Schrödinger Bridges: Old and New

IPAM workshop

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Lecture 2: Schrödinger Bridges - quantum

- **Recap on classical bridges**
  Brownian bridges & Schrödinger bridges
- **Quantum Schrödinger's Bridges (QSB)**
  Otto Bergmann 1988
- **Pre- and post-selection & Quantum bridges**
  Two-state formalism, time-reversal, measurements
- **Bridges/quantum channels in general**
  Some results and conjecture
- **Interlude on Non-commutative Transport**
- **Discussion and directions**

Brownian diffusion

B_t standard Brownian motion

$$dX_t = dB_t$$

For a Brownian particle located at time t_0 at the point x_0

$$\mathbb{P}(X_t \in [x, x + dx] | X_{t_0} = x_0) = g(t - t_0, x - x_0)dx$$

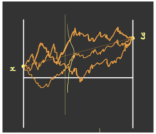
with Green's function:

$$g(t, x) := \left(\frac{1}{2\pi Dt}\right)^{-1/2} \exp(-x^2/4Dt)$$

$D = 2\sigma^2$ with D diffusion coefficient, σ^2 variance

Brownian bridge

Brownian particle located at time t_0 at the point x_0
found at the later time t_1 at x_1
(initial/final positions specified within dx_0 and dx_1)

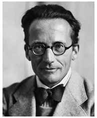


What is the probability for finding the particle in the region dx around x for some time t within the interval $[t_0, t_1]$?

Conditioning/Bayes' law

$$\mathbb{P}(X_t \in [x, x+dx] | X_{t_0} = x_0, | X_{t_1} = x_1) = \frac{g(t - t_0, x - x_0) g(t_1 - t, x_1 - x)}{g(t_1 - t_0, x_1 - x_0)} dx =: w(t, x) dx$$

Schrödinger's bridge problem



- Consider a cloud of N independent Brownian particles (N large)
- empirical distributions $\rho_0(x)$ and $\rho_1(y)$ at $t = 0$ and $t = 1$
- ρ_0 and ρ_1 not compatible with transition mechanism

$$\rho_1(y) \neq \int_0^1 \Pi(t_0, x, t_1, y)\rho_0(x)dx,$$

where

$$\Pi(t_0, y, t_1, x) = \frac{1}{\sqrt{(2\pi)^n(t_1 - t_0)}} e^{-\frac{1}{2}\frac{\|x-y\|^2}{t_1 - t_0}}, \quad t_0 < t_1$$

$\Rightarrow$ Particles have been transported in an unlikely way

Schrödinger (1931)

Of the many possible (unlikely) ways, which one is the most likely?

Schrödinger bridges

$$\mathbb{P}^\star = \operatorname{argmin} \left\{ \int_{\text{paths}} \log\left(\frac{d\mathbb{P}}{d\mathbb{W}}\right) d\mathbb{P} \;\middle|\; \mathbb{P}|_{t=0} = \rho_0,\ \mathbb{P}_{t=1} = \rho_1 \right\}$$

Disintegration of measures

$$\mathbb{P}(\text{path}) = \underbrace{\mathbb{P}(\text{path}\ |x(0) = x, x(t_f) = y)}_{\text{conditioned } = \text{ pined bridge}} \cdot \mathbb{P}_{0,t_f}(x, y)$$

$\Rightarrow$

$$\int \log\left(\frac{d\mathbb{P}}{d\mathbb{W}}\right) d\mathbb{P} = \int \log\left(\frac{d\mathbb{P}_{0,t_f}(x,y)}{d\mathbb{W}_{0,t_f}(x,y)}\right) d\mathbb{P}_{0,t_f}(x,y)$$
$$+ \underbrace{\int \log\left(\frac{d\mathbb{P}(\text{path}\ |x(0), x(t_f))}{d\mathbb{W}(\text{path}\ |x(0), x(t_f))}\right) d\mathbb{P}(\text{path}\ |x(0), x(t_f))}_{=0 \text{ for } \mathbb{P}(\text{path}\ |x(0),x(t_f)) \;=\; \mathbb{W}(\text{path}\ |x(0),x(t_f))}$$

Optimal transport

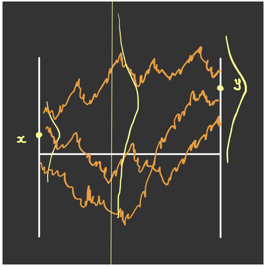
a parenthesis

$$\int c(x,y)d\mathbb{P} + \int \log\left(\frac{d\mathbb{P}}{d\mathbb{W}}\right) d\mathbb{P} = -\int \log\left(e^{-c(x,y)}\right) d\mathbb{P} + \int \log\left(\frac{d\mathbb{P}}{d\mathbb{W}}\right) d\mathbb{P}$$
$$= \int \log(\frac{d\mathbb{P}}{\underbrace{e^{-c(x,y)}d\mathbb{W}}_{\text{new ``prior''}}})d\mathbb{P}$$

