Quantum Wasserstein and Observability

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•**Schrödinger's equation**

$$i\hbar\partial_t\psi(t,x) = \underbrace{\left(-\tfrac{1}{2}\hbar^2\Delta_x + V(x)\right)}_{\text{quantum Hamiltonian } \mathcal{H}}\psi(t,x)\,, \qquad x\in\mathbb{R}^d$$

for the unknown wave function $\psi\equiv\psi(t,x)\in\mathbb{C}$ s.t. $\|\psi(t,\cdot)\|_{L^2}=1$

•**von Neumann equation**

$$i\hbar\partial_t R(t) = [\mathcal{H}, R(t)]$$

Unknown operator $R(t)=R(t)^*\geq 0$ on $L^2(\mathbb{R}^d)$ s.t. $\mathrm{tr}_{L^2}(R(t))=1$

The potential V is a **real-valued** function s.t. $\mathcal{H}$ has a **self-adjoint** extension to $\mathfrak{H} = L^2(\mathbb{R}^d)$, so that $U(t) := e^{-it\mathcal{H}/\hbar}$ is a **unitary group** on $\mathfrak{H}$ by Stone's theorem, and

$$\psi(t,\cdot) = U(t)\psi(0,\cdot)\,, \qquad R(t) = U(t)R(0)U(t)^*$$

Observing Solutions of a PDE: an Example

Let $U \subset \mathbb{C}$ be a domain, and let $\varnothing \neq \Omega \subset U$ be open.
Let $\bar{\partial}$ be the **Cauchy-Riemann operator** and let $f \in \mathcal{D}'(U)$ satisfy

$$\bar{\partial} f = 0$$

Then (by analytic continuation)

$$f\big|_{\Omega} = 0 \implies f = 0 \text{ on } U$$

Pbm Existence of a constant $C[\Omega, U] > 0$ such that

$$\bar{\partial} f = 0 \text{ on } U \implies \sup_{z \in U} |f(z)| \leq C[\Omega, U] \sup_{z \in \Omega} |f(z)| ?$$

A class $\mathcal{K}$ of **quantum states** at $t = 0$ is "observable" in $\Omega \subset \mathbb{R}^d$ (open) in time $T > 0$ for the dynamics $U(t)$ if there exists $C_{OBS} > 0$ such that

Schrödinger's dynamics For all initial (pure) state $\psi^{in} \in \mathcal{K}$

$$1 = \|\psi^{in}\|_{\mathfrak{H}}^2 \leq C_{OBS} \int_0^T \int_\Omega |\psi(t,x)|^2 dxdt$$

where $\psi(t,\cdot) := U(t)\psi^{in}$ for all $t \in \mathbb{R}$

von Neumann dynamics For all initial density operator $R^{in} \in \mathcal{K}$

$$1 = \mathrm{tr}_{\mathfrak{H}}(R^{in}) \leq C_{OBS} \int_0^T \mathrm{tr}_{\mathfrak{H}}(1_\Omega R(t))dt$$

where $R(t) := U(t)R^{in}U(t)^*$ for all $t \in \mathbb{R}$

J.L. Lions HUM for Schrödinger

Self-adjoint Hamiltonian $H = -\frac{1}{2}\Delta_x + V(x) = H^*$

Control problem

$$\begin{cases} i\partial_t \phi = H\phi + 1_{(0,T)\times\omega} f \\ \phi\big|_{t=T} = 0 \end{cases}$$

Observability problem

$$\begin{cases} i\partial_t \psi = H\psi \\ \psi\big|_{t=0} = \psi^{in} \end{cases}$$

Control operator

$$\mathcal{C} : L^2((0,T)\times\omega) \ni f \mapsto -i\phi\big|_{t=0} \in L^2(\mathbb{R}^d)$$

Observation operator

$$\mathcal{O} : L^2(\mathbb{R}^d) \ni \psi^{in} \mapsto \psi\big|_{(0,T)\times\omega} \in L^2((0,T)\times\omega)$$

For each $\psi^{in} \in L^2(\Omega)$ and each $f \in L^2((0,T) \times \omega)$, one has

