On the Ricci curvature lower bounds for quantum Markov semigroups

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Ricci curvature lower bounds beyond Riemannian manifolds

Bakry–Émery '85, Lott–Sturm–Villani '06, '09

On a complete Riemannian manifold M, TFAE

- Ric $\geq K$
- Bakry–Émery criterion for (minus) Laplace–Beltrami Δ

$$\Gamma_2(f) \geq K\Gamma(f)$$

  or equivalently the gradient estimates ($P_t = e^{t\Delta}$)

$$\Gamma P_t f \leq e^{-2Kt} P_t \Gamma f$$

- K-convexity of the (relative) entropy S along 2-Wasserstein geodesics.

Γ-calculus

$P_t = e^{-tL}$: a symmetric diffusion Markov semigroup on $L^2(\Omega, \mu)$:

- carré du champ Γ:

$$2\Gamma(f,g) = fL(g) + L(f)g - L(fg), \qquad \Gamma(f) := \Gamma(f,f)$$

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$$-L(\psi \circ f) = -\psi'(f)L(f) + \psi''(f)\Gamma(f).$$

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Example: For the Ornstein–Uhlenbeck semigroup $P_t = e^{t(\Delta - x\nabla)}$ on $(\mathbb{R}^n, \mathrm{d}\gamma)$

$$\Gamma(f,g) = \nabla f \cdot \nabla g, \qquad \Gamma_2(f) = |\nabla f|^2 + |\nabla\nabla f|^2.$$

2-Wasserstein metric

2-Wasserstein metric W_2 on Prob$(\mathbb{R}^n)$

Static definition VS Dynamical formulation: Benamou–Brenier '00

$$W_2(\mu_0,\mu_1)^2$$
$$= \inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2 d\pi(x,y)$$
$$= \inf_{A,\mu}\left\{\int_0^1\int |A_t(x)|^2\mu_t(x)dxdt : \dot{\mu}_t + \nabla\cdot(\mu_t A_t) = 0, \mu_t|_{t=0,1} = \mu_{0,1}\right\}.$$

Jordan–Kinderlehrer–Otto '98...

Heat flow is the gradient flow of the entropy w.r.t. W_2.

Markov chains on finite sets

$\mathcal{K} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ an irreducible Markov kernel on a finite set $\mathcal{X}$ and π : unique stationary probability measure on $\mathcal{X}$. Assume the detailed balance

$$\mathcal{K}(x, y)\pi(x) = \mathcal{K}(y, x)\pi(y), \qquad x, y \in \mathcal{X}.$$

Define the inner products

$$\langle \psi_1, \psi_2 \rangle_\pi = \sum_x \psi_1(x)\psi_2(x)\pi(x), \qquad \psi_1, \psi_2 : \mathcal{X} \to \mathbb{R},$$

and

$$\langle \Psi_1, \Psi_2 \rangle_\pi = \frac{1}{2} \sum_{x,y} \Psi_1(x, y)\Psi_2(x, y)\mathcal{K}(x, y)\pi(x), \qquad \Psi_1, \Psi_2 : \mathcal{X} \times \mathcal{X} \to \mathbb{R}.$$

Then the detailed balance conditions is equivalent to

$$\langle \Delta(\psi_1), \psi_2 \rangle_\pi = \langle \psi_1, \Delta(\psi_2) \rangle_\pi, \qquad \psi_1, \psi_2 : \mathcal{X} \to \mathbb{R},$$

where Δ is the Laplacian:

Laplacian and 1st order differential calculus

The Laplacian ($\rightsquigarrow P_t = e^{t\Delta}$)

$$\Delta(\psi)(x) := \sum_{y:y\sim x} \mathcal{K}(x,y)\,(\psi(y) - \psi(x))$$

satisfies

$$\Delta = \nabla \cdot \nabla = \mathcal{K} - \text{id},$$

with the discrete gradient

$$\nabla\psi(x,y) = \psi(y) - \psi(x), \qquad \psi : \mathcal{X} \to \mathbb{R},$$

and the discrete divergence

$$\nabla \cdot \Psi(x) = \frac{1}{2} \sum_{y\in\mathcal{X}} (\Psi(x,y) - \Psi(y,x))\mathcal{K}(x,y), \qquad \Psi : \mathcal{X} \times \mathcal{X} \to \mathbb{R}.$$

We have the integration by parts

$$\langle \nabla\psi, \Psi\rangle_\pi = -\langle \psi, \nabla \cdot \Psi\rangle_\pi .$$

