On the rate of approach to steady state for some Lindblad evolutions equations arising in statistical mechanics.

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Workshop II: Dynamics of Density Operators
Non-commutative Optimal Transport
IPAM, Los Angeles, April 28, 2025

April 28, 2025

Gaussian states and Gaussian quantum operations

Gaussian states and Gaussian quantum operations are ubiquitous in quantum mechanics. There are two versions: Bosonic and Fermionic.

Gaussian probability densities on $\mathbb{R}^n$ may be characterized in many ways. Two of the most important are that **orthogonality implies probabilistic independence**, and that **the entire distribution is characterized by its second moments**.

Quantum analogs of these exist on the algebras of operators on the Boson and Fermion Fock spaces. The Bosonic case is most studied and probably most familiar. Here we are largely concerned with the Fermionic case, but we begin with a few words about the Bosonic case.

Let $\mathcal{H}$ be the real Hilbert space $\mathbb{R}^n$ with its standard inner product. Let $\mathcal{F}$ denote the complex Hilbert space $L^2(\mathbb{R}^n, \gamma(x)\mathrm{d}x)$ where

$$\gamma(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$$

is the standard Gaussian distribution. Let $\mathbf{m} = (m_1, \ldots, m_n)$ where each $m_n \in \mathbb{Z}_{\geq 0}$, and let $H_{\mathbf{m}}(x)$ denote the corresponding Hermite polynomial. For $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{F}_k$ denote the space of complex polynomials of total degree k in $x_1, \ldots, x_n$. Then

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \ .$$

Let $\varphi : \mathbb{R}^n \to \mathcal{F}_1$ be given by $\varphi(f) = \sum_{j=1}^n f_j x_j$. The algebra generated by the image of φ is dense in $\mathcal{F}$. Therefore, any density matrix ρ on $\mathcal{F}$, regarded as a state on $\mathcal{B}(\mathcal{F})$, is determined by the moments

$$\rho\left(\varphi(f_1)\varphi(f_2)\cdots\varphi(f_k)\right) \ .$$

Define the $n \times n$ matrix $M_{i,j}$ by $M_{i,j} = \rho(\varphi(f_i)\varphi(f_j))$. Then ρ is a (centered) Gaussian state if and only if for even k

$$\rho\left(\varphi(f_1)\varphi(f_2)\cdots\varphi(f_k)\right) = \sum_{P \in \mathscr{P}_n} \prod_{\{i,j\} \in P} M_{i,j}$$

where $\mathscr{P}_n$ is the set of all partitions of $\{1, \dots, n\}$ into pairs, while for odd k, $\rho\left(\varphi(f_1)\varphi(f_2)\cdots\varphi(f_k)\right) = 0$.

Alternatively, Gaussian states can be characterized in terms of the Wigner transforms: **A state ρ is Gaussian if and only if its Wigner transform is a Gaussian probability density on $\mathbb{R}^{2n}$.**

A quantum operation is Gaussian if and only if it takes Gaussian states to Gaussian states. **A quantum dynamical semigroup (QDS) is Gaussian if and only if it takes Gaussian state to Gaussian states.**

For any Gaussian state σ, there is a cannonical quantum dynamical semigroup $\{\mathscr{P}_t\}_{t\geq 0}$ that has σ as its unique steady state and which satisfies the detailed balance condition with respect to σ.

There is an explicit expression not only for the generator, but the semigroup itself – a kind of **"Bosonic Mehler formula"**. (We shall give the Fermionic version soon).

Let us consider $n = 1$. then there is a one parameter family of centered Gaussian state ρ_ν, $\nu \in (0, 1)$, given by

$$\rho = (1 - \nu) \sum_{k=0}^{\infty} \nu^k |k\rangle\langle k|$$

where $|k\rangle\langle k|$ denotes the orthogonal projection onto the span on $H_k(x)$. Let $\{\mathscr{P}_t\}_{t\geq 0}$ be the canonical Gaussian QDS specified by this Gaussian state.

Let $D(\rho||\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$ be the relative entropy of ρ with respect to σ. Then by the data processing inequality

$$D(\mathscr{P}_t\rho||\sigma) = D(\mathscr{P}_t\rho||\mathscr{P}_t\sigma) \leq D(\rho||\sigma) \ ,$$

so that $t \mapsto D(\mathscr{P}_t\rho||\sigma)$ is monotonically decreasing. In fact it decreases to zero exponentially fast. With the time scale fixed in a natural way (in terms of the action on $|1\rangle\langle 1|$), it was proved by Jan Maas and myself that

$$D(\mathscr{P}_t\rho||\sigma) \leq e^{-t(1-\nu)} D(\rho|\sigma) \ .$$

This had been earlier conjectured by Huber, König and Vershynina, and they showed that this rate is best possible because there is a Gaussian state ρ that saturates the inequality.

That is, Huber, König and Vershynina showed that

$$\sup\left\{ \frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)} \ : \ \rho \text{ Gaussian} \right\} = e^{-t(1-\nu)} \ .$$

Jan Maas and I showed, using quantum optimal transport methods, that

$$\sup\left\{ \frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)} \right\} \leq e^{-t(1-\nu)} \ ,$$

as Huber, König and Vershynina had conjectured.

Consequently,

$$\sup\left\{ \frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)} \right\} = \sup\left\{ \frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)} \ : \ \rho \text{ Gaussian} \right\} \ .$$

This is an example of the “principle” that **Gaussian operators have (only) Gaussian maximizers**, thoroughly understood in the classical case through work of Lieb. Much beautiful and important work has been done proving quantum analogs in the Bosonic setting, but much remains to be done.