Structure of the law

via disintegration of measure

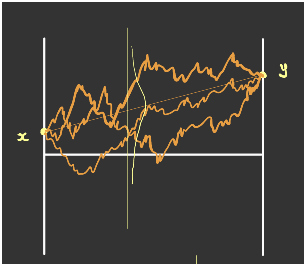
$\mathbb{P}^*(\text{path})$



Schrödinger bridge

$$= \quad \mathbb{P}^*_{0,t_f}(x,y) \times$$

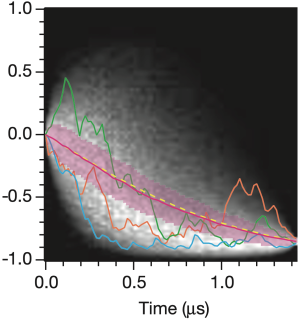
$\mathbb{P}(\text{path}|x,y)$



Pinned bridges

$\mathbb{P}^*_{0,t_f}(x,y)$: optimal end-point coupling

Quantum (pinned) bridges



Quantum trajectories

between pre- and post-selected states

Weber, Chantasri, Dressel, Jordan, Murch, Siddiqi
Mapping the optimal route between two quantum states
Nature, 2014, doi:10.1038/nature13559

Schrödinger system

Schrödinger (1931/32)

the density factors into

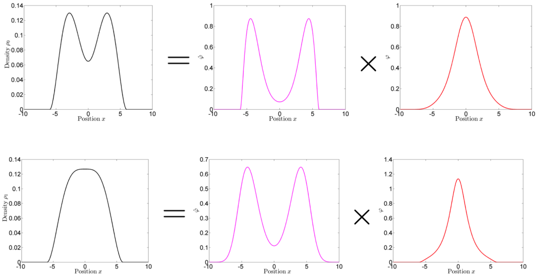
$$\rho(x,t) = \varphi(x,t)\hat{\varphi}(x,t)$$

where φ and $\hat{\varphi}$ solve (Schrödinger's system):

$$\begin{aligned} \hat{\varphi}(x,t) &= \int p(0,y,t,x)\hat{\varphi}(y,0)dy, \quad \varphi(x,1)\hat{\varphi}(x,1) = \rho_1(x). \\ \varphi(x,t) &= \int p(t,x,1,y)\varphi(y,1)dy, \quad \varphi(x,0)\hat{\varphi}(x,0) = \rho_0(x) \end{aligned} \tag{1}$$

$$\begin{array}{ccccccc} & \hat{\varphi}_0 & \xrightarrow{\frac{1}{2}\Delta} & \hat{\varphi}_1 & \\ \varphi_0\hat{\varphi}_0 = \rho_0 & \uparrow & & \downarrow & \varphi_1\hat{\varphi}_1 = \rho_1 \\ & \varphi_0 & \xleftarrow{-\frac{1}{2}\Delta} & \varphi_1 & \end{array}$$

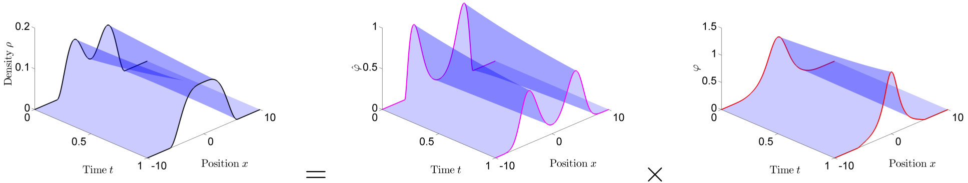
Schrödinger system



$$\frac{\partial \hat{\varphi}}{\partial t}(t,x) = \frac{1}{2}\Delta \hat{\varphi}(t,x)$$

$$-\frac{\partial \varphi}{\partial t}(t,x) = \frac{1}{2}\Delta \varphi(t,x)$$

$$\varphi(0,x)\hat{\varphi}(0,x) = \rho_0(x)$$
$$\varphi(1,x)\hat{\varphi}(1,x) = \rho_1(x)$$



Schrödinger system

Classical SBs

$$\frac{\partial\hat{\varphi}}{\partial t}(t,x) = \frac{1}{2}\Delta\hat{\varphi}(t,x)$$

$$-\frac{\partial\varphi}{\partial t}(t,x) = \frac{1}{2}\Delta\varphi(t,x)$$

$$\rho(t,x) = \hat{\varphi}(t,x)\varphi(t,x)$$

compare with

Schrödinger equation

$$\frac{1}{i\hbar}\frac{\partial\psi}{\partial t}(t,x) = \frac{1}{2}\Delta\psi(t,x)$$

$$-\frac{1}{i\hbar}\frac{\partial\psi^*}{\partial t}(t,x) = \frac{1}{2}\Delta\psi^*(t,x)$$

$$\rho(t,x) = \psi(t,x)\psi^*(t,x)$$

Quantum:
$\rho(t,x)$ a density matrix
e.g., $|\psi\rangle\langle\psi|$ or $\sum_k p_k|k\rangle\langle\psi_k|$

Quantum Schrödinger Bridges (QSB)

"The (unauthorized) extrapolation ... into the quantum mechanical domain,"

Otto Bergmann, 1988

Interpolation:

Consider initial/final states $|i\rangle$, $|f\rangle$ of an observable A (with discrete spectrum)

$$\exp\left(\frac{1}{2}\pi\frac{t-t_0}{t_1-t_0}\cdot\underbrace{(|f\rangle\langle i|+|i\rangle\langle f|)}_{S}\right)|i\rangle=\cos(\frac{\pi}{2}\tau)|i\rangle+\sin(\frac{\pi}{2}\tau)|f\rangle$$

$\tau=\frac{t-t_0}{t_1-t_0}$

For $S=|f\rangle\langle i|+|i\rangle\langle f|$, then $\exp(\alpha S)=\cos(\alpha)I+\sin(\alpha)S$