Ricci curvature lower bounds beyond continuous spaces

For any probability density ρ on $(\mathcal{X}, \pi)$ define

$$|\Psi|_\rho^2 := \frac{1}{2}\sum_{x,y} |\Psi(x,y)|^2 \hat{\rho}(x,y)\mathcal{K}(x,y)\pi(x), \qquad \hat{\rho}(x,y) = \int_0^1 \rho(x)^s \rho(y)^{1-s} \mathrm{d}s$$

$$\mathrm{Ent}_\pi(\rho) := \sum_x \pi(x)\rho(x)\log\rho(x).$$

Bakry–Émery approach	Lott–Sturm–Villani (entropic) approach
$2\Gamma(f,g) := \Delta(fg) - f\Delta(g) - g\Delta(f)$, $2\Gamma_2(f,g) := \Delta\Gamma(f,g) - \Gamma(f,\Delta g) + \Gamma(\Delta f, g)$	$\mathcal{W}_2(\rho_0,\rho_1)^2 := \inf_{\psi,\rho}\left\{\int_0^1 \|\nabla\psi_t\|_{\rho_t}^2 \mathrm{d}t : \dot{\rho}_t + \nabla\cdot(\hat{\rho}_t\nabla\psi_t), \rho_{t=0,1} = \rho_{0,1}\right\}$
$\Gamma_2 \geq K\Gamma$	K-convexity of Ent_π along $\mathcal{W}_2$ geodesics
$\langle\nabla P_t\psi, (\rho\otimes 1)\nabla P_t\psi\rangle_\pi \leq e^{-2Kt}\langle\nabla\psi, (P_t\rho\otimes 1)\nabla\psi\rangle_\pi$	$\|\nabla P_t\psi\|_\rho^2 \leq e^{-2Kt}\|\nabla\psi\|_{P_t\rho}^2$

Lin–Yau '10, Maas '11, Erbar–Maas '12, Mielke '13...and other variants...

Examples

Simple random walk on $\mathbb{Z}_N$

The simple random walk on $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ has the kernel

$$\mathcal{K}(m, m-1) = \mathcal{K}(m, m+1) = \frac{1}{2}.$$

It is known to have entropic Ricci curvature lower bound 0 (Erbar–Maas '12).

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Simple random walk on discrete hypercubes

The simple random walk on $\{0, 1\}^n$ has the kernel

$$\mathcal{K}(x, x^{\oplus j}) = \frac{1}{n}, \qquad 1 \leq j \leq n$$

where $x^{\oplus j}$ means flipping the j-th coordinate of x. It is known to have entropic Ricci curvature lower bound $\frac{2}{n}$ (Erbar–Maas '12).

Quantum Markov semigroups

$(P_t)_{t\geq 0}$ over $M_n(\mathbb{C})$ is a quantum Markov semigroup (QMS) if

- $P_0 = \text{id}$ and $P_sP_t = P_{s+t}$ for all $s, t \geq 0$
- P_t is unital completely positive
- $P_t(x) \to x, t \to 0$ for all $x \in M_n(\mathbb{C})$

The generator

$$\mathcal{L}(x) := \lim_{t\to 0} \frac{x - P_t(x)}{t}.$$

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Example: dephasing semigroup

Let σ be a density matrix in $M_n(\mathbb{C})$ and E a conditional expectation such that

$$\text{Tr}[E(A)\sigma] = \text{Tr}[A\sigma], \qquad A \in M_n(\mathbb{C}).$$

The dephasing semigroup $P_t = e^{-t\mathcal{L}}$:

$$P_t(A) = e^{-t}A + (1 - e^{-t})E(A), \qquad \mathcal{L}(A) = A - E(A).$$

When $\sigma = \mathbf{1}$, one has the depolarizing semigroup

$$P_t(a) = e^{-t}a + (1 - e^{-t})\frac{1}{n}\text{Tr}(a)\mathbf{1}, \qquad \mathcal{L}(A) = A - \frac{1}{n}\text{Tr}(a)\mathbf{1}.$$

Detailed balance condition

Recall the detailed balance condition in the discrete setting

$$\mathcal{K}(x,y)\pi(x) = \mathcal{K}(y,x)\pi(y), \qquad x,y \in \mathcal{X}$$

is equivalent to

$$\langle \Delta(\psi_1), \psi_2 \rangle_\pi = \langle \psi_1, \Delta(\psi_2) \rangle_\pi, \qquad \psi_1, \psi_2 : \mathcal{X} \to \mathbb{R}.$$

