B''$.

Examples include: $L^\infty$ spaces of measurable functions.

$L(H_B)$, in particular $M_m$.

A standard form of $B$ is a Hilbert space representation $H_B$ precisely "big" enough such that all physical states $\nu$ on $B$ can be expressed in the form

$$\nu(b) = \langle \Lambda_\nu, b\Lambda_\nu \rangle = \langle \Lambda_\nu | b | \Lambda_\nu \rangle \quad \text{for all } b \in B\,,$$

for some vector $\Lambda_\nu$ in $H_B$. For our purposes, standard forms always exist.

# Quantum transport plans: examples of standard forms

Classical. An elementary case is

$$B = \mathrm{diag}(n) = B'.$$

Indeed $$\nu(b) = \mathrm{Tr}(\rho_\nu b) = \langle \rho_\nu^{1/2}, b\rho_\nu^{1/2} \rangle$$

in the Hilbert–Schmidt inner product, that is $H_B = \mathbb{C}^n$ and $\Lambda_\nu = \rho_\nu^{1/2}$.

Quantum. For the von Neumann algebra $M_n$, set

$$B = M_n \otimes I_n$$

acting on $\psi \in H_B = M_n$ by $(b \otimes I_n)\psi = b\psi$.

Then

$$B' = I_n \otimes M_n \,,$$

acting as $$(I_n \otimes b)\psi = \psi b^T .$$

Now $$\nu(b \otimes I_n) = \mathrm{Tr}(\rho_\nu b) = \langle \rho_\nu^{1/2}, b\rho_\nu^{1/2} \rangle \qquad \text{for } b \in M_n \,.$$

Here $\Lambda_\nu = \rho_\nu^{1/2} \in M_n \cong \mathbb{C}^n \otimes \mathbb{C}^n$ is the purification of $\nu$.

# Quantum transport plans: definition of a diagonal plan

Combining the product map

$$\varpi : B \otimes B' \to L(H_B)\,,$$

in the standard form with the representation

$$\nu(b) = \langle \Lambda_\nu, b\Lambda_\nu \rangle$$

of $\nu$, we obtain a state (positivity guaranteed) on $B \otimes B'$ as

$$\delta_\nu = \langle \Lambda_\nu, \varpi(\cdot)\Lambda_\nu \rangle\,,$$

i.e., it is the extension of

$$\delta_\nu(b \otimes b') = \langle \Lambda_\nu, bb'\Lambda_\nu \rangle \qquad \text{for } b \in B, b' \in B'.$$

Central idea: The state $\delta_\nu$ is defined on

$$B \otimes B'$$

rather than on $B \otimes B$.

# Quantum transport plans: the diagonal as a transport plan

In the standard representation we have

$$\delta_\nu(b \otimes 1_{B'}) = \langle \Lambda_\nu, b\Lambda_\nu \rangle = \nu(b)$$

and

$$\delta_\nu(1_B \otimes b') = \langle \Lambda_\nu, b'\Lambda_\nu \rangle = \nu'(b')$$

in terms of the state $\nu'$ on $B'$ defined by the last equality, or equivalently by

$$\nu' = \nu \circ j_B \, ,$$

where the $*$-anti-isomorphism (from modular or Tomita-Takesaki theory)

$$j_B : B' \to B$$

preserves all structure, except that it swaps products. In this sense $B'$ is a "mirror image" of $B$, rather than an exact copy.

We need to develop the theory to verify that this is a viable approach.

# Quantum transport plans: example of a diagonal plan

Let's return to our example in standard form,

$$B = M_n \otimes I_n \quad \text{and} \quad B' = I_n \otimes M_n\,,$$

where for $b \otimes I_n \in B$,

$$\nu(b \otimes I_n) = \langle \rho_\nu^{1/2}, b\rho_\nu^{1/2} \rangle\,.$$

As before by the bimodule structure (and slight abuse of the tensor product notation on the left), we have

$$\delta_\nu(b \otimes b') = \langle \rho_\nu^{1/2}, b\rho_\nu^{1/2} b'^T \rangle = \mathrm{Tr}(\rho_\nu^{1/2} b \rho_\nu^{1/2} b'^T)$$

for $b \otimes I_n \in B$ and $I_n \otimes b' \in B'$.

