Transformations of quantum measurements and the Monge-like distance between pure states

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in collaboration with

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What this talk is about?

Two issues to be discussed:

1) Stochastic / bi-stochastic dynamics
in the space of quantum measurements,

2) Distances between quantum pure states
induced by the quantum transport problem,

1. Setting the scene: A) **classical** discrete dynamics

A) Classical states: n-point probability vectors p,

$$p = (p_1, \ldots, p_n) \text{ such that } p_i \geq 0, \ \sum_{i=1}^n p_i = 1.$$

discrete classical dynamics: $p' = Tp$, $p'_i = \sum_j T_{ij} p_j$
stochastic transition matrix, $T_{ij} \geq 0$ and $\sum_{i=1}^n T_{ij} = 1$.

In particular, for **bistochastic dynamics**,

$$B_{ij} \geq 0 \text{ and } \sum_{i=1}^n B_{ij} = 1 = \sum_{j=1}^n B_{ij}$$

the **majorization** relation (for ordered vectors) holds:

$$p' = Bp \prec p \quad \Leftrightarrow \quad \sum_{i=1}^k p'_i \leq \sum_{i=1}^k p_i \ \text{ for any } \ k = 1 \ldots n-1$$

This implies that the **Shannon entropy** does not decrease, $H(p') \geq H(p)$.

The set $\mathcal{B}_n \subset \mathbb{R}^{(n-1)^2}$ of **bistochastic matrices** of order n forms the
Birkhoff polytope = convex hull of all $n!$ permutation matrices.

B) Quantum states: density matrices ρ, of order n,

$\rho = \rho^* \geq 0$ normalized as $\mathrm{Tr}\rho = 1$.

discrete quantum dynamics: $\rho' = \Phi(\rho) = \sum_{i=1}^{M} K_i \rho K_i^\dagger$

where **stochastic map** satisfies *trace preserving* condition,

$$\sum_{i=1}^{M} K_i^\dagger K_i = \mathbb{I}$$

In particular, for a **bistochastic operation** Ψ_B satisfying the dual *unitality* condition, $\sum_{i=1}^{M} K_i K_i^\dagger = \mathbb{I}$

the **majorization** relation for density matrices (and spectra λ) holds:

$$\rho' = \Psi_B(\rho) \prec \rho \quad \Leftrightarrow \quad \lambda(\rho') \prec \lambda(\rho)$$

This implies that the **von Neumann entropy** of a quantum state does not decrease, $S(\rho') \geq S(\rho)$.

For single-qubit case, $n = 2$ any **bistochastic map** is unitarily equivalent to a **Pauli channel** (tetrahedron of rotations by 3 **Pauli** matrices and $\mathbb{I}$).


beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

C) Quantum supermap Γ

Discrete dynamics $\Phi' = \Gamma(\Phi)$ in the space of quantum maps of order d can be represented as a dynamics in the set of M **Kraus operators**,

$$\{K_1', \dots K_M'\} = \Gamma(\{K_1, \dots K_M\}).$$

Denote an **effect** by $E_i = K_i^\dagger K_i \geq 0$

Preservation of trace implies that a collection of effects, $E = (E_1, \dots, E_M)$, satisfies the **identity resolution**, $\sum_{i=1}^M E_i = \mathbb{I}_d$.

The set $\mathcal{E}$ of all such block-vectors E can be called a *'quantum simplex'* as it reduces to the standard M-point probability simplex for $d = 1$.

A particular class of supermaps: **discrete dynamics** in the set $\mathcal{E}$ of **effects**: induced by **sequential block product** $E' = \mathbf{T} * E$, so that $E_i' = \sum_{j=1}^M \sqrt{E_j} T_{ij} \sqrt{E_j}$ (**Gudder, Nagy** 2001; **Leifer** 2007), where **block-wise stochastic matrix T** with positive definite block-entries, $T_{ij} \geq 0$, satisfies a *column-wise* condition, $\sum_{i=1}^M T_{ij} = \mathbb{I}$.

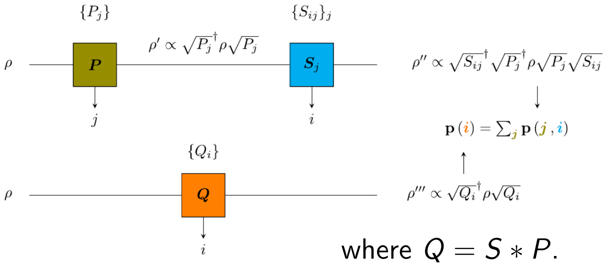
Block-wise stochastic dynamics

For any two block matrices A, B of size nd with n^2 positive blocks, $B_{ij} = B_{ij}^* \geq 0$ od order d, define **block-wise product** $A * B$,

$$(A * B)_{ik} := \left(\sum_j \sqrt{B_{jk}}\, A_{ij}\, \sqrt{B_{jk}} \right).$$

Then a quantum **sequential measurement**

(effects P_j come first, then, depending on the output, effects S_{ij})
can be described by a single measurement with effects Q_j,



where $Q = S * P$.