There is also a Fermionic notion of Gaussian states and Gaussian QDS, and Jan Maas and I proved the analogous results in this case.

In the fermionic setting, the Fock space for n degrees of freedom is finite dimensional – it has dimension 2^n. This is the dimension of both

$$(\mathbb{C}^2)^{\otimes n} \quad \text{and} \quad \bigoplus_{k=0}^{n} (\mathbb{C}^n)^{\wedge k} \ .$$

Fermionic Gaussian states arise naturally in the study of qbit systems.

Is something like

$$\sup\left\{\frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)}\right\} = \sup\left\{\frac{D(\mathscr{P}_t\rho||\sigma)}{D(\rho|\sigma)}\ :\ \rho \text{ Gaussian}\right\}$$

true for Gaussian QDS $\{\mathscr{P}_t\}_{t\geq 0}$ that do not satisfy any sort detailed balance condition?

In this general setting, $t \mapsto D(\mathscr{P}_t\rho||\sigma)$ is still monotonically decreasing, but the semigroup cannot be written as gradient flow with respect to any sort of Riemannian metric, whether a mass transport metric or not.

I do not know the answer to this question, but attached to it are a number of other interesting questions that we shall address here.

Gaussian states and processes in Fermi systems

There is a large literature on the exact solution of models of the type we considered earlier using the Jordan-Wigner transform to write a spin system in terms of fermions, and then using techniques such as Prosen's "third quantization" to solve the systems. Much of this has been focused on spin chains with reservoirs at each end of the spin chain.

We work with a class of Lindblad equations that preserve fermionic Gaussian states, also known as quasi-free states. This class of Lindbladian equations is of interest for many reasons, and so we explain some methodology and make a connection with PDE's.

Gaussian *QDS* in fermion systems

Consider the canonical anti-commutation relations (CAR) algebra for N fermion degrees of freedom acting on a Hilbert space $\mathcal{H}_N$ of dimension 2^N. That is, we have operators $\{a_1, \dots, a_N\}$ on $\mathcal{H}_N$ such that for all $1 \le i, j \le N$, the CAR are satisfied:

$$a_i^\dagger a_j + a_j a_i^\dagger = \delta_{i,j}\mathbb{1} \quad \text{and} \quad a_i a_j + a_j a_i = 0 \ .$$

One can realize this with $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ using a construction of Brauer and Weyl, or by another standard construction, with $\mathcal{H}_N$ being the fermion Fock space over $\mathbb{C}^N$.

The construction of Brauer and Weyl is also known as the Jordan-Wigner Transform. Let σ_x, σ_y and σ_z be the three Pauli matrices:

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

These are self adjoint and unitary, and they anti-commute;

$$\{\sigma_x, \sigma_y\} = \{\sigma_y, \sigma_z\} = \{\sigma_z, \sigma_x\} = 0 \ .$$

Define

$$\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

so that

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ .$$

Then

$$\sigma_+\sigma_- + \sigma_-\sigma_+ = \mathbb{1} \ .$$

Define operators $\{a_1, \ldots, a_N\}$ on $(\mathbb{C}^2)^{\otimes N}$ by

$$a_j := \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_- \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

where the σ_- is in the jth place. Since $\{\sigma_x, \sigma_\pm\} = 0$, this gives us a representation of the CAR.

A natural inner product on the algebra of operators on $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ generated by $\{a_1, \ldots, a_N, a_1^\dagger, \ldots, a_N^\dagger\}$ is

$$\langle A, B\rangle := \tau[A^* B] := 2^{-N}\mathrm{Tr}[A^* B]$$

where τ denotes the normalized trace.

There is another perhaps more familiar construction of the CAR algebra. Let $\mathcal{K}$ be an N dimensional Hilbert space. Let $\mathcal{K}^{\wedge k}$ be the k-fold anti-symmetric tensor product of $\mathcal{K}$ with itself. Then $\mathcal{F}_{\mathcal{K}}$ is the *Fock space*

$$\mathcal{F}_{\mathcal{K}} = \bigoplus_{k=1}^{N} \mathcal{K}^{\wedge k}$$

where $\mathcal{K}^{\wedge 0}$ is the span of $\mathbb{1}$ in $\mathcal{K}$, and in this context we think of it as the vacuum vector. The dimension of $\mathcal{K}^{\wedge k}$ is $\binom{N}{k}$ and hence the dimension of $\mathcal{F}_{\mathcal{K}}$ is $\sum_{k=1}^{N} \binom{N}{k} = 2^N$.

Choose an orthonormal basis $\{\mathbf{u}_, \dots, \mathbf{u_N}\}$ in $\mathcal{K}$ and for $\psi \in \mathcal{K}$ define

$$a^*(\psi) = \sum_{j=1}^{N} \langle \mathbf{u_j}, \psi \rangle \mathbf{a_j^*} \ .$$

Then we may identify $\psi_1 \wedge \cdots \wedge \psi_k$ with $a^*(\psi_1) \cdots a^*(\psi_k)\mathbb{1}$ generating a unitary relating the Fock space construction with the Jordan-Wigner construction.