For diagonal $b$, $b'$ and $\rho_\nu$ we indeed recover the classical diagonal plan

$$\delta_\nu(b \otimes b') = \mathrm{Tr}(\rho_\nu b b') = \sum_{i=1}^{n} \nu_i a_i b_i\,.$$

# Quantum transport plans: general transport plans

Consider two von Neumann algebras (always in standard form from now on) with states:

$$(A, \mu) \quad \text{and} \quad (B, \nu)\,,$$

as well as a (quantum) channel, or u.c.p. map,

$$E : A \to B \quad \text{such that} \quad \nu \circ E = \mu\,.$$

Then, paralleling the classical case, the state $\omega$ on $A \otimes B'$ defined by

$$\boxed{\omega = \delta_\nu \circ (E \otimes \mathrm{id}_{B'})\,, \quad i.e., \quad \omega(a \otimes b') = \langle \Lambda_\nu, E(a) b' \Lambda_\nu \rangle\,,}$$

is a transport plan in the sense (similar to $\delta_\nu$) that

$$\omega(a \otimes 1_{B'}) = \mu(a) \quad \text{and} \quad \omega(1_A \otimes b') = \nu'(b')$$

Since $\nu \circ E = \mu$, however, we view this as a transport plan between $\mu$ and $\nu$.

# Quantum transport plans: general transport plans

Taking this at face value, namely the channel point of view

$$\nu \circ E = \mu$$

connecting $\mu$ and $\nu$, not $\nu'$, we introduce:

**Definition**

A transport plan between $\mu$ and $\nu$ is a state on

$$A \otimes B'$$

such that

$$\omega(a \otimes 1_{B'}) = \mu(a) \quad \text{and} \quad \omega(1_A \otimes b') = \nu'(b')$$

We write this symbolically as

$$(A, \mu)\,\omega\,(B, \nu)\,.$$

# Quantum transport plans: transport plans as channels

Given any transport plan $\omega$ between $\mu$ and $\nu$, that is

$$(A, \mu)\, \omega\, (B, \nu)\,.$$

with $\mu$ and $\nu$ henceforth faithful, one can show that there is a uniquely determined channel

$$E_\omega : A \to B$$

satisfying

$$\boxed{\omega = \delta_\nu \circ (E_\omega \otimes \mathrm{id}_{B'})\,, \quad i.e., \quad \omega(a \otimes b') = \langle \Lambda_\nu, E_\omega(a) b' \Lambda_\nu \rangle\,.}$$

Then $\qquad \mu(a) = \omega(a \otimes 1_{B'}) = \langle \Lambda_\nu, E_\omega(a) \Lambda_\nu \rangle = \nu \circ E_\omega(a)\,.$

As a convention in this Heisenberg picture of $E_\omega : A \to B$ in terms of observable algebras, I'll speak as if

$\omega$ is a transport plan from $A$ to $B$,

despite $\nu \circ E_\omega = \mu$.

# Reversing transport plans

A priori our transport plans are directed, also from the channel point of view. We need to be sure we understand this properly, by reversing them.

Classically this is simple enough. The reverse of $\omega$ is simply given by the transport plan defined as

$$\omega'(V \times U) = \omega(U \times V) \quad \text{for all } U \subset X, V \subset Y\,.$$

In our noncommutative setup, there is an obvious deviation from this simple picture:

For $(A, \mu)\,\omega\,(B, \nu)$ the corresponding argument above leads to

$$\omega'(b' \otimes a) = \omega(a \otimes b')$$

which may seem suspicious: $\omega'$ is a state on $B' \otimes A$, not on $B \otimes A'$. Nevertheless, for any state $\omega$ on $A \otimes B'$, we have

$$(A, \mu)\,\omega\,(B, \nu) \iff (B', \nu')\,\omega'\,(A', \mu')\,.$$

# Reversing transport plans: a course correction

The maps

$$j_A : L(H_A) \to L(H_A) \quad \text{and} \quad j_B : L(H_B) \to L(H_B)$$

from earlier (which indeed extend as shown here) can easily convert the state $\omega'$ on $B' \otimes A$ to the required $B \otimes A'$ form by defining the state

$$\omega^\sigma = \omega' \circ (j_B \otimes j_A)\,.$$

on $B \otimes A'$, since $j_B(B) = B'$ and $j_A(A') = A$.