Block-wise bistochastic matrices

Consider block-wise matrix $B_{n,d}$ with n^2 semi-positive blocks, $B_{ij} \geq 0$, which satisfy *columnwise* and *row-wise* conditions,

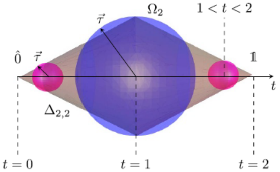
$$\sum_{i=1}^{n} B_{ij} = \mathbb{I} = \sum_{j=1}^{n} B_{ij}.$$

It froms a **blockwise bistochastic matrix**, also called *block bistochastic* **Benoist, Nechita** (2017) and *quantum magic square*, **De les Coves, Netzer, Valentiner-Branth** (2023), which for $d = 1$ it reduces to standard **bistochastic matrix**.

Simple example: $n = 2$, $d = 2$ of a block bistochastic matrix

$$B_{2,2} = \begin{bmatrix} A & \mathbb{I} - A \\ \mathbb{I} - A & A \end{bmatrix}$$

where $0 \leq A \leq \mathbb{I}$.



Birkhoff polytope and beyond

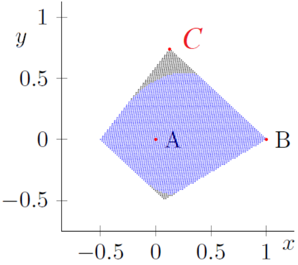
Classical case: bistochastic matrices (1946):
the set $\mathcal{B}_n = \mathcal{B}_{n,1}$ of **bistochastic** matrices, forms the **Birkhoff polytope**, convex hull of all $n!$ **permutation** matrices Π_j, so $B = \sum_j a_j \Pi_j$.

Quantum case: block-wise bistochastic matrices – an attempt to generalize Birkhoff. A convex combination of extended permutations Π_i,

$$B = \sum_j a_j \Pi_j \otimes E_j, \quad \sum_j E_j = \mathbb{I}_d, \quad \sum_j a_j = 1$$

is called **semi-classical** (SC). Observation of **De les Coves** et. al (2023):
a) for any d the set $\mathcal{B}_{2,d}$ of block-wise bistochastic matrices is equal to SC, (*Birkhoff-like statement*).
b) for any $n \geq 3$ and $d \geq 2$ the set $\mathcal{B}_{n,d}$ is *larger* than the semi-classical set SC.

Examplary cros-section of the set $\mathcal{B}_{3,2}$ determined by the center $A_{ij} = \mathbb{I}_2/3$, $B = \mathbb{I}_6$, and **non-SC matrix** C (extreme point) composed of 9 blocks of rank one.
SC matrices plotted in blue.



Bistochastic matrices: Dynamical characterisation

Classical case (1952)

Ostrowski characterization of the set B_n of **bistochastic** matrices

Square matrix B of order n with non-negative entries is **bistochastic** iff **majorization** relation, $Bp \prec p$, holds for any n-point probability vector p.

Quantum case (2024)

Definition. A vector P of effects, $P_i \geq 0$ and $\sum_{i=1}^{n} P_i = \mathbb{I}$, is called *sortable* if its effects can be sorted as $\mathbb{I} \geq P_1 \geq P_n \geq 0$.

Quantum analog of **Ostrowski characterization** of the set $B_{n,d}$ of **block-wise bistochastic** matrices:

Block-wise matrix B of order n with positive blocks $B_{ij} \geq 0$ is **block-wise bistochastic** iff **block majorization** relation

$$Q = B * P \prec P \quad \Leftrightarrow \quad \sum_{j=1}^{k} Q_j \leq \sum_{j=1}^{k} P_j$$

holds for $k = 1, \ldots n$ and any *sortable* vector P of n positive blocks P_j suming to identity, **A. Rico, K.Ż**, *J. Phys.* **A** (2024)