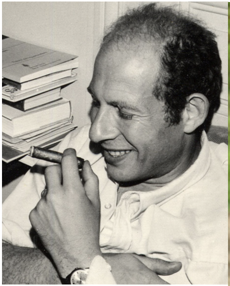
$$\int_0^T \int_\omega \overline{\mathcal{O}\psi^{in}}(t,x) f(t,x) dxdt = \int_0^T \int_{\mathbb{R}^d} \overline{\psi(t,x)}(i\partial_t \phi - H\phi)(t,x) dxdt$$
$$= \int_0^T \int_{\mathbb{R}^d} (\overline{\psi} i\partial_t \phi - \overline{H\psi}\phi)(t,x) dxdt$$
$$= \int_0^T \int_{\mathbb{R}^d} (\overline{\psi} i\partial_t \phi + \phi i\partial_t \overline{\psi})(t,x) dxdt$$
$$= \left[i \int_{\mathbb{R}^d} \overline{\psi}\phi(t,x) dx \right]_{t=0}^{t=T}$$
$$= \int_{\mathbb{R}^d} \overline{\psi^{in}} \mathcal{C} f(x) dx$$

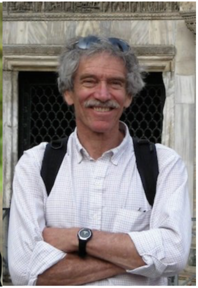
Conclusion

(1) One has $\mathcal{O} = \mathcal{C}^*$ so that $\overline{\operatorname{ran}(\mathcal{C})} = \ker(\mathcal{O})^\perp$

(2) The existence of $C_{OBS} > 0$ implies that $\operatorname{ran}(\mathcal{C}) = \ker(\mathcal{O})^\perp$

In 1992, C. Bardos, G. Lebeau and J. Rauch provided a necessary and sufficient condition for controllability/observability of solutions of the wave equations in a bounded domain Ω by a subset of its (smooth) boundary $\partial\Omega$.





Their proof uses microlocal analysis based on an important earlier observation by J. Ralston (CPAM1969), and on the Melrose-Sjöstrand (CPAM1978-1982) propagation of singularities in domains with boundaries

Let $(X,\Xi)(t;x,\xi)\in\mathbb{R}^{2d}$ be the flow of the classical Hamiltonian

$$H(x,\xi):=\tfrac{1}{2}|\xi|^2+V(x)$$

In other words, $(X,\Xi)(t;x,\xi)$ is the solution at time t of

$$\begin{cases}\dot{X}=\frac{\partial H}{\partial \Xi}(X,\Xi)=\Xi\,, \\ \dot{\Xi}=-\frac{\partial H}{\partial X}(X,\Xi)=-\nabla V(X)\,,\end{cases}\qquad (X,\Xi)\big|_{t=0}=(x,\xi)$$

Geometric condition satisfied by a triple (K,Ω,T) with $K\subset\mathbb{R}^{2d}$ **compact**, $\Omega\subset\mathbb{R}^d$ **open** and $T>0$:

$$(GC)\qquad \begin{aligned}&\text{for each } (x,\xi)\in K \text{ there exists } t\in(0,T)\\ &\qquad\qquad \text{such that } X(t;x,\xi)\in\Omega\end{aligned}$$

Illustration for the Geometric Condition

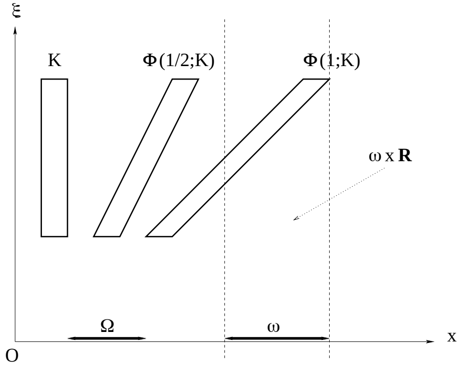


Figure: The geometric condition in space dimension $d = 1$, with $V \equiv 0$. The classical free flow is $\Phi(t; x, \xi) := (X(t; x, \xi), \Xi(t; x, \xi)) = (x + t\xi, \xi)$. The picture represents the image of the closed phase-space rectangle K by the map $(x, \xi) \mapsto \Phi(t; x, \xi)$ at time $t = \frac{1}{2}$ and $t = 1$. The interval Ω satisfies the geometric condition with $T = 1$, at variance with ω. Indeed, phase-space points on the bottom side of K stay out of the strip $\omega \times \mathbb{R}$ for all $t \in [0, 1]$.