Bridge interpolates $|i\rangle\langle i|$ and $|f\rangle\langle f|$

QSB - Bergmann

Restores resemblance with the Brownian bridge, i.e.,

$$\frac{g(t-t_0, x-x_0)g(t_1-t, x_1-x)}{Z}$$

Measuring $|k\rangle$ at t, for evolution $U(t) = \exp(-\frac{i}{\hbar}Ht)$,

$$\frac{\sum_k |k\rangle\langle k| \cdot \overbrace{\left(|\langle f|U(t_1-t)|k\rangle|^2 \cdot |\langle k|U(t-t_0)|i\rangle|^2\right)}^{\text{probability}}}{Z_t}$$

- $|i\rangle$ evolves to a mixed state and back to $|f\rangle$, a decrease in entropy at some point
- The normalization depends on the time t when a projective measurement takes place normalize by $|\langle f|U(t_1-t_0)|i\rangle|^2$ to restore analogy classical at end-points
- Discusses non-selective measurements at t, writes the law in product form

QSB - Bergmann

Bergmann:

• "Schrödinger's main interests... statistical mechanics and the interpretation of quantum mechanics"

• "inspired by the old and almost unknown paper by Schrödinger and no attempt was made to draw any conclusions about its impact, if any, on the theory of measurements. It was written primarily as a historical study"

• "a referee .. informed the author [Bergmann] of"
– Y. Aharonov, P.G. Bergmann and J.L. Lebowitz
– F.J. Belinfante
"contributions to the same problem .. written without Schrödinger's inspiration"

Two-state vector formalism[1]

- pre- and post- selected quantum systems
- a time-symmetric description of QM in which the present is caused by states

$$\langle\psi_1| \cdot |\psi_0\rangle,$$

evolving backwards from the future ($\langle\psi_1|$) and forward from the past ($|\psi_0\rangle$)

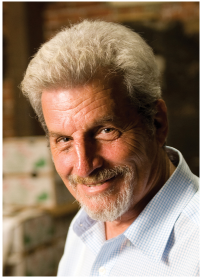
- time-symmetry by construction

Figure: Watanabe and his son, 1949

[1] Watanabe, S. (1955). Symmetry of physical laws. Part III. Prediction and retrodiction. Reviews of Modern Physics, 27(2), 179.

Two-state vector formalism[2]

- Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz in 1964
- "indubitable asymmetry in time direction" is not due to the principles of QM but from the intrusion of the macroscopic world



Yakir Aharonov

Peter Bergmann

(1915–2002)

Joel Lebowitz

[2] Aharonov, Y., Bergmann, P. G., and Lebowitz, J. L. (1964). Time symmetry in the quantum process of measurement. Physical Review, 134(6B), B1410.

Aharonov-Bergmann-Lebowitz (1964)

Time symmetry in the quantum process of measurement (1964)

- ensembles with histories that begin and end at particular states

$$\text{pre-select } |\psi_0\rangle \to \text{measure } |x\rangle\langle x| \to \text{post-select } |\psi_1\rangle$$

- given a **sequence of measurements** of arbitrary observables, a **time-symmetric expression is obtained** for the probabilities of measurement outcomes:

$$P(x|\psi_0, \psi_1) = \frac{P(\psi_0, x, \psi_1)}{P(\psi_0, \psi_1)} = \frac{|\langle\psi_1|x\rangle\langle x|\psi_0\rangle|^2}{\sum_{x'} |\langle\psi_1|x'\rangle\langle x'|\psi_0\rangle|^2}.$$

[A-B-L]: If this time-symmetric expression is taken as the fundamental algorithm in QM, what assumptions are required to retrieve the standard predictive probability expression?

Measurements and time-reversal[3]

Prediction vs Postdiction: apparent asymmetry

- F.J. Belifante extends Aharonov etal. non-ideal measurement processes
- presence of inter-measurement processes breaks time-symmetry

Frederik Jozel Belinfante (1913–1991)

[3] Belinfante, F.J. (1975). Measurements and Time Reversal in Objective Quantum Theory: International Series in Natural Philosophy (Vol. 75)

Measurements and time-reversal

intervening quantum dynamics & time-symmetry, pre-/post-selection

$\Pi_0^i = |\psi_0^i\rangle\langle\psi_0^i|, \quad \Pi_1^j = |\psi_1^j\rangle\langle\psi_1^j|,$ and $\mathcal{E}^\dagger(\cdot) = \sum_k L_k(\cdot)L_k^\dagger$

- start at 0, measure x, end at 1

$$P(x|\psi_0, \psi_1) = \frac{|\langle\psi_1|x\rangle\langle x|\psi_0\rangle|^2}{\sum_{x'} |\langle\psi_1|x'\rangle\langle x'|\psi_0\rangle|^2} \qquad \text{(reads forward and backward)}$$

- start at 0, pass through $\mathcal{E}^\dagger$, measure x, end at 1

$$P(x|\psi_0, \psi_1) = \frac{\sum_k |\langle\psi_1|x\rangle\langle x|L_k|\psi_0\rangle|^2}{\sum_{k,x'} |\langle\psi_1|x'\rangle\langle x'|L_k|\psi_0\rangle|^2} \qquad \text{(only reads forward)}$$

Next:
Schrödinger's question with pre/post selection & time-symmetry

Pre/post-selected outcomes & large deviations[4]

- fix two bases $|\psi_0^i\rangle$ and $|\psi_1^j\rangle$
- assistant prepares ensembles and reports pre/post-selected marginals

Olga Movilla Miangolarra
Universidad de La Laguna, Spain

Bridge problem: What is the most likely coupling of initial and final outcomes?