n-point probability simplex Δ_n	Set of quantum measurements $\Delta_{n,d}$
Probability vector $p=(p_1,\ldots,p_n)^T\in\Delta_n$, $p_j\geqslant 0,\quad \sum_{j=1}^n p_j=1$	Blockwise probability vector (POVM) [13] $\boldsymbol{P}=(P_1,\ldots,P_n)^\dagger\in\Delta_{n,d}$ $P_j\geqslant 0,\quad \sum_{j=1}^n P_j=\mathbb{1}_d$
Stochastic matrix $S=(s_1,\ldots,s_n)\in\Delta_n{}^{\times n}$ $s_j=(s_{1j},\ldots,s_{nj})\in\Delta_n$	Blockwise stochastic matrix $\boldsymbol{S}=(\boldsymbol{S_1},\ldots,\boldsymbol{S_n})\in\Delta_{n,d}{}^{\times n}$ $\boldsymbol{S_j}=(S_{1j},\ldots,S_{nj})\in\Delta_{n,d}$
Transformations within Δ_n $Sp=q$ $q_i=\sum_{j=1}^n s_{ij}p_j$	Transformations within $\Delta_{n,d}$ $\boldsymbol{S}*\boldsymbol{P}=\boldsymbol{Q}$ $Q_i=\sum_{j=1}^n\sqrt{P_j}S_{ij}\sqrt{P_j}$
Allowed transformations $p\underset{S}{\rightarrow}q\in\Delta_n$ always possible.	Allowed transformations $\boldsymbol{P}\underset{\boldsymbol{S}}{\rightarrow}\boldsymbol{Q}\in\Delta_{n,d}\iff$ Jointly measurable
Bistochastic $B=(b_{ij})$ $b_{ij}\geqslant 0;\quad \sum_i b_{ij}=\sum_j b_{ij}=1$	Blockwise bistochastic [14, 21] $\boldsymbol{B}=(B_{ij})$ $B_{ij}\geqslant 0;\quad \sum_{i=1}^n B_{ij}=\sum_{j=1}^n B_{ij}=\mathbb{1}_d$
Sortability Nonincreasing order, $1\geqslant p_1\geqslant\ldots\geqslant p_n\geqslant 0$	Sortability Sortable subset, $\mathbb{1}\geqslant P_1\geqslant\ldots\geqslant P_n\geqslant 0$
Majorization for all $p\in\Delta_n$ $p\succ q=Bp$: $\sum_{j=1}^k p_j\geqslant\sum_{j=1}^k q_j$	Majorization for $\boldsymbol{P}\in$ sortable $\subset\Delta_{n,d}$ $\boldsymbol{P}\succ\boldsymbol{Q}=\boldsymbol{B}*\boldsymbol{P}$: $\sum_{j=1}^k P_j\geqslant\sum_{j=1}^k Q_j$

a comparison: **classical** – **quantum**

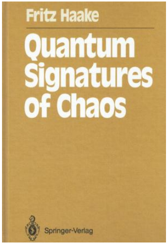
Wawel castle in Cracow

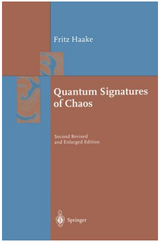
D.& K. Ciesielscy theorem

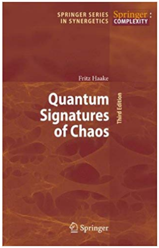
D.& K. Ciesielscy theorem: For any $\epsilon > 0$ there exist $\eta > 0$ such that with **probability** $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in **1916** was localized in η-**neighbourhood** of the **red arrow**.

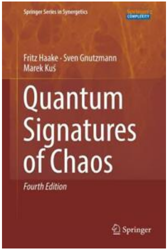
Quantum Signatures of Chaos: **Fritz Haake**, 1941 – 2019

Four editions (1991 – 2018) of the key reference on **quantum chaos**



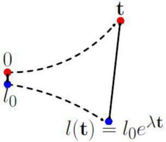






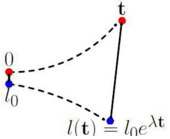
Quantum Signatures of Chaos:

How to define a quantum analogue of the **Lyapunov exponent** ?



but $D_{HS}(\rho, \sigma) = D_{HS}(U\rho U^\dagger, U\sigma U^\dagger)$.

How to define a quantum analogue of the **Lyapunov exponent** ?



but $D_{HS}(\rho, \sigma) = D_{HS}(U\rho U^\dagger, U\sigma U^\dagger)$.

Ann. Physik 1 (1992) 531 – 539

At the basis of our study lies a generalization of the Lyapunov exponent,

$$\lambda = \lim_{t\to\infty} \lambda(t), \quad \lambda(t) = \lim_{d(0)\to 0} \frac{1}{t} \ln\left(\frac{d(t)}{d(0)}\right),$$

Annalen der Physik

Johann Ambrosius Barth 1992

which distance d ?

Lyapunov exponents from quantum dynamics

Fritz Haake, Harald Wiedemann, and Karol Życzkowski*

Vistas in Astronomy, Vol. 37, pp. 153–156, 1993
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HOW TO GENERALIZE THE LAPUNOV EXPONENT FOR QUANTUM MECHANICS

Karol Życzkowski, *† Harald Wiedemann† and Wojciech Słomczyński‡

Monge distance between both Q - functions

Are all 'reasonable' distances between quantum states unitarily invariant,

$$D(\rho, \sigma) = D(U\rho U^\dagger, U\sigma U^\dagger) \ ?$$

a counter example: the **Monge distance**

J. Phys. A: Math. Gen. **31** (1998) 9095–9104. Printed in the UK PII: S0305-4470(98)93137-7

The Monge distance between quantum states

Karol Życzkowski†§ and Wojeciech Słomczyński‡‖

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 6689–6722 PII: S0305-4470(01)18080-7

The Monge metric on the sphere and geometry of quantum states

Karol Życzkowski[1,2] and Wojciech Słomczyński[3]

defined between the corresponding Q-functions, $Q_i(\alpha) = \langle\alpha|\rho_i|\alpha\rangle$,

$$D_M(\rho_1, \rho_2) = D_M\big(Q_1(\alpha), Q_2(\alpha)\big)$$

Monge problem (1781)

An optimal scheme of translocation of soil between the initial shape $Q_1(x_1, x_2)$ and the final one $Q_2(x_1, x_2)$ gives the **Monge distance** between both probability distributions, $D_M(Q_1, Q_2)$.