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In the quantum setting: For a fixed faithful state σ we say $(P_t) = (e^{-t\mathcal{L}})$ is

- GNS(Gelfand–Naimark–Segal)-symmetric if for all A, B

$$\mathrm{Tr}[\mathcal{L}(A)^* B\sigma] = \mathrm{Tr}[A^* \mathcal{L}(B)\sigma].$$

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- KMS(Kubo-Martin-Schwinger)-symmetric if for all A, B

$$\mathrm{Tr}[\mathcal{L}(A)^* \sigma^{1/2} B \sigma^{1/2}] = \mathrm{Tr}[A^* \sigma^{1/2} \mathcal{L}(B) \sigma^{1/2}].$$

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- BKM(Bogoliubov-Kubo-Mori)-symmetric if for all A, B

$$\int_0^1 \mathrm{Tr}[\mathcal{L}(A)^* \sigma^s B \sigma^{1-s}] \mathrm{d}s = \int_0^1 \mathrm{Tr}[A^* \sigma^s \mathcal{L}(B) \sigma^{1-s}] \mathrm{d}s.$$

Structure of QMS generators with GNS symmetry

Theorem: Alicki '76

A linear operator $\mathcal{L}$ on $M_n(\mathbb{C})$ generates a QMS that is GNS-symmetric with respect to σ iff there exist non-zero $\{V_j\}_{j=1}^d \subset M_n(\mathbb{C})$ and $\{\omega_j\}_{j=1}^d \subset \mathbb{R}$ s.t.

(a) $\mathrm{Tr}(V_j^* V_k) = 0$ for $j \neq k$,

(b) $\mathrm{Tr}(V_j) = 0$ for $1 \leq j \leq d$,

(c) for every $j \in \{1, \ldots, d\}$ there exists $j^* \in \{1, \ldots, d\}$ such that $V_j^* = V_{j^*}$,

(d) $\sigma V_j \sigma^{-1} = e^{-\omega_j} V_j$,

(e) the operator $\mathcal{L}$ acts as

$$\mathcal{L}(A) = \sum_{j=1}^{d} e^{-\omega_j/2}(V_j^*[V_j, A] + [A, V_j^*]V_j), \qquad A \in M_n(\mathbb{C}).$$

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In particular, when $\sigma = \mathbf{1}$, one may write ($\mathrm{Tr}[T^\dagger(A)B^*] = \mathrm{Tr}[AT(B^*)]$)

$$\mathcal{L}(A) = \partial^\dagger \partial(A) = \sum_{1 \leq j \leq d} [V_j^*, [V_j, A]] = \sum_{1 \leq j \leq d} \partial_j^\dagger \partial_j(A).$$

Ricci curvature lower bound for quantum Markov semigroups

For any density matrix $\rho \in M_n(\mathbb{C})$

$$\hat{\rho}A := \int_0^1 \rho^s A \rho^{1-s} \mathrm{d}s, \qquad D(\rho||\sigma) := \frac{\mathrm{Tr}}{n}[\rho(\log\rho - \log\sigma)].$$

$(P_t) = e^{-t\mathcal{L}}$: GNS-symmetric with respect to σ. For simplicity: $\sigma = \mathbf{1}$.

Bakry–Émery approach	Lott–Sturm–Villani approach
$2\Gamma(a,b) := a^*\mathcal{L}b + (\mathcal{L}a)^*b - \mathcal{L}(a^*b)$, $2\Gamma_2(a,b) := \Gamma(a,\mathcal{L}b) + \Gamma(\mathcal{L}a,b) - \mathcal{L}\Gamma(a,b)$	$\mathcal{W}_2(\rho_0,\rho_1)^2 := \inf_{A,\rho}\left\{\int_0^1 \langle A_t, \hat{\rho}_t A_t\rangle \mathrm{d}t : \dot{\rho}_t = \partial^\dagger(\hat{\rho}_t A_t), \rho_{t=0,1} = \rho_{0,1}\right\}$
$\Gamma_2 \geq K\Gamma$	K-convexity of $D(\cdot||\sigma)$ along $\mathcal{W}_2$ geodesics
$\langle \partial P_t a, \rho \partial P_t a\rangle \leq e^{-2Kt}\langle \partial a, P_t\rho\partial a\rangle$	$\int_0^1 \langle \partial P_t a, \rho^s(\partial P_t a)\rho^{1-s}\rangle \mathrm{d}s \leq e^{-2Kt}\int_0^1 \langle \partial a, (P_t\rho)^s \partial a (P_t\rho)^{1-s}\rangle \mathrm{d}s$
call it $\mathrm{BE}(K,\infty)$	call it Ric $\geq K$
easier to compute	has interesting applications e.g. exponential decay of relative entropy