[4] Quantum Schrödinger bridges: large deviations and time-symmetric ensembles, Olga et al., arXiv.

Pre/post-selected states

hidden realizations performed by the assistant

reported initial state

$$\tilde{\rho}_0 = \sum_i \tilde{\alpha}_i |x_0^i\rangle\langle x_0^i| \neq \rho_0$$

pre-select $\qquad\qquad\qquad\qquad$ post-select

$$|x_0^i\rangle \longrightarrow \sum_{i,k,j} L_{ikj}\,(\cdot)\,L_{ikj}^\dagger \longrightarrow |y_1^j\rangle$$

$\tilde{p}_{ij}$ often

reported final state

$$\tilde{\rho}_1 = \sum_j \tilde{\beta}_j |y_1^j\rangle\langle y_1^j| \neq \rho_1$$

Pre/post-selected states

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\rho_0 = \sum_i \alpha_i |\psi_0^i\rangle\langle\psi_0^i| \arrow[r, "\sum_{ikj} L_{ikj}(\cdot) L_{ikj}^\dagger"'] \arrow[d, "/" marking] & \rho_1 = \sum_j \beta_j |\psi_1^j\rangle\langle\psi_1^j|, \arrow[d, "/" marking] \\
\tilde{\rho}_0 = \sum_i \tilde{\alpha}_i |\psi_0^i\rangle\langle\psi_0^i| \arrow[r, "?"] & \tilde{\rho}_1 = \sum_j \tilde{\beta}_j |\psi_1^j\rangle\langle\psi_1^j|,
\end{tikzcd}
$$

w/ $L_{ikj} = |\psi_0^i\rangle\langle\psi_0^i| L_k |\psi_1^j\rangle\langle\psi_1^j|$ & fixed endpoint bases

Notation: $\tilde{}$ denotes new marginals, updated law

Pre/post-selected states

Coupling $\tilde{p}_{ij}$ between states that minimizes the entropy relative to the prior

$$p_{ij} = \alpha_i \sum_k |\langle \psi_1^j | L_{ikj} | \psi_0^i \rangle|^2.$$

Optimal coupling $\tilde{p}_{ij}$:

$$\arg\min_{\tilde{p}_{ij}} \sum_{i,j} \tilde{p}_{ij} \log \frac{\tilde{p}_{ij}}{p_{ij}} \quad \text{s.t} \quad \sum_j \tilde{p}_{ij} = \tilde{\alpha}_i \ , \ \sum_i \tilde{p}_{ij} = \tilde{\beta}_j.$$

$$\Rightarrow \quad \tilde{p}_{ij} = \frac{b_j}{a_i} \frac{\tilde{\alpha}_i}{\alpha_i} p_{ij}, \qquad a_i, b_j > 0.$$

Pre/post-selected states

Define

$$\phi_0 := \sum_i a_i |\psi_0^i\rangle\langle\psi_0^i|, \qquad \phi_1 := \sum_j b_j |\psi_1^j\rangle\langle\psi_1^j|,$$

the updated Kraus map takes the form

$$\sum_{ikj} \tilde{L}_{ikj}(\cdot)\tilde{L}_{ikj}^\dagger = \sum_{ikj} {\phi_1}^{1/2} L_{ikj}({\phi_0}^{-1/2}(\cdot){\phi_0}^{-1/2}) L_{ikj}^\dagger {\phi_1}^{1/2}.$$

The Schrödinger system

$$\hat\phi_0 := \phi_0^{-1/2}\tilde\rho_0\phi_0^{-1/2} \qquad \begin{tikzcd} \hat\phi_0 \arrow[rr, "\sum_{ikj} L_{ikj}(\cdot)L_{ikj}^\dagger"'] & & \hat\phi_1, \arrow[d] \\ & \circlearrowright & \\ \phi_0 \arrow[uu] & & \phi_1, \arrow[ll, "\sum_{ikj} L_{ikj}^\dagger(\cdot)L_{ikj}"'] \end{tikzcd} \qquad \hat\phi_1 := \phi_1^{-1/2}\tilde\rho_1\phi_1^{-1/2}$$

- $\phi, \tilde\rho$ diagonalized wrt same basis at the two ends
- Fortet-Sinkhorn algorithm converges (classical)
- existence of ϕ_0, ϕ_1

Time-reversal of classical and quantum channels

interlude

For **vectors** $\rho_0(i), \rho_1(j) \neq 0$, Π **row stochastic** matrix

Classical Markov kernel

$$\begin{array}{ccc} \rho_0 & \xrightarrow{\Pi^T} & \rho_1 \\ \mathbb{1} & \xleftarrow{\Pi} & \mathbb{1} \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{(\mathrm{diag}(\rho_1)^{-1}\,\Pi^T\,\mathrm{diag}(\rho_0))} & \mathbb{1} \\ \rho_0 & \xleftarrow{\mathrm{diag}(\rho_0)\,\Pi\,\mathrm{diag}(\rho_1)^{-1}} & \rho_1 \end{array}$$

Time-reversal

$$\Pi^T \xrightarrow{R} (\Pi^{\mathrm{rev}})^T := \mathrm{diag}(\rho_0)\,\Pi\,\mathrm{diag}(\rho_1)^{-1}$$

Depends on ρ_0, ρ_1

Time-reversal of classical and quantum channels

interlude

For ρ_0, ρ_1 **invertible density matrices,** $\mathcal{E}^\dagger(\cdot) = \sum_k L_k(\cdot)L_k^\dagger$ **with** $\mathcal{E}(I) = \sum_k L_k^\dagger L_k = I$