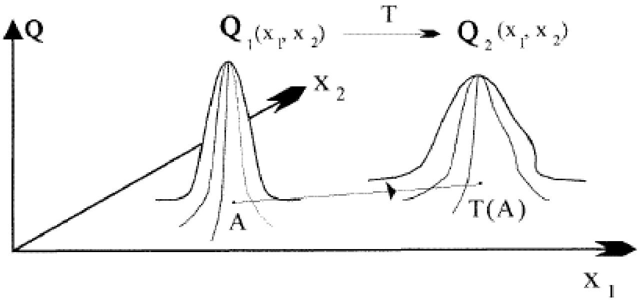
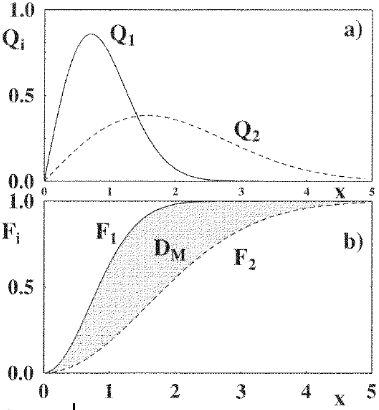


Figure 1. Monge transport problem: how to move a pile of sand $Q_1(x_1, x_2)$ to a new location $Q_2(x_1, x_2)$ minimizing the work done?

we minimize the **total work** against **friction**,
(vertical component is neglected!)

1D problem – solution of T. Salvemini
Sul calcolo degli indici di concordanza... (1943)

For any two 1D probability distributions $Q_1(t)$ and $Q_2(t)$,
represented by their cummulative distributions, $F_i(x) = \int_{-\infty}^{x} Q_i(t)\, dt$,



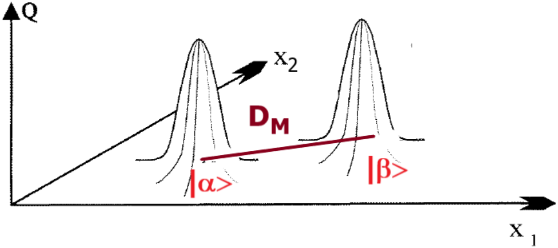
their **Monge distance** reads,

$$D_M(Q_1, Q_2) = \int_{-\infty}^{+\infty} |F_1(x) - F_2(x)|\, dx.$$

Monge metric & quantum states: a) infinite space

natural choice: harmonic oscillator **coherent states** $|\alpha\rangle$ for $\alpha \in \mathbb{C}$

Monge distance between any two *coherent states* satisfies classical property :



$$D_M(|\alpha\rangle, |\beta\rangle) = |\alpha - \beta|$$

2D problems with radial symmetry $\Rightarrow$ **1D** solution of Salvemini works!

Fock states $|n\rangle$ with $n = 0, 1, 2 \ldots$ *with* $D_{HS}(|i\rangle, |j\rangle) = \sqrt{2} = \text{const}$

$D_M(|0\rangle, |1\rangle) << D_M(|1\rangle, |100\rangle)$ (*as desired*)

thermal states $|\bar{n}\rangle$ with mean number of photons equal to $\bar{n}$

$D_M(|\bar{n}\rangle, |\bar{m}\rangle) \approx |\sqrt{n} - \sqrt{m}|.$

Wawel Castle in Cracow

Plate commemorating the discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

transport problem – **Kantorovich** formulation (1939)

Mathematical Methods in the Organization and Planning of Production

Transport plan

A **transport plan** is a measure ω on $X \times Y$ such that

$$\omega(A \times Y) = \mu(A),\ \omega(X \times B) = \nu(B), \text{ for any } A \subset X,\ B \subset Y.$$

Kantorovich optimal transport problem (1942)

Denote by $\Gamma(\mu, \nu)$ the set of all transference plans for fixed μ, ν.

$$\text{Find } \gamma, \text{ which realises } \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x, y) d\gamma(x, y).$$

Wasserstein p–distances (1969)

Let $Y = X$ and take c to be a **distance function**. Then, for any $p \geq 1$,

$$W_{c,p}(\mu, \nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x, y)^p d\gamma(x, y) \right)^{1/p}$$

is a distance on $\mathcal{P}(X) \simeq S(\mathcal{C}(X))$.