Junge et al, Carlen–Maas... Goal: derive Ric $\geq K$ from $\Gamma_2 \geq K\Gamma$ type computation

The intertwining condition

Ric $\geq K$ would follow from an intertwining condition

$$\partial_j P_t = e^{-Kt} P_t \partial_j$$

The heat semigroup on $(\mathbb{R}^n, \mathrm{d}x)$	$\partial_j P_t = P_t \partial_j$	Ric ≥ 0
The Ornstein–Uhlenbeck semigroup on $(\mathbb{R}^n, \mathrm{d}\gamma)$	$\partial_j P_t = e^{-t} P_t \partial_j$	Ric ≥ 1

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The Bose Ornstein–Uhlenbeck	$\partial_j P_t = e^{-\sinh(\beta/2)t} P_t \partial_j$	Ric $\geq \sinh(\beta/2)$

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The Bose Ornstein–Uhlenbeck	$\partial_j P_t = e^{-\sinh(\beta/2)t} P_t \partial_j$	Ric $\geq \sinh(\beta/2)$
Simple random walk on $\mathbb{Z}_N, N \geq 3$	$\partial P_t = \vec{P}_t \partial$	Ric ≥ 0
Simple random walk on $\{0,1\}^n$	$\partial P_t = \vec{P}_t \partial$	Ric $\geq \frac{2}{n}$

The intertwining condition can be relaxed (Münch–Wirth–Z. '24)

Idea of relaxing intertwining: Wirth–Z. '21

Say we want to prove Ric $\geq K$, and let us consider the gradient estimate form

$$\int_0^1 \langle \partial P_t a, \rho^s (\partial P_t a) \rho^{1-s} \rangle \mathrm{d}s \leq e^{-2Kt} \int_0^1 \langle \partial a, (P_t \rho)^s \partial a (P_t \rho)^{1-s} \rangle \mathrm{d}s$$

which we reformulate as

$$\langle \partial P_t a, \Lambda(L_\rho, R_\rho)(\partial P_t a) \rangle \leq e^{-2Kt} \langle \partial a, \Lambda(L_{P_t \rho}, R_{P_t \rho})(\partial a) \rangle.$$

Here $L_A(X) = AX, R_B(X) = XB$ and $\Lambda(L_\rho, R_\rho)(X) = \int_0^1 \rho^s X \rho^{1-s} ds$.

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Consider a stronger estimate that is equivalent to the operator inequality

$$\vec{P}_t^{\dagger} \Lambda(L_\rho, R_\rho) \vec{P}_t \leq e^{-2Kt} \Lambda(L_{P_t \rho}, R_{P_t \rho}).$$

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By nice properties of Λ, it suffices to show the “linearized"

$$\vec{P}_t^{\,\dagger} L_\rho \vec{P}_t \leq e^{-2Kt} L_{P_t\rho}, \qquad \vec{P}_t^{\,\dagger} R_\rho \vec{P}_t \leq e^{-2Kt} R_{P_t\rho}.$$

Idea of relaxing intertwining: (Münch–Wirth–Z. '24)

So far, we only assume $\partial P_t = \vec{P}_t \partial$, so $\vec{P}_t$ is semi-flexible:

$$\vec{P}_t|_{\mathrm{ran}(\partial)} = \text{fixed} : \qquad \vec{P}_t \partial(a) = \partial P_t(a)$$

while

$$\vec{P}_t|_{\mathrm{ran}(\partial)^\perp} = \text{anything}.$$

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But we need $\vec{P}_t|_{\mathrm{ran}(\partial)^\perp} =$ something that yields

$$\vec{P}_t^{\,\dagger} L_\rho \vec{P}_t \le e^{-2Kt} L_{P_t\rho}, \qquad \vec{P}_t^{\,\dagger} R_\rho \vec{P}_t \le e^{-2Kt} R_{P_t\rho}. \qquad (*)$$

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$$\vec{P}_t|_{\mathrm{ran}(\partial)^\perp} = \text{anything}.$$