Then

$$(A, \mu)\, \omega\, (B, \nu) \iff (B, \nu)\, \omega^\sigma\, (A, \mu)\,.$$

Thus, $\omega$ and $\omega^\sigma$ appear to be sensible reverses of one another.

But don't discount $\omega'$ just yet.

## Reversing transport plans: in terms of channels

We have $(A,\mu)\,\omega\,(B,\nu)$ corresponding to $E_\omega : A \to B$

$(B',\nu)\,\omega'\,(A',\mu)$ corresponding to $E_{\omega'} : B' \to A'$

$(B,\nu)\,\omega^\sigma\,(A,\mu)$ corresponding to $E_{\omega^\sigma} : B \to A$

The latter two channels can be viewed as duals of $E_\omega$ defined by

$$E'_\omega = E_{\omega'} \quad \text{and} \quad E^\sigma_\omega = E_{\omega^\sigma}$$

The dual $E^\sigma_\omega$ arose in other contexts and goes by many names:
In quantum information it is the Petz recovery map.
In quantum detailed balance it is the KMS dual or standard dual.
Petz simply called it the dual in his early work (1984).
At its inception in Accardi and Cecchini (1982), it was called the bidual, in reference to the formula

$$E^\sigma_\omega = j_A \circ E'_\omega \circ j_B$$

in terms of $E'_\omega$, which in turn we'll refer to as the AC-dual.

# Reversing transport plans: intuition about classical duals

The duals $E'_\omega$ and $E^\sigma_\omega$ are clearly related to reversing $E\omega$.

In the classical case $j_A = \mathrm{id}$ and $j_B = \mathrm{id}$, hence

$$E^\sigma_\omega = E'_\omega \, .$$

As an example, consider an $n \times m$ transition matrix and two probability distributions

$$\tau = [\,\tau_{rs}\,], \quad \mu = [\,\mu_1 \cdots \mu_m\,] \quad \text{and } \nu = [\,\nu_1 \cdots \nu_n\,]\,, \quad \text{with} \quad \nu\tau = \mu \, .$$

The reverse $\tau'$ of $\tau$ as a transition matrix, is required to satisfy

$$\mu_s \tau'_{sr} = \nu_r \tau_{rs} \, .$$

Exercise: When viewing $\tau$ as representing a transport plan, this $\tau'$ is indeed exactly the AC-dual (thus also KMS-dual) of $\tau$.

# Reversing transport plans: intuition about quantum duals

In the quantum case the duals $E'_\omega : B' \to A'$ and $E^\sigma_\omega : B \to A$ are distinct.

In our standard example, $A = M_m \otimes I_m$ and $B = M_n \otimes I_n$

$$A' = I_m \otimes M_m \quad \text{and} \quad B' = I_n \otimes M_n\,,$$

we can identify the $M_m$'s in $A$ and $A'$, etc, and write

$$E_\omega : M_m \to M_n \quad \text{and} \quad E'_\omega : M_n \to M_m\,.$$

One can then argue that $E'_\omega$ is in fact the reverse of $E_\omega$ in a conceptually and physically sensible way, due to the simple swapping of $M_m$ and $M_n$ in the definition of $\omega'$. (Duvenhage, Oerder, van den Heuvel, 2024.)
For example, a standard form of quantum detailed balance for a quantum channel $\tau : M_n \to M_n$, can be expressed as $\tau' = \tau$ in exact analogy to classical detailed balance in a Markov chain.

Nevertheless, $E^\sigma_\omega$ fits better in the grand transport scheme of things, including in relation to cost, as will be seen later on.