Discrete optimal transport

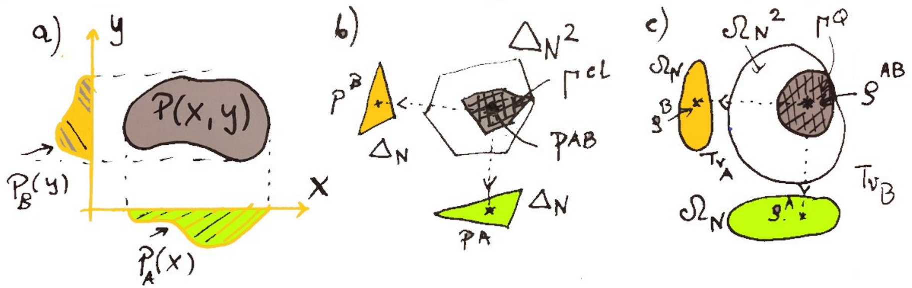
- Take an N point set $X = Y = \{x_i\}_{i=1}^N$.
- Consider two probability vectors p^A, p^B of length N,
  which can be seen as *classical states* $p^A, p^B \in \mathcal{P}(X)$.
- A transport plan $P^{AB} \in \Gamma^{cl}(p^A, p^B)$ is a classical state $\mathcal{P}(X \times X)$.
- P^{AB} is identified with the probability vector $\widetilde{P}^{AB}$ of length N^2.
- Define a *diagonal* **coupling matrix** $\rho^{AB}_{\mu\nu} := \widetilde{P}^{AB}_{\mu}\delta_{\mu\nu}$, for $\mu, \nu = 1, \ldots, N^2$.

- Take a distance function d on X and define a matrix $E_{ij} := d(x_i, x_j)$.
- Recast E into a vector $\widetilde{E}$ of length N^2.
- Define a *diagonal* **cost matrix** $C^{cl}_{\mu\nu} := \widetilde{E}_{\mu}\delta_{\mu\nu}$.
- The **classical optimal transport problem** then reads

$$T^{cl}_C(p^A, p^B) := \min_{P^{AB} \in \Gamma^{cl}(p^A, p^B)} \operatorname{Tr} C^{cl} \rho^{AB}.$$

Quantum optimal transport – idea

Kantorovich formulation of transport problem for:



a) continuous 1D probabilities $p_A(x)$ and $p_B(y)$
coupled by a joint distribution $P(x,y)$;

b) two N-point classical states $p^A, p^B \in \Delta_N$ coupled by a joint state
$P^{AB} \in \Gamma^{cl} \subset \Delta_{N^2}$ with adjusted marginals;

c) two quantum states $\rho^A, \rho^B \in \Omega_N$ coupled by a bipartite state
$\rho^{AB} \in \Gamma^Q \subset \Omega_{N^2}$ such that $\mathrm{Tr}_A\, \rho^{AB} = \rho^B$ and $\mathrm{Tr}_B\, \rho^{AB} = \rho^A$.

Quantum optimal transport – brief history

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  - Friedland, Eckstein, Cole, K. Ż. *Phys. Rev. Lett.* (2022)
  - *several other recent papers,* (2022-2025)

Quantum optimal transport – definition

- $\Omega_N := \{\rho \in \mathcal{B}(\mathbb{C}^N) \mid \rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{Tr}\,\rho = 1\}$
  density matrices of order N.
- Fix two states $\rho^A, \rho^B \in \Omega_N$.
- Consider a **coupling matrix** (or "**quantum transport plan**")
  $$\rho^{AB} \in \Omega_{N^2}, \text{ such that } \mathrm{Tr}_A\,\rho^{AB} = \rho^B \text{ and } \mathrm{Tr}_B\,\rho^{AB} = \rho^A.$$
- Denote by $\Gamma^Q(\rho^A, \rho^B) \subset \Omega_{N^2}$ the set of all coupling matrices.
  - Note that $\rho^A \otimes \rho^B \in \Gamma^Q(\rho^A, \rho^B)$.
- Take a **quantum cost matrix** $C = C^\dagger \in \mathcal{B}(\mathbb{C}^{N\times N})$.
- The **quantum optimal transport problem** defined by the **minimum**
  $$T_C^Q(\rho^A, \rho^B) := \min_{\rho^{AB} \in \Gamma^Q(\rho^A, \rho^B)} \mathrm{Tr}\, C\rho^{AB}.$$

How to select a suitable **cost matrix C**?

Quantum cost matrix C^Q: $\mathrm{diag}(C^Q) = C^{cl}$.

Motivations:

- semi-classical limit of QM (∞ dim) [Golse, Mouhot, Paul, Caglioti]
- quantum transport plans $\leftrightarrow$ quantum channels [De Palma, Trevisan (2019)]
- Hamming distance [De Palma, Marvian, Trevisan, Lloyd (2019)]

Our motivation: (coherification of the diagonal classical cost matrix C^{cl})

- Find cost matrices, which yield an analogue of **Wasserstein** distances.

Projective cost matrix C^Q – antisymmetric subspace – singlet state

Take a computational basis $\{|i\rangle\}_{i=1}^N$ and set $|\psi_{ij}^-\rangle = \frac{1}{\sqrt{2}}(|i,j\rangle - |j,i\rangle)$.

$$C^Q = \sum_{j>i=1}^{N} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| = \tfrac{1}{2}\left(\mathbb{1}_{N^2} - \mathrm{SWAP}\right) = (C^Q)^2.$$

The same idea explored in: Reira (2018); Yu, Zhou, Ying, Ying (2018) and Chakrabarti, Huang, Li, Feizi, Wu (2019).