Classical-to-Quantum Couplings [FG-TP ARMA 2017]

Coupling of $f \in \mathcal{P}^{ac}(\mathbb{R}^{2d})$ (probability density on $\mathbb{R}^{2d}$) with a density operator $R \in \mathcal{D}(\mathfrak{H}) := \{0 \leq T = T^* \in \mathcal{L}(\mathfrak{H}) \text{ s.t. } \operatorname{tr}_{\mathfrak{H}}(T) = 1\}$

$$\mathbb{R}^{2d} \ni (x,\xi) \mapsto Q(x,\xi) = Q(x,\xi)^* \in \mathcal{L}(\mathfrak{H}) \text{ s.t. } Q(x,\xi) \geq 0 \text{ a.e.}$$

$$\text{s.t. } \operatorname{tr}_{\mathfrak{H}}(Q(x,\xi)) = f(x,\xi) \text{ a.e., and } \iint_{\mathbb{R}^{2d}} Q(x,\xi)dxd\xi = R$$

Set of couplings of f and R denoted $\mathcal{C}(f,R)$; one has $\mathcal{C}(f,R) \neq \varnothing$ since $Q = f \otimes R$ defined by

$$Q: \ (x,\xi) \mapsto Q(x,\xi) := f(x,\xi)R \quad \text{belongs to } \mathcal{C}(f,R)$$

Classical to Quantum Transport Cost a (differential) operator parametrized by the classical phase-space variables

$$c_\hbar^\lambda(x,\xi) \equiv c_\hbar(x,\xi,y,D_y) := \lambda^2|x-y|^2+|\xi+i\hbar\nabla_y|^2$$

Shifted harmonic oscillator; in particular (by Heisenberg's uncertainty inequality)

$$c_\hbar^\lambda(x,\xi) \geq \lambda d\hbar\, I_{\mathfrak{H}}$$

Metric if $R\in\mathcal{D}_2(\mathfrak{H}):=\{T\in\mathcal{D}(\mathfrak{H}) \text{ s.t. } \operatorname{tr}_{\mathfrak{H}}(T^{\frac{1}{2}}(|x|^2-\Delta)T^{\frac{1}{2}})<\infty\}$ while $f\in\mathcal{P}_2^{ac}(\mathbb{R}^{2d}):=\{\phi\in\mathcal{P}^{ac}(\mathbb{R}^{2d}) \text{ s.t. } (|x|^2+|\xi|^2)\phi\in L^1(\mathbb{R}^{2d})\}$

$$\mathfrak{d}_\lambda(f,R)^2 := \underbrace{\inf_{Q\in\mathcal{C}(f,R)}}_{\geq\lambda d\hbar} \iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}(Q(x,\xi)^{\frac{1}{2}}c_\hbar^\lambda(x,\xi)Q(x,\xi)^{\frac{1}{2}})dxd\xi$$

Thm [F.G.-T. Paul (ARMA2017)] Assume $V \in C^{1,1}(\mathbb{R}^d)$ such that $\mathcal{H} := -\frac{1}{2}\hbar^2\Delta_x + V(x)$ has a self-adjoint extension to $\mathfrak{H}$, and denote its quantum dynamics by $U(t) := e^{-it\mathcal{H}/\hbar}$.

Let $\Phi(t; x, \xi) = (X, \Xi)(t; x, \xi)$ is the classical Hamiltonian flow of $H(x, \xi) := \frac{1}{2}|\xi|^2 + V(x)$.

Then, for each $f^{in} \in \mathcal{P}_2^{ac}(\mathbb{R}^{2d})$ and each $R^{in} \in \mathcal{D}_2(\mathfrak{H})$

$$\mathfrak{d}_\lambda(f^{in} \circ \Phi(t, \cdot, \cdot), U(t)R^{in}U(t)^*) \leq \mathfrak{d}_\lambda(f^{in}, R^{in})e^{L|t|}$$

with

$$L := \frac{1}{2}\left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)$$

Since

$$\dot{X}_t - \dot{Y}_t = \Xi_t - H_t\,, \qquad \dot{\Xi}_t - \dot{H}_t = -(\nabla V(X_t) - \nabla V(Y_t))$$

one has

$$\begin{aligned}
&\tfrac{d}{dt}(\lambda^2|X_t - Y_t|^2 + |\Xi_t - H_t|^2)\\
&= 2\lambda^2(X_t - Y_t)\cdot(\Xi_t - H_t) - 2(\Xi_t - H_t)\cdot(\nabla V(X_t) - \nabla V(Y_t))\\
&\le 2\lambda^2|X_t - Y_t||\Xi_t - H_t| + 2|\Xi_t - H_t||\nabla V(X_t) - \nabla V(Y_t)|\\
&\le 2\lambda^2|X_t - Y_t||\Xi_t - H_t| + 2\frac{\mathrm{Lip}(\nabla V)}{\lambda}|\Xi_t - H_t|\lambda|X_t - Y_t|\\
&\le \left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)(\lambda^2|X_t - Y_t|^2 + |\Xi_t - H_t|^2)
\end{aligned}$$