Kraus maps

$$\begin{array}{ccc} \rho_0 & \xrightarrow{\mathcal{E}^\dagger} & \rho_1 \\ I & \xleftarrow{\mathcal{E}} & I \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} I & \xrightarrow{\rho_1^{-1/2}\mathcal{E}^\dagger\left(\rho_0^{1/2}(\cdot)\rho_0^{1/2}\right)\rho_1^{-1/2}} & I \\ \rho_0 & \xleftarrow{\rho_0^{1/2}\mathcal{E}\left(\rho_1^{-1/2}(\cdot)\rho_1^{-1/2}\right)\rho_0^{1/2}} & \rho_1 \end{array}$$

Time-reversal

$$\mathcal{E}^\dagger \xrightarrow{R} (\mathcal{E}^{\mathrm{rev}})^\dagger := \rho_0^{1/2}\mathcal{E}\left(\rho_1^{-1/2}(\cdot)\rho_1^{-1/2}\right)\rho_0^{1/2}$$

Depends on ρ_0, ρ_1

Time-symmetry of the Schrödinger bridge

The optimal coupling for the time-reversed Kraus map

$$R: \; L_{ijk} \mapsto M_{ijk} = {\rho_0}^{1/2} L_{ijk}^{\dagger} {\rho_1}^{-1/2}$$

gives the same bridge

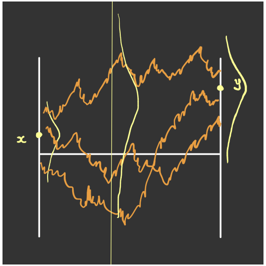
$$\begin{tikzcd}
\sum_{ikj} L_{ikj}(\cdot)L_{ikj}^{\dagger} \arrow[r, "\mathrm{R}"] \arrow[d, "\sim"'] & \sum_{ikj} M_{ikj}(\cdot)M_{ikj}^{\dagger}, \arrow[d, "\sim"] \\
\sum_{ikj} \tilde{L}_{ikj}(\cdot)\tilde{L}_{ikj}^{\dagger} \arrow[r, "\mathrm{R}"] & \sum_{ikj} \tilde{M}_{ikj}(\cdot)\tilde{M}_{ikj}^{\dagger},
\end{tikzcd}$$

- $\sim$: **most likely update**
- R: Kraus time-reversal

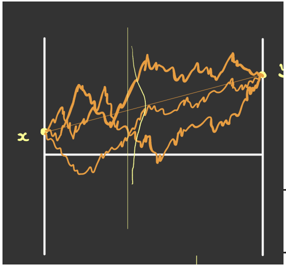
The diagram commutes

Intervening measurements

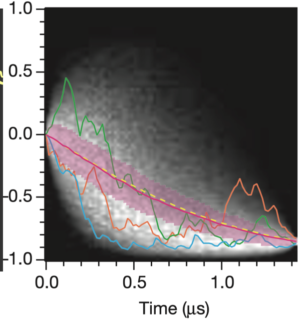
Statistics, path probabilities



Schrödinger bridge



Pinned bridges



Quantum trajectories of Weber etal.

Intervening outcomes

- The assistant non-selectively measures $\hat{Z} = \sum_z |z\rangle\langle z|$ at $\tau \in (0,1)$:

$$0 \xrightarrow{\hspace{6em}\text{at } \tau\hspace{6em}} 1$$

$$|x_0^i\rangle\langle x_0^i| \to \sum_k L_k(\cdot)L_k^\dagger \to \sum_z \Pi_z(\cdot)\Pi_z \to \sum_l E_l(\cdot)E_l^\dagger \to |y_1^j\rangle\langle y_1^j|$$

- Reports only initial and final density matrices

$$\tilde{\rho}_0 = \sum_i \tilde{\alpha}_i |x_0^i\rangle\langle x_0^i| \quad \text{and} \quad \tilde{\rho}_1 = \sum_j \tilde{\beta}_j |y_1^j\rangle\langle y_1^j|,$$

which do not match the expected ($\tilde{\alpha}_1 \neq \alpha_i$, $\tilde{\beta}_j \neq \beta_j$).

Schrödinger's dictum: What is the most likely distribution $\tilde{P}(z)$ of measurement outcomes?

Most likely intervening outcomes

Probability of observing a (simple) eigenvalue z at τ with pre/post-selection

$$P_\tau(z|x_0^i, y_1^j) = \frac{\sum_{k,l} |\langle y_1^j|E_l|z\rangle\langle z|L_k|x_0^i\rangle|^2}{\sum_{k,l,z'} |\langle y_1^j|E_l|z'\rangle\langle z'|L_k|x_0^i\rangle|^2} \qquad \textit{(pinned bridges)}$$

Coupling $= \mathrm{Prob}$(starting at $|x_0^i\rangle$ and ending at $|y_1^j\rangle$) given $\tilde{\rho}_0$ and $\tilde{\rho}_1$

$$\tilde{p}_{ij} \qquad \textit{(determined via Sinkhorn)}$$

$$\tilde{P}_\tau(z) = \sum_{i,j} \tilde{p}_{ij} P_\tau(z|y_1^j, x_0^i) \qquad \textit{(posterior)}$$

Optimal coupling: $\tilde{p}_{ij} = \frac{b_j}{a_i}\frac{\tilde{\alpha}_i}{\alpha_i} p_{ij}$