Let U be an $N \times N$ matrix. Define operators $\{\hat{a}_1, \dots, \hat{a}_N\}$ by

$$\hat{a}_i = \sum_{k=1}^{N} U_{i,k} a_k \ .$$

Then $\hat{a}_i \hat{a}_j + \hat{a}_j \hat{a}_i = \sum_{k,\ell=1}^{N} U_{i,k} U_{j,\ell} (a_k a_\ell + a_\ell a_k) = 0$, and

$$\begin{aligned} \hat{a}_i^\dagger \hat{a}_j + \hat{a}_j \hat{a}_i^\dagger &= \sum_{k,\ell=1}^{N} \overline{U_{i,k}} U_{j,\ell} (a_k^\dagger a_\ell + a_\ell a_k^\dagger) \\ &= \sum_{k,\ell=1}^{N} \overline{U_{i,k}} U_{j,\ell} \delta_{k,\ell} \mathbb{1} = \sum_{k=1}^{N} \overline{U_{i,k}} U_{j,k} \mathbb{1} \end{aligned}$$

Therefore, the condition $U^* U = \mathbb{1}$ is necessary and sufficient for the operators $\{\hat{a}_1, \dots, \hat{a}_N\}$ to satisify the CAR.

Gaussian states

In our fermionic setting, a density matrix is a non-negative operator ρ on $\mathcal{H}_N$ such that $\tau[\rho] = 0$. A **Gaussian state** is a density matrix ρ on $\mathcal{H}_N$ that is **in the closure** of the set of density matrices of the form

$$\rho = \frac{1}{Z} e^{-\widehat{K}}$$

for some quadratic $\widehat{K}$. That is,

$$\widehat{K} = \sum_{i,j-1}^{N} K_{i,j} a_i^\dagger a_j \ .$$

Gaussian states are also known as **quasi-free** states.

Using the normal modes coming from diagonalizing K, define

$$n_\alpha := c_\alpha^\dagger c_\alpha \quad \text{and} \quad n_\alpha^\perp = c_\alpha c_\alpha^\dagger$$

so that $n_\alpha + n_\alpha^\perp = \mathbb{1}$. It is easy to see that n_α and $n_\alpha^\perp$ commute with each c_β and $c_\beta^\dagger$ for $\beta \neq \alpha$.
Then $\widehat{K} = \sum_{\alpha=1}^N \lambda_\alpha(n_\alpha - \frac{1}{2}\mathbb{1})$, and

$$e^{-\widehat{K}} = \prod_{\alpha=1}^N (\mathbb{1} + (e^{-\lambda_\alpha} - 1)n_\alpha)e^{\frac{1}{2}\lambda_\alpha} \ .$$

The general Gaussian state, written in terms of its preferred modes, is

$$\rho = \prod_{\alpha=1}^N \frac{2}{1 + e^{-\lambda_\alpha}}(n_\alpha^\perp + e^{-\lambda_\alpha} n_\alpha) \ ,$$

Theorem

Let ρ be a Gaussian state. Define the $N \times N$ matrix M_ρ by

$$(M_\rho)_{i,j} = \tau(\rho a_i^\dagger a_j) \ .$$

Then

$$0 \leq M_\rho \leq \mathbb{1} \ .$$

Moreover defining

$$K_\rho = \log\left(\frac{\mathbb{1} - M_\rho}{M_\rho}\right) \quad \text{and} \quad \widehat{K_\rho} = \sum_{j,k=1}^{n} [K_\rho]_{j,k} a_j^\dagger a_k \ ,$$

$$\rho = \frac{1}{\tau\left[e^{-\widehat{K}_\rho}\right]} e^{-\widehat{K}_\rho} \ .$$

Strictly speaking K_ρ is only defined when for some $\epsilon > 0, \epsilon\mathbb{1} \leq M_\rho \leq (1-\epsilon)\mathbb{1}$.

Wick's Theorem

Gaussian states are precisely the states whose non-vanishing moments are given by

$$\tau[\rho c^\dagger(\psi_1)\cdots c^\dagger(\psi_k)c(\varphi_1)\cdots c(\varphi_k) = \det\Big(\langle\psi_i, M_\rho\varphi_j\rangle\Big)\ .$$

While one can reconstruct ρ given M_ρ, to compute expectations of polynomials in generators of the algebra, one does not have to.

The following is well-known and easy to prove:

Theorem

Let $\widehat{H} = \sum_{i,j=1}^{N} H_{i,j} a_i^\dagger a_j$. *be a quadratic Hamiltonian, and let* ρ *be a Gaussian state. Define*

$$\rho(t) = e^{it\widehat{H}} \rho e^{-it\widehat{H}} \ .$$

Then for all t, $\rho(t)$ *is a Gaussian state, and*

$$M_{\rho(t)} = e^{itH} M_\rho e^{-itH} \ .$$

We are now ready to define a class of Lindblad equations that take Gaussian states to Gaussian states. The construction is based on an idea of Gustav Mehler.

The classical Mehler formula

Let γ denote the standard unit Gaussian density on $\mathbb{R}$:

$$\gamma(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \ .$$

For any continuous bounded, or even polynomially bounded, function f on $\mathbb{R}$, define

$$P_t f(x) = \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(y)\mathrm{d}y \ .$$

The operators $\{P_t\}_{t\geq 0}$ form a semigroup.