From the drei Männer Arbeit [Born-Heisenberg-Jordan Z. Phys. **35** (1926) 557–615, eq. (15)]

Let $Q, P :=$ **position** and **momentum** operators (assuming $d = 1$)

$$(Q\psi)(x) := x\psi(x)\,, \qquad (P\psi)(x) := -i\hbar\psi'(x)\,, \qquad x \in \mathbf{R}$$

and let $U(t) := e^{-it\mathcal{H}/\hbar}$, where

$$\mathcal{H} := -\tfrac{1}{2}\hbar^2\partial_x^2 + V(x) = \mathcal{H}^*$$

Set

$$Q(t) := U(t)QU(t)^*\,, \qquad P(t) := U(t)PU(t)^*$$

Then

$$\tfrac{d}{dt}Q(t) = P(t)\,, \qquad \tfrac{d}{dt}P(t) = -V'(Q(t))$$

Replace (Y_t, H_t) with $(Q(t), P(t))$ in the pair dispersion inequality.

Proof of the Propagation Theorem

(1) Pick $Q^{in} \in \mathcal{C}(f^{in}, R^{in})$, set $Q(t;x,\xi) = U(t)Q^{in}(\Phi(t,x,\xi))U(t)^*$;

$$\partial_t Q + \{\tfrac{1}{2}|\xi|^2 + V(x); Q\} + \tfrac{i}{\hbar}\left[-\tfrac{1}{2}\hbar^2\Delta + V; Q\right] = 0$$

Then

$$\mathfrak{d}_\lambda(f(t,\cdot), R(t))^2 \le \iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}(Q(t,x,\xi)^{\frac{1}{2}} c_\hbar^\lambda(x,\xi) Q(t,x,\xi)^{\frac{1}{2}}) dx d\xi$$

(2) Moreover

$$\begin{aligned} &\frac{d}{dt}\iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}(Q(t,x,\xi)^{\frac{1}{2}} c_\hbar^\lambda(x,\xi) Q(t,x,\xi)^{\frac{1}{2}}) dx d\xi \\ &\le \iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}(Q(t,x,\xi)^{\frac{1}{2}} \{\tfrac{1}{2}|\xi|^2 + V(x); c_\hbar^\lambda(x,\xi)\} Q(t,x,\xi)^{\frac{1}{2}}) dx d\xi \\ &+ \iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}\left(Q(t,x,\xi)^{\frac{1}{2}} \tfrac{i}{\hbar}\left[-\tfrac{\hbar^2}{2}\Delta + V; c_\hbar^\lambda(x,\xi)\right] Q(t,x,\xi)^{\frac{1}{2}}\right) dx d\xi \end{aligned}$$

(3) With the notation $a \vee b := ab + ba$, elementary computations show that one has

$$\{\tfrac{1}{2}|\xi|^2; c_\hbar^\lambda(x,\xi)\} + \tfrac{i}{\hbar}\left[-\tfrac{\hbar^2}{2}\Delta; c_\hbar^\lambda(x,\xi)\right]$$
$$= \lambda(\xi + i\hbar\nabla_y) \vee \lambda(x-y) \le \lambda c_\hbar^\lambda(x,\xi)$$

and

$$\{V(x); c_\hbar^\lambda(x,\xi)\} + \tfrac{i}{\hbar}\left[V; c_\hbar^\lambda(x,\xi)\right]$$
$$= \tfrac{\mathrm{Lip}(\nabla V)}{\lambda}(\xi + i\hbar\nabla_y) \vee \tfrac{\lambda}{\mathrm{Lip}(\nabla V)}(\nabla V(x) - \nabla V(y))$$
$$\le \tfrac{\mathrm{Lip}(\nabla V)}{\lambda} c_\hbar^\lambda(x,\xi)$$

by using the elementary inequality

$$\left.\begin{array}{l} A^* = A \\ B^* = B \end{array}\right\} \implies A \vee B = AB + BA \le A^2 + B^2$$

(4) Hence

$$\frac{d}{dt}\iint_{\mathbb{R}^{2d}} \mathrm{tr}_{\mathfrak{H}}(Q(t,x,\xi)^{\frac{1}{2}} c_\hbar^\lambda(x,\xi) Q(t,x,\xi)^{\frac{1}{2}}) dx d\xi$$
$$\leq (\lambda + \tfrac{\mathrm{Lip}(\nabla V)}{\lambda}) \iint_{\mathbb{R}^{2d}} \mathrm{tr}_{\mathfrak{H}}(Q(t,x,\xi)^{\frac{1}{2}} c_\hbar^\lambda(x,\xi) Q(t,x,\xi)^{\frac{1}{2}}) dx d\xi$$

which implies the announced inequality.