# Transport plans between systems

By a system we mean a triple

$$(A, \alpha, \mu) \quad \text{with a channel} \quad \alpha : A \to A \quad \text{such that} \quad \mu \circ \alpha = \mu\,,$$

that is, the state $\mu$ is stationary under the dynamics $\alpha$. This is motivated in part by the original goal of this work related to deviation from quantum detailed balance.

Given a second system $(B, \beta, \nu)$, we are interested in similar structure or properties in $\alpha$ and $\beta$. An approach to this question of much value in classical and noncommutative ergodic theory via joinings, in essence has been to consider the condition

$$E_\omega \circ \alpha = \beta \circ E_\omega$$

given that $(A, \mu)\,\omega\,(B, \nu)$. Equivalently,

$$\omega(\alpha(a) \otimes b') = \omega(a \otimes \beta'(b'))$$

which is closer to the usual joining condition in ergodic theory.

# Transport plans between systems: the definition

Formalizing this discussion, we state an extension of the definition of transport plans between states:

**Definition**

Given $(A, \mu)\,\omega\,(B, \nu)$, we call $\omega$ a transport plan from $(A, \alpha, \mu)$ to $(B, \beta, \nu)$, and write

$$(A, \alpha, \mu)\,\omega\,(B, \beta, \nu)\,,$$

if

$$\omega(\alpha(a) \otimes b') = \omega(a \otimes \beta'(b'))$$

or equivalently,

$$E_\omega \circ \alpha = \beta \circ E_\omega\,.$$

This can alternatively be viewed as a restriction on the allowed transport plans between states.

# Transport plans between systems: lack of symmetry

Unlike

$$(A,\mu)\,\omega\,(B,\nu) \iff (B,\nu)\,\omega^{\sigma}\,(A,\mu)\,,$$

we don't have equivalence between

$$(A,\alpha,\mu)\,\omega\,(B,\beta,\nu) \quad \text{and} \quad (B,\beta,\nu)\,\omega^{\sigma}\,(A,\alpha,\mu)\,.$$

Illustration. Let $(B,\beta,\nu)$ be a trivial 1-point system. Then:

$(A,\alpha,\mu)\,\omega\,(B,\beta,\nu)$ merely says that $\mu\circ\alpha=\mu\,.$

$(B,\beta,\nu)\,\omega^{\sigma}\,(A,\alpha,\mu)$ is equivalent to $\alpha(1_A)=1_A\,.$

If we ultimately want to ensure symmetry of Wasserstein distance between systems, we'll need to require both

$$(A,\alpha,\mu)\,\omega\,(B,\beta,\nu) \quad \text{and} \quad (B,\beta,\nu)\,\omega^{\sigma}\,(A,\alpha,\mu)\,.$$

That is, we'll restrict the allowed transport plans to those for which the reverse is also a transport plan.

# Cost

Finally the bottom line: there will be a cost attached to a transport plan.

Returning to the classical case for the moment, recall that

$$\omega(a \otimes b) = \nu(E_\omega(a)b)$$

for a transport plan $\omega$ between its two marginals $\mu$ and $\nu$.

As an illustrative example, specialize this to

$$\omega(a^* \otimes a) = \nu(E_\omega(a)^* a)\,,$$

where we now consider transport within the same space $X$.

Think of $\nu(E_\omega(a)^* a)$ as a measure of correlation between $E_\omega(a)$ and $a$.

We expect higher correlation to correspond to lower cost.

That is: if $E_\omega$ does less, then $\omega$ costs less.

# Cost: symmetry of correlation

Let's accept the correlation view of cost, even in the quantum case. Expressing cost in our general setup in terms of correlations

$$\nu(E_\omega(a)b) \quad \text{or} \quad \nu(bE_\omega(a))\,,$$

we should expect sufficient symmetry of correlation under reversal to lead to symmetry of Wasserstein distance. Plausibly we could require

$$\nu(E_\omega(a)b) = \mu(aE_\omega^\sigma(b))\,.$$

Interestingly, this requirement can be shown to be equivalent to the transport condition

$$(A, \sigma^\mu, \mu)\,\omega\,(B, \sigma^\nu, \nu)$$

between systems, in terms of the modular dynamics $\sigma^\mu$ and $\sigma^\nu$.