An easy way to see this, and more, is to apply P_t to the generating function of the Hermite polynomials.

$$g_\lambda(x) := e^{\lambda x - \frac{1}{2}\lambda^2} = \sum_{n=0} \frac{\lambda^n}{n!} H_n(x) \ .$$

A simple calculation shows that

$$P_t g_\lambda(x) = \sum_{n=0} \frac{(e^{-t}\lambda)^n}{n!} H_n(x) \quad \text{and hence} \quad P_t H_n(x) = e^{-tn} H_n(x) \ .$$

Next, noting that $e^{-t}x = x(1-t) + o(t)$ and $\sqrt{1-e^{-2t}}y = y\sqrt{2}\sqrt{t+t^2} + o(t)$, for small t, defining $f(x,t) := P_t f(x)$,

$$\frac{\partial}{\partial t} f(x,t) = -x\frac{\partial}{\partial x} f(x,t) + \frac{\partial^2}{\partial x^2} f(x,t) \quad \text{with} \quad f(x,0) = f(x) \ .$$

Let ρ be a probability density on $\mathbb{R}$. Define $P_t^*\rho$ by

$$\int_{\mathbb{R}} (P_t^*\rho)(x) f(x) \mathrm{d}x = \int_{\mathbb{R}} \rho(x) P_t f(x) \mathrm{d}x \ .$$

It follows that

$$P_t^*\rho(x) = \int_{\mathbb{R}} \rho(e^{-t}x - \sqrt{1-e^{-2t}}y)\gamma(\sqrt{1-e^{-2t}}x + e^{-t}y)\mathrm{d}x \ .$$

Then $\rho(x,t) := P_t^*\rho$ satisfies

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}\left(x\rho(x,t) + \frac{\partial}{\partial x}\rho(x,t)\right) \quad \text{with} \quad \rho(x,0) = \rho(x) \ .$$

If $\rho_0(x)$ is Gaussian, so is $\rho(x,t)$.

A Mehler formula for fermions

First, double the variables. Extend $\{a_1, \ldots, a_N\}$ to a representation $\{a_1, \ldots, a_N, b_1, \ldots, b_N\}$ in $2N$ variables. Let $\mathcal{A}$ denote the CAR algebra generated by $\{a_1, \ldots, a_N, b_1, \ldots, b_N\}$ and define $\mathcal{A}$ to be the subalgebra generated by $\{a_1, \ldots, a_N\}$ and $\mathcal{B}$ to be the subalgebra generated by $\{b_1, \ldots, b_N\}$.

Fix constants $\beta_j > 0$, $j = 1, \ldots, N$. For $t > 0$, and $1 \leq j \leq N$, define

$$\hat{a}_j(t) = e^{-t\beta_j} a_j + \sqrt{e^{-2t\beta_j}} b_j \quad \text{and} \quad \hat{b}_j(t) = -\sqrt{1 - e^{-2t\beta_j}} a_j + e^{-t\beta_j} b_j \ .$$

Write this in the form

$$\hat{a}_j(t) = e^{i\widehat{H}(t)} a_j e^{-i\widehat{H}(t)} \quad \text{and} \quad \hat{b}_j(t) = e^{i\widehat{H}(t)} b_j e^{-i\widehat{H}(t)} \ .$$

Note that ρ and σ, considered as elements of the CAR in $2N$ variables, commute and the product is a state in $\mathcal{A}$.

Therefore, we may define

$$P_t^*\rho := \mathbb{E}_{\mathcal{A}}\left(e^{i\widehat{H}(t)}\rho\sigma e^{-i\widehat{H}(t)}\right) \ .$$

Then if ρ is Gaussian, so is $\rho\sigma$, and hence so is $P_t^*\rho$. Therefore, P_t^* is a trace preserving CP semigroup with σ as invariant state.
There is a natural inner product on the CAR defined in terms of σ:

$$\langle A, B\rangle_\sigma = \tau[A^*B\sigma] \ ,$$

It turns out that P_t^* is self adjoint with respect to this inner product, and so this semigroup of quantum operations satisfies the σ detailed balance condition.

Thus, given any Gaussian state σ, the Mehler construction provides is a semigroup P_t^* of quantum operations satisfying the σ detailed balance condition and in particular, $P_t^*\sigma = \sigma$ for all t. Moreover, the Mehler construction gives a Stinespring dilation of this semigroup.

The generators of these semigroups are positive linear combinations of the generators

$$\mathcal{K}_L(\rho) = L^\dagger \rho L - \frac{1}{2}(LL^\dagger \rho + \rho LL^\dagger)$$

and $L = Wa_j$ where $W := \prod_{j=1}^N (a_j^\dagger a_j - a_j a_j^\dagger)$, noting that all factors commute. W is self-adjoint and unitary, and for each j,

$$Wa_j W = -a_j \ .$$

Fermion hypercontractivity

The normalized trace τ is a Gaussian state, and taking all rate $\beta_j = 1$, one gets $P_t^* = e^{-tN}$, where N is the fermion number operator. Gross' conjecture, proved by myself and Lieb in 1992 was that for all X in the CAR, and all $1 < p < q < \infty$,

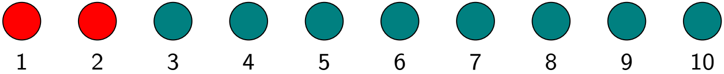
$$\|e^{-tN}X\|_q \leq \|X\|_p \quad \text{for} \quad e^{-2t} \leq \frac{p-1}{q-1} \ .$$

The Stinespring factorization provided by the Mehler formula plays a role in our proof.

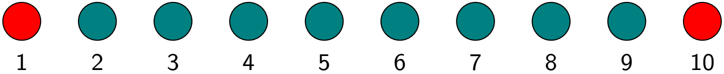
Boundary driven systems

Let $\mathcal{H}$ be the tensor product of two other Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \ .$$



or,



Each dot corresponds to a component system, say as spin in a spin chain, with a single particle Hilbert space $\mathcal{K}$.