Lemma
For (K,Ω,T) with $K\subset\mathbb{R}^{2d}$ compact, $\Omega\subset\mathbb{R}^d$ open and $T>0$

$$(GC)\implies C[K,\Omega,T]:=\inf_{(x,\xi)\in K}\int_0^T 1_\Omega(X(t;x,\xi))dt>0$$

Proof Since Ω is open, 1_Ω is l.s.c. By the (GC), for each $x,\xi\in\mathbb{R}^d$, there exists $t_{x,\xi}\in(0,T)$ and $\eta_{x,\xi}>0$ such that, for all $(x,\xi)\in K$

$$|t-t_{x,\xi}|<\eta_{x,\xi}\implies 1_\Omega(X(t;x,\xi))=1$$
$$\implies\int_0^T 1_\Omega(X(t;x,\xi))dt\geq 2\eta_{x,\xi}>0$$

By Fatou's lemma, the positive function $(x,\xi)\mapsto\int_0^T 1_\Omega(X(t;x,\xi))dt$ is l.s.c. on K compact, and therefore attains its minimum on K. □

Thm Let $V \in C^{1,1}(\mathbb{R}^d)$ and (K, Ω, T) satisfying (GC). Then, for all $R^{in} \in \mathcal{D}_2(\mathfrak{H})$ and all $\delta > 0$, denoting $\Omega_\delta := \Omega + B(0,\delta)$, one has

$$\int_0^T \mathrm{tr}_{\mathfrak{H}}(1_{\Omega_\delta} U(t) R^{in} U(t)^*) dt \geq \underbrace{C[K,\Omega,T]}_{\text{geometric}}$$

$$\underbrace{- \frac{1}{\delta} \inf_{\lambda>0} \frac{1}{\lambda} \frac{\exp\left(\frac{1}{2}T\left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)\right) - 1}{\frac{1}{2}\left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)} \inf_{\mathrm{supp}(f^{in}) \subset K} \mathfrak{d}_\lambda(f^{in}, R^{in})}_{\text{semiclassical correction}}$$

Example 1: Toeplitz Initial Data

Assume that R^{in} is of the form

$$R^{in} := \int_{\mathbb{R}^{2d}} |q,p\rangle\langle q,p|\mu(dqdp), \quad \mu \in \mathcal{P}_2(\mathbb{R}^{2d})$$

$$\text{where } |q,p\rangle(x) := (\pi\hbar)^{-d/4} e^{-|x-q|^2/2\hbar} e^{ip\cdot x/\hbar}$$

In that case (see [FG-T. Paul, ARMA2017] Thm. 2.4)

$$\lambda d\hbar \le \mathfrak{d}_\lambda(f^{in}, R^{in})^2 \le \max(1,\lambda^2) W_2(f^{in}, \mu^{in})^2 + \lambda d\hbar$$

so that $\operatorname{supp}(\mu) \subset K \implies \inf_{\operatorname{supp}(f^{in}) \subset K} \mathfrak{d}_\lambda(f^{in}, R^{in}) = \sqrt{\lambda d\hbar}$

Therefore, in that case, one arrives at

$$\frac{1}{C_{OBS}} = C[K,\Omega,T] - \frac{1}{\delta} \inf_{\lambda>0} \sqrt{\frac{d\hbar}{\lambda} \frac{\exp\left(\frac{1}{2}T\left(\lambda + \frac{\operatorname{Lip}(\nabla V)}{\lambda}\right)\right) - 1}{\frac{1}{2}\left(\lambda + \frac{\operatorname{Lip}(\nabla V)}{\lambda}\right)}}$$

Example 2: Pure State

Assume that $R(t) = |U(t)\psi^{in}\rangle\langle U(t)\psi^{in}|$, where $U(t) = e^{-it\mathcal{H}/\hbar}$ is the Schrödinger group.