Properties of the quantum optimal transport cost

$$T^Q(\rho^A,\rho^B) := \min_{\rho^{AB}\in\Gamma^Q} \operatorname{Tr} C^Q \rho^{AB}, \quad W_p := \left(T^Q\right)^{1/p}, \quad W := W_2 = \sqrt{T^Q}$$

Theorem

The optimal quantum transport cost T^Q on N-level systems is

- convex,
- symmetric,
- non-negative,
- $T^Q(\rho^A,\rho^B) = 0$ if and only if $\rho^A = \rho^B$,
- $T^Q\left(\rho^A,\rho^B\right) = T^Q\left(U\rho^A U^\dagger, U\rho^B U^\dagger\right)$ for any $U \in \mathcal{U}(N)$.

Corollary

For any $p \geq 1$, W_p is a unitarily invariant semidistance on Ω_N.

Bounds on quantum optimal transport, $W = \sqrt{T^Q}$

Fidelity $F(\rho^A, \rho^B) := \left(\mathrm{Tr}\left|\sqrt{\rho^A}\sqrt{\rho^B}\right|\right)^2$.

Quantum distances:

$$I := \sqrt{1 - F}, \qquad \text{root infidelity,}$$

$$B := \sqrt{2\left(1 - \sqrt{F}\right)} \qquad \text{Bures distance.}$$

Bounds on quantum optimal transport, $W = \sqrt{T^Q}$

Fidelity $F(\rho^A, \rho^B) := \left(\mathrm{Tr}\left|\sqrt{\rho^A}\sqrt{\rho^B}\right|\right)^2$.

Quantum distances:

$$I := \sqrt{1 - F}, \qquad \text{root infidelity},$$

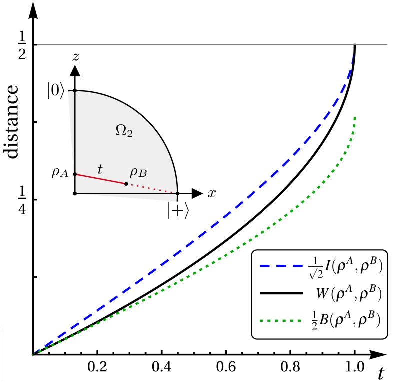
$$B := \sqrt{2\left(1 - \sqrt{F}\right)} \qquad \text{Bures distance}.$$

Theorem: bounds for $W = \sqrt{T^Q}$
(based on [Yu, Zhou, Ying, Ying (2018)])

For any $\rho^A, \rho^B \in \Omega_N$ we have

$$\frac{1}{\sqrt{2}} I(\rho^A, \rho^B) \geq W(\rho^A, \rho^B) \geq \frac{1}{2} B(\rho^A, \rho^B).$$

Left ineqality is saturated
if ρ^A or ρ^B is pure.



comparison of distances for an exemplary trajectory

$$\rho^A = \frac{9}{20}\mathbb{1} + \frac{1}{10}|0\rangle\langle 0|,$$

$$\rho^B = (1-t)\rho^A + t(|+\rangle\langle +|)$$

Transport metric for $N \geq 2$

Theorem 1. concerning $N = 2$ and the Bloch ball

For $N = 2$, W_p satisfies the **triangle inequality** iff $p \geq 2$:
For any **mixed states** $\rho^A, \rho^B, \rho^C \in \Omega_2$ one has

$$W_p(\rho^A, \rho^B) + W_p(\rho^B, \rho^C) \geq W_p(\rho^A, \rho^C).$$

Thus, W_p for $p \geq 2$ forms a **distance** on the **Bloch ball** Ω_2.

Theorem 2. concerning pure states of any $N \geq 2$ quantum system

Root optimal transport, $W_2 = \sqrt{T_E^Q}$, related to the cost matrix

$$C_E^Q = \sum_{j>i=1}^{N} E_{ij} |\psi_{ij}^-\rangle\langle\psi_{ij}^-|$$

corresponding to any classical Euclidean distance function $E_{ij} = d(x_i, x_j)$ for pure states of any $N \geq 2$ system satisfies the **triangle inequality** and forms a **Wasserstein distance** (on the set of pure quantum states).

Energy distance for pure quantum states, $N \geq 2$

1. **Monge distance** defined by coherent states is not easy to compute...
hard **optimization problem** (*even for two pure states*)

2. For pure states the **Wasserstein distance** determined by any classical Euclidean distance matrix $E_{ij} = d(x_i, x_j)$ is given **explicitely** !