There is a given Hamiltonian H acting on $\mathcal{H}$, governing the time evolution of the isolated system. This is the **system Hamiltonian**. We take H to be traceless and write

$$H = H_A \otimes \mathbb{1}_B + H_{AB} + \mathbb{1}_A \otimes H_B \ ,$$

where $H_A = d_B^{-1}\mathrm{Tr}_B[H]$ and $H_B = d_A^{-1}\mathrm{Tr}_A[H]$. $H_{AB} = 0$ then describes the dynamics of separately isolated A and B systems.

We wish to model the interaction of the system with a heat bath, or some other type of reservoir. This is often done in classical statistical mechanics by coupling degrees of freedom at the boundary to a **driving Langevin equation**; i.e., an Ornstein-Uhlenbeck process with equilibrium being a Gaussian of prescribed temperature.

In the quantum setting it is natural to replace the driving Langevin equation with a **driving Lindblad equation**.

The evolution equation for our system then is

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \mathscr{D}\rho \ ,$$

where $\mathscr{D}$ is the dissipator acting on density matrices ρ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Here we shall take $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ with $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes m}$ and $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes (N-m)}$, and shall identify $\mathcal{H}$ with the Fermion Fock space for N degrees of freedom.

Take σ to be a given Gaussian state on $\mathcal{H}_A$, and take $\mathscr{D}_A$ to be the canonical Gaussian generator associated to σ. Take H to be a quadratic Hamiltonian on $\mathcal{H}$.

This specifies a Gaussian QDS. Let $\bar{\rho}$ denotes its steady state, assuming H is such that this is unique. Then $\bar{\rho}$ is the non-equilibrium steady state (NESS) of the system. What can we say about it?

Currents in NESS

Consider a situation in which A consists of several sites. Even though there is only a single connection to a single reservoir, in such cases, there can be current through the bulk in the NESS.
Consider an observable K, maybe the energy or particle number at a site. The corresponding current operator is

$$C_K := -i[H, K] \ .$$

If the state ρ is a generalized Gibbs state, then

$$\mathrm{Tr}[\rho C_K] = i\mathrm{Tr}[\rho(HK - KH)] = i\mathrm{Tr}[[H, \rho]K] = 0 \ .$$

Moreover, if K_1 and K_2 are such that $[K_1, C_{K_2}] = [K_1, C_{K_1}] = 0$, Then correlations are reciprocal: $\mathrm{Tr}[\rho K_1 C_{K_2}] = -\mathrm{Tr}[\rho K_2 C_{K_1}]$.

Finding current with non-zero steady state expectations, or finding non-reciprocal correlations, is a signature of broken time reversal invariance in the steady state.

One expects to find currents in steady state when one connects the bulk up to two reservoirs at two different temperatures. Then energy flows through the system from the hotter reservoir to the colder reservoir.

It is less obvious that coupling to a single reservoir can produce currents in the bulk.

Exact NESS with non-zero currents and one reservoir

The following is joint work with David Huse and Joel Lebowitz. We consider steady state solutions of the following Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -i[\widehat{H}, \rho] + \gamma \mathscr{L} \rho$$

where $\gamma > 0$ is a coupling constant. The Hamiltonian is given by

$$\widehat{H} := -\sum_{j=1}^{N-1} \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right) ,$$

We take A to corrspond to the first two sites, and the bulk B is the rest. Then

$$\widehat{H}_A = -(a_2^\dagger a_1 + a_1^\dagger a_2) = -c_1^\dagger c_1 + c_2^\dagger c_2$$

where

$$c_1 = \frac{1}{\sqrt{2}}(a_1 + a_2) \ , \quad c_2 = \frac{1}{\sqrt{2}}(a_1 - a_2)], \quad n_j = c_j^\dagger c_j \quad \text{and} \quad n_j^\perp = c_j c_j^\dagger$$

for $j = 1, 2$.

The density matrix for the ground state is

$$\sigma := 4 n_1 n_2^{\perp} \ .$$

The state σ is the normalized (for the normalized trace) rank one projector onto the state

$$\frac{1}{\sqrt{2}}(a_1^{\dagger} + a_2^{\dagger})|\mathrm{vac}\rangle \ .$$

We choose $\mathscr{L}$ to be the Mehler semigroup with this steady state, so that

$$\mathscr{L}(\rho) = 2\left(c_1^{\dagger} W \rho W c_1 - \frac{1}{2}(n_1^{\perp}\rho + \rho n_1^{\perp})\right) + 2\left(c_2 W \rho W c_2^{\dagger} - \frac{1}{2}(n_2\rho + \rho n_2)\right) \ .$$

The Hamiltonian will be the hopping Hamiltonian,

$$\widehat{H} := (a_1^\dagger a_2 + a_2^\dagger a_1) + (a_2^\dagger a_3 + a_3^\dagger a_2) + \cdots + (a_{N-1}^\dagger a_N + a_N^\dagger a_{N-1})$$

and the corresponding $N \times N$ matrix has the form

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

Let $\bar{\rho}$ denote the steady state. Then $M_{\bar{\rho}}$ satisfies

$$i[H, M_{\bar{\rho}}] = \gamma M'_{\bar{\rho}}$$

where

$$(M'_{\bar{\rho}})_{i,j} = \tau[(\mathscr{L}\rho)a_i^\dagger a_j] = \tau[\rho\mathscr{L}^\dagger(a_i^\dagger a_j)] \ .$$