But we need $\vec{P}_t|_{\mathrm{ran}(\partial)^\perp} =$ something that yields

$$\vec{P}_t^\dagger L_\rho \vec{P}_t \leq e^{-2Kt} L_{P_t \rho}, \qquad \vec{P}_t^\dagger R_\rho \vec{P}_t \leq e^{-2Kt} R_{P_t \rho}. \qquad (*)$$

Not so clear, so let's consider the infinitesimal form:

$$\partial P_t = \vec{P}_t \partial \qquad \Leftrightarrow \qquad \partial \mathcal{L} = \vec{\mathcal{L}} \partial$$

and $\left((\xi|\eta) := \sum_i \xi_i^* \eta_i\right.$ for $\xi = (\xi_i)$ and $\left.\eta = (\eta_i)\right)$

$$(*) \qquad \Leftrightarrow \qquad \frac{1}{2}\Big((\vec{\mathcal{L}}\xi|\xi) + (\xi|\vec{\mathcal{L}}\xi) - \mathcal{L}(\xi|\xi)\Big) \geq K(\xi|\xi).$$

Idea of relaxing intertwining: (Münch–Wirth–Z. '24)

So far, we only assume $\partial P_t = \vec{P}_t\partial$, so $\vec{P}_t$ is semi-flexible:

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Now: choose a good $\vec{\mathcal{L}} \rightsquigarrow \vec{P}_t$ by optimizing λ in

$$\vec{\mathcal{L}} : \mathsf{ran}(\partial) \oplus \mathsf{ran}(\partial)^\perp \to \mathsf{ran}(\partial) \oplus \mathsf{ran}(\partial)^\perp, \qquad \partial a + \eta \mapsto \partial\mathcal{L}a + \lambda\eta.$$

Example: complete graph

Let $\mathcal{X}$ be a set of n points. Consider

$$L(\psi)(x) = \frac{1}{n} \sum_{y} (\psi(x) - \psi(y)).$$

For any $\xi \in \ell^2(\mathcal{X} \times \mathcal{X})$ with $\xi = \partial f + \eta$, our $\vec{L}$ is of the form

$$\vec{L}\xi = \partial L f + 2K\eta$$

where $K \geq \frac{1}{2} + \frac{1}{n}$ is the Bakry–Émery curvature lower bound.

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Münch–Wirth–Z. '24

This choice of $\vec{L}$ yields Ric $\geq K$ via intertwining. In particular, Ric $\geq \frac{1}{2} + \frac{1}{n}$, recovering a result of Mielke ('13).

Example: dephasing semigroups

The dephasing semigroup $P_t = e^{-t\mathcal{L}}$ on $M_n(\mathbb{C})$ with $E^\dagger(\sigma) = \sigma$

$$P_t(A) = e^{-t}A + (1 - e^{-t})E(A), \qquad \mathcal{L}(A) = A - E(A).$$

For any $\xi \in L^2(M_n(\mathbb{C}), \sigma)^{\oplus d}$ with $\xi = \partial(A) + \eta$ with $\eta \perp \mathrm{ran}\partial$, our $\vec{\mathcal{L}}$ is

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$$C(E) := \sup\{C > 0 : CA \leq E(A), \forall \text{ positive } A \in M_n(\mathbb{C})\}.$$

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Münch–Wirth–Z. '24

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Example: depolarizing semigroups

The depolarizing semigroup $P_t = e^{-t\mathcal{L}}$ on $M_n(\mathbb{C})$ (with $\sigma = \mathbf{1}$)

$$P_t(a) = e^{-t}a + (1 - e^{-t})\frac{1}{n}\mathrm{Tr}(a)\mathbf{1}, \qquad \mathcal{L}(A) = A - \frac{1}{n}\mathrm{Tr}(a)\mathbf{1}.$$

In this case, our result reads

Münch–Wirth–Z. '24

The above intertwining gives Ric $\geq \frac{1}{2} + \frac{1}{n+1}$, improving previous estimate Ric $\geq \frac{1}{2} + \frac{1}{2n}$ (...).

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Münch–Wirth–Z. '24

For depolarizing semigroup, one has Ric $\geq \frac{1}{2} + \frac{1}{n}$ via direct but slightly involved computations.