$p_{ij} = \alpha_i \sum_{k,z,l} |\langle y_1^j|E_l \Pi_z L_k|x_0^i\rangle|^2$

a_i, b_j are such that $\sum_j \tilde{p}_{ij} = \tilde{\alpha}_i$, $\sum_i \tilde{p}_{ij} = \tilde{\beta}_j$

Intervening outcomes

The **most likely** distribution of outcomes z at time $\tau \in (0,1)$ factors

$$\tilde{P}^*_\tau(z) = \sum_{i,j} \tilde{p}_{ij} P_\tau(z|y_1^j, x_0^i) = \varphi(\tau,z)\hat{\varphi}(\tau,z)$$

with "forward" and "backward" evolved factors to τ:

$$\hat{\varphi}(\tau,z) = \sum_{i,k} \frac{\tilde{\alpha}_i}{a_i} |\langle z|L_k|x_0^i\rangle|^2, \qquad \varphi(\tau,z) = \sum_{j,l} b_j |\langle y_1^j|E_l|z\rangle|^2.$$

Likewise, the **most likely** state at τ factors

$$\tilde{\rho}_\tau = \sum_z \tilde{P}_\tau(z)|z\rangle\langle z| = \phi_\tau^{1/2}\hat{\phi}_\tau\phi_\tau^{1/2}$$

- The solution is identical under time-reversal

Inference of generalized intervening measurements

Inference of most likely generalized measurement oucome

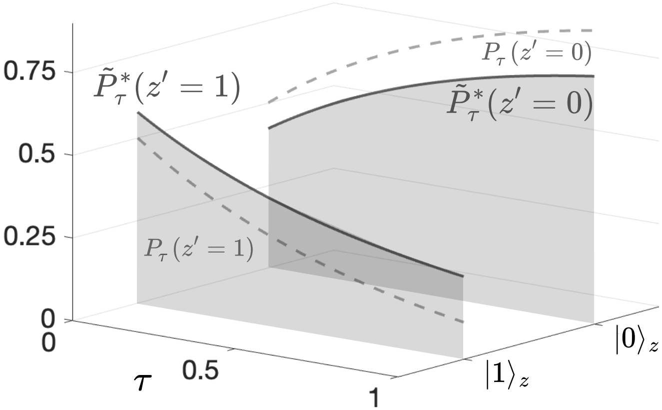
Assistant weakens the measurement of $\hat{Z}$ at $\tau \in (0,1)$, corresponding Kraus map is

$$\rho \mapsto \int_{\mathbb{R}} \Omega_z^{\delta} \rho \Omega_z^{\delta} \mathrm{d}z, \quad \Omega_x^{\delta} = \left(\frac{\delta}{\pi}\right)^{1/4} e^{-\delta(\hat{Z}-z)^2}, \quad \delta: \text{ strength parameter}$$

- projective measurement $(\delta \to \infty) \longrightarrow z \in \sigma(\hat{Z})$
- weak measurement $(0 < \delta < \infty) \longrightarrow z \in \mathbb{R}$
- infinitesimal measurement $(\delta \to 0) \longrightarrow z \in \mathbb{C}$ (weak value)[5]

[5] Aharonov, Y., Albert, D. Z., and Vaidman, L. (1988). Physical review letters, 60(14), 1351.

Amplitude damping experiment



Probability $P_\tau(z')$ of measuring $z' \in \{0,1\}$ at times τ
Most likely = solid, prior = dashed

Pre/post-selected quantum bridges

With specified bases & intervening measurements:

$$\{|\psi_0^i\rangle \mid i \in \mathrm{Index}\} \to \{|\psi_1^j\rangle \mid j \in \mathrm{Index}\}$$

Schrödinger bridges are manifestly classical

Large deviations' interpretation
Most likely weak value
Update of Kraus maps

Next:
QSBs & non-commutative Sinkhorn, redux

QSBs & non-commutative Sinkhorn, redux

Michele Pavon

Leonid Gurvits

https://en.wikipedia.org/wiki/Sinkhorn%27s_theorem

Georgiou T.T. and M. Pavon, Positive contraction mappings for classical and quantum Schrödinger systems, (2015) Journal of Mathematical Physics 56.3

Gurvits, L. Classical complexity and quantum entanglement, (2004) J.Comp. Syst. Sci. 69

Quantum Schrödinger Bridges, redux

$\mathcal{E}^\dagger$ prior, $\mathcal{F}^\dagger$ posterior

Given maps $\{\mathcal{E}_t^\dagger; 0 \leq t \leq T-1\}$:

$$\mathcal{E}_{0:T}^\dagger := \mathcal{E}_{T-1}^\dagger \circ \cdots \circ \mathcal{E}_0^\dagger$$

that are not consistent with initial and final densities ρ_0 and ρ_T, determine:

$$\mathcal{F}_{0:T}^\dagger = \mathcal{F}_{T-1}^\dagger \circ \cdots \circ \mathcal{F}_0^\dagger$$

such that

$$\mathcal{F}_{0:T}^\dagger(\rho_0) = \rho_T$$

Is there a natural notion of distance between $\mathcal{F}$ and $\mathcal{E}$?