Simple computations give

$$\begin{aligned}
\mathscr{L}^\dagger(a_1^\dagger a_1) &= \mathbb{1} - \tfrac{3}{2}a_1^\dagger a_1 - \tfrac{1}{2}a_2^\dagger a_2 \\
\mathscr{L}^\dagger(a_2^\dagger a_2) &= \mathbb{1} - \tfrac{1}{2}a_1^\dagger a_1 - \tfrac{3}{2}a_2^\dagger a_2 \\
\mathscr{L}^\dagger(a_1^\dagger a_2) &= \mathbb{1} - \tfrac{3}{2}a_1^\dagger a_2 - \tfrac{1}{2}a_2^\dagger a_1 \\
\mathscr{L}^\dagger(a_2^\dagger a_1) &= \mathbb{1} - \tfrac{3}{2}a_2^\dagger a_1 - \tfrac{1}{2}a_1^\dagger a_2 \ .
\end{aligned}$$

For all $m \geq 3$ that

$$\mathscr{L}^\dagger(a_1^\dagger a_m) = -a_1^\dagger a_m \quad \text{and} \quad \mathscr{L}^\dagger(a_2^\dagger a_m) = -a_2^\dagger a_m$$

Define m_j to be the $j^{\rm th}$ diagonal entry of $M_{\bar\rho}$, and for $k \geq j$, define $z_{j,k}$ to be the $(j,k)^{\rm th}$ entry of $M_{\bar\rho}$, with

$$z_{j,k} = x_{j,k} + iy_{j,k}$$

being the decomposition of $z_{j,k}$ into its real and imaginary parts.
Therefore,

$$M_{\bar\rho} = \begin{bmatrix} m_1 & z_{1,2} & z_{1,3} & z_{1,4} & \cdots \\ * & m_2 & z_{2,3} & z_{2,4} & \cdots \\ * & * & m_3 & z_{3,4} & \cdots \\ * & * & * & m_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

Then

$$M'_{\bar\rho} = \frac{1}{2}\begin{bmatrix} 1 - \frac{3}{2}m_1 - \frac{1}{2}m_2 & 1 - \frac{3}{2}z_{1,2} - \frac{1}{2}\overline{z_{1,2}} & -z_{1,3} & -z_{1,4} & -z_{1,5} \\ * & 1 - \frac{1}{2}m_1 - \frac{3}{2}m_2 & -z_{2,3} & -z_{2,4} & -z_{2,5} \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} .$$

Lemma

The commutator $[H, M_{\bar{\rho}}]$ *has the following form*

$$([H, M_{\bar{\rho}}])_{i,j} = z_{i,j+1} + z_{i,j-1} - z_{i+1,j} - z_{i-1,j} \ .$$

Thus $[H, M_\rho]$ is given by

$$\begin{bmatrix} -2iy_{1,2} & m_2 - m_1 - z_{1,3} & z_{2,3} - z_{1,2} - z_{1,4} & z_{2,4} - z_{1,3} - z_{1,5} \\ * & 2i(y_{1,2} - y_{2,3}) & m_3 - m_2 + z_{1,3} - z_{2,4} & z_{1,4} + z_{3,4} - z_{2,3} - z_{2,5} \\ * & * & 2i(y_{2,3} - y_{3,4}) & m_4 - m_3 + z_{2,4} - z_{3,5} \\ * & * & * & 2i(y_{3,4} - y_{4,5}) \\ * & * & * & * \end{bmatrix}$$

The equation we have to solve is

$$i[H, M_{\bar{\rho}}] = \gamma M'_{\bar{\rho}} \ .$$

Extracting systems of equations for each super-diagonal, and deducing from these equations for the entries in the first row leads to a complete description of $M_{\bar{\rho}}$ in several stages.

We first use an inductive computation along the diagonal and each of the super-diagonals. We start from the far end where we can rake advantage of $x_{j,N+1} = y_{j,N+1} = 0$. The diagonal and the first diagonal are special since they run into the upper left 2×2 block. After that, there is a general pattern given in the following lemma:

Lemma

For all integers $n \geq 2$. the following identities are valid:

$$y_{j-n-1,j} = 0 \quad \text{for } j = n+3, \ldots, N \qquad (*)$$

$$x_{j-n,j+1} - x_{j-n-1,j} = x_{j-n+1,j} - x_{j-n,j-1} \quad \text{for } j = n+2, \ldots, N$$

$$y_{1,n+2} = -\gamma x_{2,n+2} \qquad (**)$$

$$x_{1,n} + x_{1,n+2} = -\frac{1+\gamma^2}{\gamma} y_{1,n+1}$$

$$y_{1,n} + y_{1,n+2} = \gamma x_{1,n+1}$$

Taking $j = n+3$ in $(*)$, we see that second row is real. (Higher j shows that the same is true for further rows). Then (**) determined the second row in terms of the first. After the second row we can simply use

$$([H, M_{\bar{\rho}}])_{i,j} = z_{i,j+1} + z_{i,j-1} - z_{i+1,j} - z_{i-1,j}$$

Now working back in from the end of the first row, using $x_{1,N+1} = y_{1,N+1} = 0$, yields.