Example - N points on an (energy) **line**: $E_{ij} = d(x_i, x_j) = |x_i, x_j|$

For a *given* **Hamiltonian** H with non-degenerate eigenvalues E_i and eigenvectors $|i\rangle$, so that $H|i\rangle = E_i|i\rangle$, we set $E_{ij} = |E_i - E_j|$ and obtain

$$W_H^2(|\psi\rangle, |\phi\rangle) = \sum_{j>i=1}^{N} |E_i - E_j|^2 \, |\psi_i\phi_j - \phi_i\psi_j|^2$$

where the analyzed states are expanded in eigenbasis of Hamiltonian, $|\psi\rangle = \sum_i \psi_i |i\rangle$ and $|\phi\rangle = \sum_j \phi_j |j\rangle$.

Energy distance determined by a Hamiltonian H

1. **Energy distance** for any two eigenstates of H are equal to the energy difference

$$W(|i\rangle, |j\rangle) = |E_i - E_j| \quad (**)$$

2. For any to pure states $|\psi\rangle$ and $|\phi\rangle$ their **Energy distance** satisfies the bounds

$$|\langle\phi|H|\phi\rangle - \langle\phi|H|\phi\rangle|^2 \leq W^2(|\phi\rangle, |\psi\rangle) \leq |\langle\phi|H|\phi\rangle - \langle\phi|H|\phi\rangle|^2 + 2(\Delta_\phi^2 + \Delta_\psi^2)$$

where the variance read $\Delta_\phi^2 = \langle\phi|H^2|\phi\rangle - \langle\phi|H|\phi\rangle^2$.
which for two eigenstates $(\Delta_\phi = \Delta_\psi^2 = 0)$ implies Eq. (**).

Example: 1D **Hydrogen atom**, $H = p^2/2m - e^2/r$ and its eigenstates $|n\rangle$: any *standard distance* D_x (trace, HS, Bures) imply equilateral triangle,

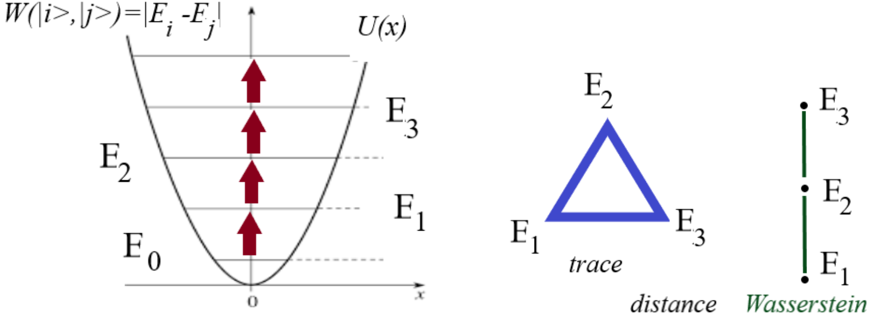
$$D_x(|0\rangle, |1\rangle) \;=\; D_x(|1\rangle, |100\rangle) = D_x(|0\rangle, |100\rangle) \text{ for all eigenstates,}$$

while the **energy** (**Wasserstein**) distance reveals the energy difference:

$$W(|0\rangle, |1\rangle) << W(|1\rangle, |100\rangle) < D_x(|0\rangle, |100\rangle).$$

Trace distance & Energy distance

For eigenstates of H the *energy distance* is equal to the number of **resonant photons** absorbed during the transition



In such a case the **trace distance** between orthogonal states forms an **equilateral triangle**, $D_{tr}(|1\rangle, |3\rangle) = D_{tr}(|1\rangle, |2\rangle) = D_{tr}(|2\rangle, |3\rangle)$,

while the **Energy distance** forms a **metric line**'

$W(|1\rangle, |3\rangle) = W(|1\rangle, |2\rangle) + W(|2\rangle, |3\rangle)$.

Quantization of a classical distance: a general approach

Consider a set of N points $x_i \in \mathbb{R}^m$, $k = 1, \ldots, N$.
Denote distances between them by $d_{ij} = d(x_i, x_j)$, also not Euclidean !

Theorem: (**Bistroń, Miller**, 2025 *to appear*). For any chosen classical distance matrix, $d_{ij} = d_{ji} \geq 0$, the map acting on the space of pure quantum states of size N,

$$D_W^2(|\psi\rangle, |\phi\rangle) := \sum_{j>i=1}^{N} d_{ij}^2 \; |\psi_i \phi_j - \phi_i \psi_j|^2,$$

satisfies the triangle inequality and induces a quantum distance in the complex projective space $\mathbb{C}P^{N-1}$.

Here ψ_i and ϕ_j denote complex expansion coefficients,

$$|\psi\rangle = \sum_{i=1}^{N} \psi_i |i\rangle \text{ and } |\phi\rangle = \sum_{j=1}^{N} \phi_j |j\rangle.$$

Proof is based on a generalized Cauchy - Schwarz inequality

generalized **Cauchy - Schwarz** inequality (complex case),
(coefficients ω_{ijk} can be negative!)