Example. In finite dimensions,

$$\sigma_t^\mu(a \otimes I_m) = (\rho_\mu^{it}\, a\, \rho_\mu^{-it}) \otimes I_m \quad \text{for } a \in M_m \text{ and } t \in \mathbb{R}.$$

# Cost: the classical quadratic case

The cost of a transport plan $\omega$ is built as the sum of terms of the form

$$Q_{a,b}(\omega) = \int_{X\times Y} |a \otimes 1_Y - 1_X \otimes b|^2 \, d\omega \,,$$

where $a \otimes 1_Y(x,y) = a(x)1_Y(y) = a(x)$ and $1_X \otimes b(x,y) = b(y)$.

Example. For $X, Y \subset \mathbb{R}^d$, $a_i(x) = x_i$, and $b_i(y) = y_i$, for some $i$,

$$Q_{a_i,b_i}(\omega) = \int_{X\times Y} |x_i - y_i|^2 \, d\omega(x,y) \,.$$

In general,

$$Q_{a,b}(\omega) = \mu(a^* a) + \nu(b^* b) - \nu(E_\omega(a^*) b) - \nu(b^* E_\omega(a))$$

as is easily verified, in line with our view of higher correlation corresponding to lower cost.

# Cost: the quantum quadratic case

In the quantum case we use exactly the same form

$$Q_{a,b}(\omega) = \mu(a^* a) + \nu(b^* b) - \nu(E_\omega(a^*) b) - \nu(b^* E_\omega(a))$$

where we've written $a^* = a^\dagger$ as is usual in $*$-algebraic notation (to have a unified framework and notation).

The full quadratic cost is of the form

$$\sum_{i=1}^{d} Q_{k_i, l_i}(\omega)$$

in terms of chosen "coordinate systems", each of length $d$,

$$k = (k_1, ..., k_d), \quad k_i \in A \quad \text{and} \quad l = (l_1, ..., l_d), \quad l_i \in B.$$

Remark. When $A = B$ and $k = l$, it generalizes the usual classical quadratic cost when working on a bounded subset of $\mathbb{R}^d$.

# Cost: symmetry and zero cost

Symmetry. Imposing our "correlation symmetry" condition

$$(A,\sigma^{\mu},\mu)\,\omega\,(B,\sigma^{\nu},\nu)\,, \quad \text{or equivalently} \quad \nu(E_{\omega}(a)b)=\mu(aE_{\omega}^{\sigma}(b))\,,$$

we immediately find symmetry of the general cost term

$$Q_{a,b}(\omega)=\mu(a^{*}a)+\nu(b^{*}b)-\nu(E_{\omega}(a^{*})b)-\nu(b^{*}E_{\omega}(a))=Q_{b,a}(\omega^{\sigma})\,.$$

Since this holds for all $a$ and $b$, it is strictly speaking more than we need, but in the system context the condition $(A,\sigma^{\mu},\mu)\,\omega\,(B,\sigma^{\nu},\nu)$ is very natural.

Zero cost. For the case $(A,\mu)=(B,\nu)$ the cost of the diagonal transport plan $(B,\nu)\,\delta_{\nu}\,(B,\nu)$ is

$$Q_{b,b}(\delta_{\nu})=0$$

for each term in the cost, as $E_{\delta_{\nu}}=\mathrm{id}_{B}$. This corresponds to the classical

$$Q_{b_i,b_i}(\delta_{\nu})=\int_{Y\times Y}|x_i-y_i|^{2}\,d\delta_{\nu}(x,y)=\int_{Y}|y_i-y_i|^{2}\,d\nu(y)=0.$$

# Wasserstein distance between systems

First, we equip each system to be considered with its very own coordinate system of length $d$, writing

$$\mathbf{A} = (A, \alpha, \mu, k) \quad \text{and} \quad \mathbf{B} = (B, \beta, \nu, l)$$

for tuples $k = (k_1, ..., k_d)$ and $l = (l_1, ..., l_d)$ from $A$ and $B$ respectively.