“1-term” local-transformations

$$\mathcal{F}_t^\dagger(\cdot) = \phi_{t+1}^{1/2}\left(\mathcal{E}_t^\dagger(\phi_t^{-1/2}(\cdot)\phi_t^{-1/2})\right)\phi_{t+1}^{1/2}$$

i.e., $\mathcal{F}_t^\dagger = \Phi_{t+1} \circ \mathcal{E}_t^\dagger \circ \Phi_t^{-1}$ where Φ Kraus with one coefficient

Quantum Schrödinger bridge

for uniform marginals $\boxed{\rho_0 = \rho_1 = I}$

Thm: Suppose $\mathcal{E}_{0:T}^\dagger$ is positivity improving[a], then $\exists\ \phi_0, \phi_T > 0$ s.t. for any factorization

$$\begin{aligned} \phi_0 &= \chi_0^\dagger \chi_0, \text{ and} \\ \phi_T &= \chi_T^\dagger \chi_T, \end{aligned}$$

$$\mathcal{F}_{0:T}^\dagger(\cdot) := \chi_T \left(\mathcal{E}_{0:T}^\dagger(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$

is a *doubly stochastic* Kraus map, i.e., $\mathcal{F}(I) = I$ as well as $\mathcal{F}^\dagger(I) = I$

[a] $\mathcal{E}^\dagger$ is positivity improving if $\rho \geq 0 \Rightarrow \mathcal{E}^\dagger(\rho) > 0$

A sequence of factors $\phi_t = \chi_t \chi_t^\dagger$, update the $\mathcal{E}$'s in $\mathcal{E}_{T-1}^\dagger \circ \cdots \circ \mathcal{E}_0^\dagger$ into the $\mathcal{F}$'s

Analog of diagonal scaling

Proof

$$
\begin{array}{ccccc}
 & \hat{\phi}_0 & \xrightarrow{\mathcal{E}_{0,T}^{\dagger}} & \hat{\phi}_T & \\
\hat{\phi}_0 = \phi_0^{-1} & \uparrow & & \downarrow & \phi_T = \hat{\phi}_T^{-1} \\
 & \phi_0 & \xleftarrow{\mathcal{E}_{0,T}} & \phi_T &
\end{array}
$$

The composition map

$$
\mathcal{C} \ : \ \left(\hat{\phi}_0\right)_{\text{starting}} \xrightarrow{\mathcal{E}_{0,T}^{\dagger}} \hat{\phi}_T \xrightarrow{(\cdot)^{-1}} \phi_T \xrightarrow{\mathcal{E}_{0,T}} \phi_0 \xrightarrow{(\cdot)^{-1}} \left(\hat{\phi}_0\right)_{\text{next}}
$$

is strictly contractive in the Hilbert metric.

steps identical to classical case

Quantum version of Schrödinger's system

Given $\mathcal{E}_{0:T}^{\dagger}$ and ρ_0 and ρ_T, $\boxed{\textbf{if } \exists\ \phi_0,\ \phi_T,\ \hat{\phi}_0,\ \hat{\phi}_T}$ solving

$$\begin{aligned} \mathcal{E}_{0:T}(\phi_T) &= \phi_0, \\ \mathcal{E}_{0:T}^{\dagger}(\hat{\phi}_0) &= \hat{\phi}_T, \\ \rho_0 &= \chi_0 \hat{\phi}_0 \chi_0^{\dagger}, \\ \rho_T &= \chi_T \hat{\phi}_T \chi_T^{\dagger}, \end{aligned}$$

then for any factorization

$$\begin{aligned} \phi_0 &= \chi_0^{\dagger}\chi_0, \text{ and} \\ \phi_T &= \chi_T^{\dagger}\chi_T, \end{aligned}$$

the map

$$\mathcal{F}_{0:T}^{\dagger}(\cdot) := \chi_T \left(\mathcal{E}_{0:T}^{\dagger}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^{\dagger}$$

is a quantum bridge for $(\mathcal{E}_{0:T}^{\dagger}, \rho_0, \rho_T)$, namely $\mathcal{F}(I) = I$ and $\mathcal{F}^{\dagger}(\rho_0) = \rho_T$.

Conjecture

The quantum Schrödinger system has a solution for arbitrary ρ_0, ρ_T.

Snag in the proof:
$\phi \to \hat{\phi}$ and $\hat{\phi} \to \phi$ are not isometries,

$$\begin{aligned} D_T &: \quad \hat{\phi}_T \mapsto \phi_T = \left(\rho_T^{1/2}\left(\rho_T^{-1/2}\hat{\phi}^{-1}\rho_T^{-1/2}\right)^{1/2}\rho_T^{1/2}\right)^2 \\ \hat{D}_0 &: \quad \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{-1/2}\rho(\phi_0)^{-1/2} \end{aligned}$$

Extensive numerical evidence that the composition $\hat{D}_0 \circ \mathcal{E}_{0:T} \circ D_T \circ \mathcal{E}_{0:T}^{\dagger}$ has a fixed point

---

Solving $\rho_T = \phi_T^{1/2}\hat{\phi}_T\phi_T^{1/2}$ for ϕ_T gives $D_T(\hat{\phi}_T)$

Similarly, solving $\rho_0 = \phi_0^{1/2}\hat{\phi}_0\phi_0^{1/2}$ gives $\hat{D}_0(\phi_0)$

Example

$$\mathcal{E}^\dagger(\cdot) = E_1(\cdot)E_1^\dagger + E_2(\cdot)E_2^\dagger + E_3(\cdot)E_3^\dagger$$

$$E_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{bmatrix}.$$

$$\rho_0 = \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix} \text{ and } \rho_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\begin{aligned} \phi_0 &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, & \hat{\phi}_0 &= \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix} \\ \phi_1 &= \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}, & \hat{\phi}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow F_1 = \begin{bmatrix} \sqrt{2/3} & 0 \\ 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1/3} \end{bmatrix}, \; F_3 \begin{bmatrix} 0 & \sqrt{2/3} \\ \sqrt{1/3} & 0 \end{bmatrix}.$$