Theorem 1. *Fix $n \geq 3$ and an orthonormal system $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \subset \mathbb{C}^n$, and define ω_{ijk} as*

$$\omega_{ijk} := \overline{x}_i \overline{y}_j \overline{z}_k \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix}. \tag{1}$$

Then for any symmetric matrix $(A_{ij}) \in \mathbb{M}_n(\mathbb{R})$

$$\left| \sum_{ijk} A_{ik} A_{jk} \omega_{ijk} \right| \leq \sqrt{\sum_{ijk} A_{ik}^2 \omega_{ijk}} \sqrt{\sum_{ijk} A_{jk}^2 \omega_{ijk}}. \tag{2}$$

Without loss of generality, we can assume that $A_{ii} = 0$ for all i.

Rafał Bistroń and **Tomasz Miller** (2025)

Quantum Hamming distance

Consider two pure states of n-qubit system, $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_2^n$ represented by 2^n coeficients, $\psi_{i_1 \dots i_n}$ and $\phi_{j_1 \dots j_n}$.

Find a true **distance** D_H such that for any two states in the computational basis, $|\Psi\rangle = |i_1 i_2 \dots i_n\rangle$ and $|\Psi\rangle = |j_1 j_2 \dots j_n\rangle$
the distance $D_H(|\Psi\rangle, |\Phi\rangle)$ is equal to the **classical Hamming** distance $d_H(i_k, j_k)$ between the bit strings i_k and j_k,
i.e. the minimal number of NOT gates to transform string i_k into j_k.

Related problem was studied by **Chau** (1999); **De Palma, Marvian, Trevisan, Lloyd** (2019); **Girolami, Anza**, Phys Rev. Lett. (2021); **Grudka, Kurzyński, Sajna, Wójcik**[2], Phys. Rev. A (2024).

Our explicit solution (no optimization needed!) reads

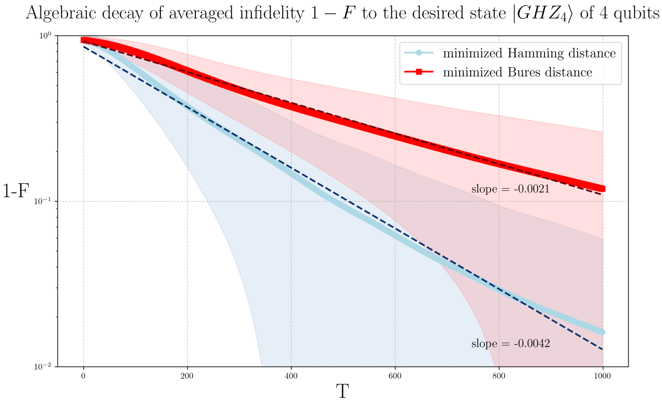
$$D_H^2(|\psi\rangle, |\phi\rangle) := \sum_{i_1,\dots i_n=0}^{1} d_H^2(i_k, j_k)\ |\psi_{i_k}\phi_{k_j} - \phi_{i_k}\psi_{j_k}|^2,$$

and forms a true distance, as the triangle inequality holds.

Quantum Hamming distance & applications

Random search procedure: we wish to get close to a given desired state by minimization a *distance* to the goal: 4-qubit state

$$|GHZ_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}.$$



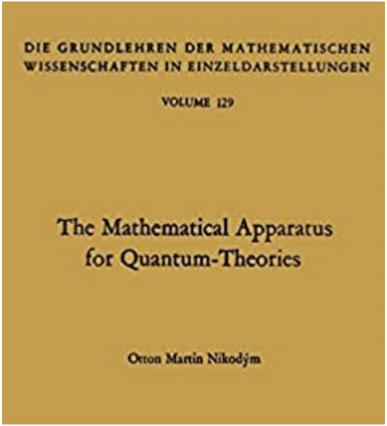
Minimization of **quantum Hamming** distance converges much faster than minimization of *unitarily invariant* **Bures** distance.

Bench commemorating the discussion between
Otton Nikodym and **Stefan Banach** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016

50 years after the discussion at the bench in Cracow, in 1966, **Otton Nikodym** published the book



The Mathematical Apparatus for Quantum-Theories

Concluding Remarks

- Discrete dynamics in the space of quantum measurements can be described by **blockwise** **stochastic** matrices and sequential product.
- **Blockwise** **bistochastic** matrices lead to dynamics characterized by **operator majorization** of *sorted* vectors of effects $E_i = E_i^* \geq 0$.
- A simple generalization of **Birkhoff** polytope is correct for the set $\mathcal{B}_{2,d}$ only. For a larger $n \geq 3$ there exist non-semi-classical blockwise bistochastic matrices outside this set.

- **Monge distance** between two **Husimi functions** satisfy semiclassical property: distance between **coherent states** is equal to the classical distance between the points in the phase space they are localized.
- **Monge-Kantorovich-Wasserstein** approach can be applied for any two states of an arbitrary size N. For a **cost matrix** induced by any classical distance it gives a true distance between any two pure states.
- Examples include **energy distance**, applicable in quantum physics, and **quantum** **Hamming** **distance**, useful for quantum search.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