Second, we define our set of transport plans

$$T(\mathbf{A}, \mathbf{B})$$

from $\mathbf{A}$ to $\mathbf{B}$ to consist of all states $\omega$ on $A \otimes B'$ satisfying the requirements we have been developing:

$(A, \alpha, \mu)\,\omega\,(B, \beta, \nu)$, the basic transport plan requirement from $\mathbf{A}$ to $\mathbf{B}$,

$(B, \beta, \nu)\,\omega^{\sigma}\,(A, \alpha, \mu)$, the reverse is also a transport plan, from $\mathbf{B}$ to $\mathbf{A}$,

$(A, \sigma^{\mu}, \mu)\,\omega\,(B, \sigma^{\nu}, \nu)$, ensuring equal cost in the two directions.

# Wasserstein distance between systems: the definition

Given our set $T(\mathbf{A}, \mathbf{B})$ as just described, we define

**Wasserstein distance between systems A and B**

$$W_2(\mathbf{A}, \mathbf{B}) = \inf_{\omega \in T(\mathbf{A},\mathbf{B})} I_{\mathbf{A},\mathbf{B}}(\omega)^{1/2}$$

for quadratic cost of the form

$$I_{\mathbf{A},\mathbf{B}}(\omega) = \sum_{i=1}^{d} Q_{k_i, l_i}(\omega)\,.$$

Here, as before,

$$\mathbf{A} = (A, \alpha, \mu, k) \quad \text{and} \quad \mathbf{B} = (B, \beta, \nu, l)\,,$$

and

$$Q_{a,b}(\omega) = \mu(a^* a) + \nu(b^* b) - \nu(E_\omega(a^*) b) - \nu(b^* E_\omega(a))\,.$$

# The triangle inequality

In terms of the (cyclic) representation obtained from a transport plan,

$$I_{\mathbf{A},\mathbf{B}}(\omega) = \left\| \pi_\mu^\omega(k)\Omega - \pi_\nu^\omega(l)\Omega \right\|_{\oplus\omega}^2 .$$

This should already make the triangle inequality plausible. However, we need a noncommutative substitute for the gluing lemma used in the classical case. A natural bimodule structure on $H_\omega$ given by,

$$a\psi b = \pi_\omega(a \otimes j_B(b))\psi \quad \text{for all } \psi \in H_\omega$$

in our standard form setup, provides a relative tensor product $H_\omega \otimes_\nu H_\varphi$ of such bimodules to play the role of the gluing lemma, leading to

$$W_2(\mathbf{A}, \mathbf{C}) \leq W_2(\mathbf{A}, \mathbf{B}) + W_2(\mathbf{B}, \mathbf{C}) ,$$

using the above along with

$$\mathbf{A}\, \omega\, \mathbf{B} \quad \text{and} \quad \mathbf{B}\, \varphi\, \mathbf{C} \quad \Longrightarrow \quad \mathbf{A}\, \omega \circ \varphi\, \mathbf{C}$$

in terms of $\omega \circ \varphi$ given by $E_{\omega\circ\varphi} = E_\varphi \circ E_\omega$ .

# Faithfulness

Classically, all coordinates of $\mathbb{R}^d$ need to be used to obtain faithfulness of $W_2$.

Analogously, in our general framework, we need to assume that the coordinates generate the algebra. That is, for a system $\mathbf{A}$:

Assume that $k_1, ..., k_d$ generate the von Neumann algebra $A$.

In addition, assume that

$$\{k_1^*, ..., k_d^*\} = \{k_1, ..., k_d\}$$

that is the set of coordinates as a whole is self-adjoint.

If satisfied for all systems being considered, then:

$$W_2(\mathbf{A}, \mathbf{B}) = 0 \implies \mathbf{A} \cong \mathbf{B} \, .$$

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