Lemma

For N odd, $N \geq 3$, define $k_* := \frac{N-3}{2}$. For N even, $N \geq 4$, define $k_* := \frac{N-4}{2}$. Then

$$x_{1,N-2k-1} = -\frac{a_k}{\gamma} y_{1,N-2k} \quad \text{for all} \quad 0 \leq k \leq k_*$$

where $a_0 = 1 + \gamma^2$, and for $0 \leq k < k_*$ $a_{k+1} = \left(\frac{1}{a_k} + 1\right)^{-1} + 1 + \gamma^2$. Likewise,

$$y_{1,N-2k-1} = -b_k x_{1,N-2k} \quad \text{for all} \quad 0 \leq k \leq k_*$$

where $b_0 = \frac{\gamma}{1+\gamma^2}$, and for $0 \leq k < k_*$ $b_{k+1} = \left(\frac{1}{b_k} + \frac{\gamma}{1+\gamma^2}\right)^{-1} + \gamma$.

With these recursive formulas in hand, it remains to determine $x_{1,2}$ and $y_{1,3}$. For N odd, with $k_* = \frac{N-3}{2}$,

$$x_{1,2} = -\frac{a_{k_*}}{\gamma} y_{1,3} \ .$$

From and equation arising from the analysis of the first super-diagonal $2\gamma x_{1,2} - y_{1,3} = \gamma$. Thus we have the system

$$\begin{bmatrix} 2\gamma & -1 \\ 1 & \frac{a_{k_*}}{\gamma} \end{bmatrix} \begin{pmatrix} x_{1,2} \\ y_{1,3} \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix} \ .$$

Since $a_{k_*} > 0$, this has a unique solution which is given by

$$x_{1,2} = \frac{a_{k_*}}{1 + 2a_{k_*}} \qquad \text{and} \qquad y_{1,3} = -\frac{\gamma}{1 + 2a_{k_*}} \ .$$

Using these formulas yields the following expression for $M_{\bar{\rho}}$ when $N = 7$, in which

$$2c = 2\gamma^6 + 15\gamma^4 + 33\gamma^2 + 21 \ :$$

$$\frac{1}{2c}\left[\begin{matrix} c & \gamma^6 + 7\gamma^4 + 14\gamma^2 + 8 & -i\left(\gamma^5 + 5\gamma^3 + 5\gamma\right) & -\left(\gamma^4 + 4\gamma + 3\right) & i\left(\gamma^3 + 2\gamma\right) \\ * & c & \gamma^4 + 5\gamma^2 + 5 & 0 & -\gamma^2 - 2 \\ * & * & c & 2\gamma^4 + 8\gamma^2 + 6 & 0 \\ * & * & * & c & 2\gamma^4 + 7\gamma^2 + 6 \\ * & * & * & * & c \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}\right.$$

The last two columns do not fit on the slide, but you see the pattern.

As one can see, the large γ limit of $M_{\bar{\rho}}$ is quite simple for $N = 7$:

$$\lim_{\gamma\to\infty} M_{\bar{\rho}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

and one can also readily determine $\lim_{\gamma\to 0} M_{\bar{\rho}}$. These calculations concerning the limiting values of $M_{\bar{\rho}}$ for $N = 7$ are not misleading. In fact, one can explicitly compute $\lim_{N\to\infty} M_{\bar{\rho}}$ as since $\lim_{k\to\infty} a_k = a_*$ is easily calculated, as is $\lim_{k\to\infty} b_k = b_*$.

Steady state currents

The imaginary entries give rise to non-aero currents, or current correlations. Let $n_j = a_j^\dagger a_j$. Then

$$[\widehat{H}, n_j] = (a_{j-1}^\dagger a_j - a_j^\dagger a_{j-1}) + (a_{j+1}^\dagger a_j - a_j^\dagger a_{j+1}) \ .$$

Call $a_{j-1}^\dagger a_j - a_j^\dagger a_{j-1}$ the particle current across the $j^{\rm th}$ bond.
Since all entries immediately above and below the main diagonal are real, these all have zero expectation in the steady state. However,

$$[\widehat{H}, c_1^\dagger c_1] = \frac{1}{2}((a_3^\dagger a_1 - a_1^\dagger a_3) + (a_3^\dagger a_2 - a_2^\dagger a_3)) \ .$$

Therefore

$$i\tau(\bar{\rho}[\widehat{H}, c_1^\dagger c_1]) = -2\Im((M_{\bar{\rho}})_{1,3} + (M_{\bar{\rho}})_{2,3}) > 0 \ .$$

There are also non-zero correlations between bond currents at $j-1, j$ and particle numbers at k for certain of k near but no equal to either $j-1$ of j. One uses Wick's Theorem to reduce the expected values of the quartics involved to expressions involving the entries of $M_{\bar{\rho}}$. This give an interpretation of the meaning of the remaining imaginary entries.

Thus, we have a genuine NESS with currents, and hence broken time-reversal invariance. **How fast is the steady state approached?**

This is a hard problem, and we will address it directly **only in the large γ-limit**. Note that in the infnite γ limit, the steady state is rather simple, and is an equilibrium state with no currents.

The quantum Zeno effect

In the physics literature,, the large γ limit in our type of system is known as the quantum Zeno regime. **The Zeno effect was first pointed out by Alan Turing in 1954**. There is no published paper, 1954 was the year of his death. His observation, quoted in the book *Alan Turing: Life and Legacy of a Great Thinker* by Hodges is:

It is easy to show using standard theory that if a system starts in an eigenstate of some observable, and measurements are made of that observable N times a second, then, even if the state is not a stationary one, the probability that the system will be in the same state after, say, one second, tends to one as N tends to infinity; that is, that continual observations will prevent motion. . .

All measurement occurs through interaction with an environment, and so if a system is coupled to the environment through a Lindblad dissipative term $\mathscr{D}$, this can nearly "freeze" the evolution generated by $-i[H, \cdot]$ if the coupling is very strong. This "stabilization" has very important applications, and is much studied.