General picture: 1st order differential calculus

Theorem: Wirth '22

If $\mathcal{L}$ generates a QMS on $M_n(\mathbb{C})$ that is GNS-symmetric with respect to σ, then there exists a finite-dimensional Hilbert C^*-bimodule F over $M_n(\mathbb{C})$, a strongly continuous group of isometries (V_t) on F, an anti-linear operator $\mathcal{J}\colon F \to F$ and a derivation $\partial\colon M_n(\mathbb{C}) \to F$ such that

(a) $V_t(A\xi B) = \sigma^{it}A\sigma^{-it}(V_t\xi)\sigma^{it}B\sigma^{-it}$ for all $A, B \in M_n(\mathbb{C})$, $\xi \in F$,

(b) $\mathcal{J}(A\xi B) = \sigma^{1/2}B^*\sigma^{-1/2}(\mathcal{J}\xi)\sigma^{1/2}A^*\sigma^{-1/2}$ for all $A, B \in M_n(\mathbb{C})$, $\xi \in F$,

(c) $\tau((\mathcal{J}\xi|\mathcal{J}\eta)\sigma) = \tau((\eta|\xi)\sigma)$ for all $\xi, \eta \in F$,

(d) $\mathcal{J}V_t = V_t\mathcal{J}$ for all $t \in \mathbb{R}$,

(e) $\partial(\sigma^{it}A\sigma^{-it}) = V_t\partial(A)$ for all $A \in M_n(\mathbb{C})$, $t \in \mathbb{R}$,

(f) $\partial(\sigma^{1/2}A^*\sigma^{-1/2}) = \mathcal{J}\partial(A)$ for all $A \in M_n(\mathbb{C})$,

(g) $F = \mathrm{lin}\{\partial(A)B \mid A, B \in M_n(\mathbb{C})\}$,

(h) $\Gamma(A, B) = (\partial(A)|\partial(B))$ for all $A, B \in M_n(\mathbb{C})$.

Alicki's theorem: $F = M_n(\mathbb{C})^d$ with $((A_j)|(B_j)) = \sum_j A_j^* B_j$, $(V_t\xi)_j = e^{i\omega_j t}\sigma^{it}\xi_j\sigma^{-it}$, $(\mathcal{J}\xi)_j = \sigma^{1/2}\xi_{j^*}^*\sigma^{-1/2}$ and $(\partial A)_j = e^{-\omega_j/4}[V_j, A]$.

General picture: gradient estimate $\mathrm{GE}(K, \infty)$

Let Λ be an operator mean function with $f(t) = \Lambda(1, t)$, e.g.

$$\Lambda(a, b) = \int_0^1 a^s b^{1-s} \mathrm{d}s, \qquad f(t) = \frac{t-1}{\log(t)}.$$

Define

$$\|\xi\|^2_{\Lambda,\rho} = \tau\left[(\xi | f(V^\rho_{-i})\xi)\rho\right].$$

In Alicki's picture:

$$\|\partial A\|^2_{\Lambda,\rho} = \sum_{j=1}^{d} \tau\left[[V_j, A]^* \Lambda(e^{\omega_j/2} L_\rho, e^{-\omega_j/2} R_\rho)[V_j, A]\right].$$

Definition

We say that a GNS-symmetric QMS with first-order differential calculus $(F, (V_t), \mathcal{J}, \partial)$ satisfies the gradient estimate $\mathrm{GE}_\Lambda(K, \infty)$ if

$$\|\partial(P_t(A))\|^2_{\Lambda,\rho} \leq e^{-2Kt} \|\partial(A)\|^2_{\Lambda, P_t^\dagger \rho}$$

for all self-adjoint $A \in M_n(\mathbb{C})$, positive definite $\rho \in M_n(\mathbb{C})$ and $t \geq 0$.

General picture: intertwining curvature lower bound

Theorem: Münch–Wirth–Z. '24

Suppose that a GNS-symmetric QMS (P_t) has first-order differential calculus $(F, \mathcal{J}, (V_t), \partial)$. If there exists a linear operator $\vec{\mathcal{L}}$ on F such that

(a) $\vec{\mathcal{L}}\partial = \partial\mathcal{L}$,

(b) $\vec{\mathcal{L}}\mathcal{J} = \mathcal{J}\vec{\mathcal{L}}$ and

(c) for all $\xi \in F$,

$$\frac{1}{2}\Big((\vec{\mathcal{L}}\xi|\xi) + (\xi|\vec{\mathcal{L}}\xi) - \mathcal{L}(\xi|\xi)\Big) \geq K(\xi|\xi),$$

then it satisfies $\mathrm{GE}_\Lambda(K, \infty)$ for every operator mean function Λ.

When (a)-(c) are satisfied, we say that (P_t) with first-order differential calculus $(F, \mathcal{J}, (V_t), \partial)$ has intertwining curvature bounded below by K.

Thank you very much!