Choosing $f^{in}(q,p) := \frac{|\langle q,p|\psi^{in}\rangle|^2}{(2\pi\hbar)^d}$ = **Husimi function** of ψ^{in} leads to

$$\frac{1}{C_{OBS}} = C[K,\Omega,T]\iint_K |\langle q,p|\psi^{in}\rangle|^2 \frac{dqdp}{(2\pi\hbar)^d} - D[T,\mathrm{Lip}(\nabla V)]\frac{\Sigma[\psi^{in}]}{\delta}$$

where

$$\begin{aligned} D[T,L] :=& 4\frac{e^{(1+L)T/2}-1}{1+L} \\ \Sigma[\psi^{in}]^2 :=& \langle\psi^{in}|\,|x|^2|\psi^{in}\rangle - |\langle\psi^{in}|x|\psi^{in}\rangle|^2 \\ &+ \langle\psi^{in}|-\hbar^2\Delta_x|\psi^{in}\rangle - |\langle\psi^{in}|-i\hbar\nabla_x|\psi^{in}\rangle|^2 \end{aligned}$$

Proof

Call $f(t,\cdot,\cdot) := f^{in} \circ \Phi(t;\cdot,\cdot)$ and $R(t) := U(t)R^{in}U(t)^*$. For all $Q(t) \in \mathcal{C}(f(t,\cdot,\cdot), R(t))$, one has

$$\begin{aligned}
&\left|\operatorname{tr}_{\mathfrak{H}}(\chi R(t)) - \iint_{\mathbb{R}^{2d}} \chi(x) f(t,x,\xi)dxd\xi\right| \\
&= \left|\iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}((\chi(x)-\chi(y))Q(t,x,\xi)dxd\xi\right| \\
&\le \frac{\operatorname{Lip}(\chi)}{\lambda}\left(\iint_{\mathbb{R}^{2d}} \operatorname{tr}_{\mathfrak{H}}(Q_t^{\frac{1}{2}}(\lambda^2|x-y|^2+|\xi+i\hbar\nabla_y|^2)Q_t^{\frac{1}{2}})dxd\xi\right)^{\frac{1}{2}}
\end{aligned}$$

so that

$$\begin{aligned}
\left|\operatorname{tr}_{\mathfrak{H}}(\chi R(t)) - \iint_{\mathbb{R}^{2d}} \chi(x) f(t,x,\xi)dxd\xi\right| &\le \frac{\operatorname{Lip}(\chi)}{\lambda}\mathfrak{d}_\lambda(f(t,\cdot,\cdot),R(t)) \\
&\le \frac{\operatorname{Lip}(\chi)}{\lambda}\mathfrak{d}_\lambda(f^{in},R^{in})\exp\left(\tfrac{1}{2}t\left(\lambda+\frac{\operatorname{Lip}(\nabla V)}{\lambda}\right)\right)
\end{aligned}$$

Since

$$\iint_{\mathbb{R}^{2d}} \chi(x) f(t,x,\xi) dx d\xi = \iint_{\mathbb{R}^{2d}} \chi(X(t;x,\xi)) f^{in}(x,\xi) dx d\xi$$

one has

$$\int_0^T \mathrm{tr}_{\mathfrak{H}}(\chi R(t)) dt \geq \inf_{(x,\xi)\in K} \int_0^T \chi(X(t;x,\xi)) dt \iint_K f^{in}(x,\xi) dx d\xi$$
$$- \frac{\mathrm{Lip}(\chi)}{\lambda} \frac{\exp\left(\frac{1}{2} T \left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)\right) - 1}{\frac{1}{2}\left(\lambda + \frac{\mathrm{Lip}(\nabla V)}{\lambda}\right)} \mathfrak{d}_\lambda(f^{in}, R^{in})$$

Conclude by choosing $\chi(x) := \left(1 - \frac{\mathrm{dist}(x,\Omega)}{\delta}\right)_+$, so that $\mathrm{Lip}(\chi) = \frac{1}{\delta}$

□

Conclusions/Perspectives

•We have proved an **observation inequality** for the quantum dynamics under the only assumption that V is regular enough for the existence and uniqueness of the classical dynamics ($C^{1,1}$ potential)

•The **observation constant is explicit** in terms of the geometric data of the Bardos-Lebeau-Rauch controllability condition

•Approach based on a quantum analogue of the Wasserstein distance to measure the difference between a classical and a quantum density

Possible extensions

•Obtaining a controllability statement (by duality as in HUM)
•Including magnetic fields (I. Ben Porath's PhD thesis)
•Other dispersive dynamics? Klein-Gordon?