An important contribution was made by Zanardi and Venuti in 2014. To describe it, we return to our general setting of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with

$$\mathscr{L} = \mathcal{K} + \mathscr{D}$$

where $\mathcal{K} := -i\frac{1}{\gamma}[H, \cdot]$ and $\mathscr{D} := \mathbb{1}_B \otimes \mathscr{D}_A$. Assume $\mathscr{D}_A$ has the unique steady state σ and satisfies the detailed balance condition with respect to it. The nullspace of $\mathscr{D}$ is the span of operators of the form $\sigma \otimes X$, $X \in \mathcal{B}(\mathcal{H}_B)$.

Define the projectors

$$\mathscr{P}_0 := \lim_{t\to\infty} e^{t\mathscr{D}} \quad \text{and} \quad \mathscr{Q}_0 := \mathbb{1} - \mathscr{P}_0 \ .$$

Let $\{\varphi_j\}_{0\le j\le d_A^2-1}$ be an orthonormal basis of eigenvectors of $\mathscr{D}_A$ with respect to the σ-GNS inner product.

Theorem

Let ρ be a density matrix on $\mathcal{H}$ such that $\mathscr{P}_0\rho = \rho$, so that $\mathscr{D}\rho = 0$. Then the "leakage" of $e^{t\mathscr{L}}\rho$ outside of the range of $\mathscr{P}_0$ is controlled uniformly in time by

$$\|\mathscr{Q}_0 e^{t\mathscr{L}}\rho\|_1 \le \frac{2}{\gamma}\|K\|_\infty \sum_{j=1}^{d_A^2-1} \|\varphi_j\|_1 \|\varphi_j\|_\infty \ .$$

The proof uses du Hamel's formula. By Theorem 0.5, for large γ and ρ such that $\mathscr{P}_0\rho = \rho$, we have that uniformly in t,

$$e^{t\mathscr{L}}\rho \approx \mathscr{P}_0 e^{t\mathscr{L}}\rho \ .$$

$\mathscr{P}_0 e^{t\mathscr{L}}\rho := \sigma \otimes \tau(t)$ is a positive operator, and $\mathrm{Tr}[\mathscr{P}_0 e^{t\mathscr{L}}\rho] = \mathrm{Tr}[\tau(t)]$. Evidently, $|\mathrm{Tr}[\tau(t)] - 1|$ is of order $\frac{1}{\gamma}$ uniformly in t. Define

$$\rho_B(t) := \frac{1}{\mathrm{Tr}[\tau(t)]}\tau(t) \ .$$

Therefore, uniformly in time,

$$e^{t\mathscr{L}}\rho \approx \sigma \otimes \rho_B(t) \ .$$

The error, in the trace norm, is bounded uniformly in time by a constant multiple of $\frac{1}{\gamma}$.

Corollary

There is a finite constant C such that for all ρ such that $\mathscr{P}_0\rho = \rho$, and all $s, t > 0$,

$$\|e^{(t+s)\mathscr{L}}\rho - e^{t\mathscr{L}}\rho\|_1 \leq C\frac{s}{\gamma} \ .$$

This is the called the quantum Zeno effect: after the "initial layer" enforced by the condition $\mathscr{P}_0\rho = \rho$ or at least $\mathscr{P}_0\rho \approx \rho$, the evolution is very slow for large γ. However, the factor of $\frac{1}{\gamma}$ corresponds to the factor of $\frac{1}{\gamma}$ in $\mathcal{K}$.

How does $\rho_B(t)$ evolve? This question was taken up in 2018 by Popov, Essink, Presilla and Schütz.

Define operators $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{B}(\mathcal{H}_B)$ by

$$\sigma \otimes \mathcal{A}\rho_B = \mathscr{P}_0 \mathcal{K} \mathscr{P}_0(\sigma \otimes \rho_B)$$

and

$$\sigma \otimes \mathcal{B}\rho_B = \mathscr{P}_0 \mathcal{K} \mathcal{S} \mathscr{Q}_0 \mathcal{K} \mathscr{P}_0(\sigma \otimes \rho_B) \ .$$

Then

$$\begin{aligned} \rho_B(t) &= \rho_B(0) + t(\mathcal{A}+\mathcal{B})\rho_B(0) + \int_0^t (t-r)(\mathcal{A}+\mathcal{B})^2 \rho_B(r)\mathrm{d}r \\ &+ \mathcal{O}(\gamma^{-3}) \ . \end{aligned}$$

This comes from iterating du Hamel's formula; i.e., going out to second order in a Dyson expansion in their terminology.

They then show that $\mathcal{A}+\mathcal{B}$ is a Lindbladian operator of the form

$$\gamma(\mathcal{A}+\mathcal{B})\rho_B = -i[\widetilde{H},\rho_B] + \frac{1}{\gamma}\widetilde{\mathscr{D}}\rho_B$$

with explicit formulas for $\widetilde{H}$ and $\widetilde{\mathscr{D}}$. Notice that in this reduced dynamics in the rescaled time, the dissipation is of order $\frac{1}{\gamma}$ instead of γ, as initially.

David Huse, joel Lebowitz and I are currently investigating this in the context of our model. There are two aspects to this. One is to properly derive the effective Linbladian using an addaptation of Davies' analysis of thee weak coupling limit. The second is to understand the the approach to the NESS described by the effective Linbladian and to relate this to the approach to the NESS for the original Lindbladian.

Thank you for your attention!