Pinned Quantum Schrödinger bridge

With $\mathcal{E}_{0:T}^\dagger$ positivity improving and two pure states

$$\rho_0 = v_0 v_0^\dagger \text{ and } \rho_T = v_T v_T^\dagger$$

(i.e., v_0, v_T are unit norm vectors), define

$$\begin{aligned} \phi_0 &:= \mathcal{E}(v_T v_T^\dagger) \\ \phi_T &:= v_T v_T^\dagger, \end{aligned}$$

and

$$\mathcal{F}^\dagger(\cdot) := \phi_T^{1/2} \mathcal{E}^\dagger(\phi_0^{-1/2}(\cdot)\phi_0^{-1/2})\phi_T^{1/2}.$$

Then, $\mathcal{F}^\dagger$ is TPTP and satisfies the marginal conditions

$$\rho_T = \mathcal{F}^\dagger(\rho_0).$$

Next:
– epilogue on non-commutative transport
– discussion & questions

Non-commutative transport

epilogue

- **Kantorovich formulation**
  given ρ_A, ρ_B, determine ρ, on a tensor product space, to
  minimize $\{\text{trace}(\rho H) \mid \text{trace}_B(\rho) = \rho_A,\ \text{trace}_A(\rho) = \rho_B\}$.
  Is there a natural choice[6] for H?

- **Dynamic formulation:** formulate[7] for matrices ρ, v, x,

$$\int_0^1 \overbrace{\int_{\text{space}} \rho(t,x)\|v(t,x)\|^2\, dx}^{\text{kinetic energy}}\, dt$$

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho v) = 0$$

$$\rho(0,\cdot) = \rho_0,\ \rho(1,\cdot) = \rho_1$$

[6] Feliciangeli, Gerolin, Portinale, A non-commutative entropic optimal transport approach to quantum composite systems at positive temperature, 2023

[7] Carlen/Maas, Mittnenzweig/Mielke, Chen/Georgiou/Tannenbaum, Chen/Georgiou/Gangbo/Tannenbaum (all in 2016)

Some calculus

with ordinary functions:

$f(x) \ : \ g(x) \mapsto f(x)g(x)$

$\partial_x \ : \ g(x) \mapsto \partial_x g(x)$

$[\partial_x, f(x)] \ : \ g(x) \mapsto \partial_x f(x)g(x) - f(x)\partial_x g(x) = (\partial_x f(x))g(x)$

with matrices:

$\partial_{L_i} X = [L_i, X] = [L_i X - X L_i]$ and $\nabla_L : X \mapsto \begin{bmatrix} L_1 X - X L_1 \\ \vdots \\ L_N X - X L_N \end{bmatrix}$

gradient

$$\nabla_L : X \mapsto \begin{bmatrix} L_1 X - X L_1 \\ \vdots \\ L_N X - X L_N \end{bmatrix}$$

divergence

$$\nabla_L^* : \ Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} \mapsto \sum_k^N L_k Y_k - Y_k L_k .$$

$\langle X, Y \rangle = \sum_{k=1}^N \mathrm{trace}(X_k^* Y_k)$

$\langle \nabla_L X, Y \rangle = \langle X, \nabla_L^* Y \rangle$

$\nabla_L(XY) = (\nabla_L X) Y + X(\nabla_L Y)$

Non-commutative continuity equation

Schrödinger & Lindblad equations

$L = L^\dagger$ for simplicity

Lindblad equation

$$
\begin{aligned}
\dot\rho &= -[iH,\rho] + \sum_{k=1}^{N}\left(L_k\rho L_k - \frac{1}{2}\rho L_k L_k - \frac{1}{2}L_k L_k\rho\right) \\
&= -\nabla^*_{iH}\rho + \underbrace{\left(-\nabla^*_L\nabla_L\rho\right)}_{\Delta\rho}
\end{aligned}
$$

Schrödinger's (Liouville, von Neumann) term: $\nabla^*_{iH}\rho = \nabla^*_{iH}(\overbrace{\rho \times \underbrace{v_1}_{I}}^{\text{momentum}})$

Lindbladian as a Laplacian: need to write $\nabla_L\rho$ as "$\rho \times \underbrace{\nabla \log \rho}_{v_2}$"

Non-commutative continuity equation

$$\dot{\rho} + \nabla_L^* \overbrace{(\rho \circ v)}^{\substack{\text{momentum} \\ m}} = 0$$

choices of non-commutative "momentum"

$(\rho \circ v) =$

$\frac{1}{2}(\rho v + v\rho)$ ("anti-commutator")

$\int_0^1 \rho^s v \rho^{1-s} ds$ (Kubo-Mori)

$\rho^{1/2} v \rho^{1/2}$

Kubo-Mori momentum - quantum Wasserstein metric

$$W_2(\rho_0,\rho_1)^2 := \inf_{\rho,v} \int_0^1 \int_0^1 \mathrm{trace}(v^* \rho^s v \rho^{1-s}) ds\, dt$$

$$\dot{\rho} = \nabla_L^* \int_0^1 \rho^s v \rho^{1-s} ds,$$

$$\rho(0) = \rho_0, \quad \rho(1) = \rho_1.$$

- duality
- geodesic space:

$$W_2(\rho_0,\rho_1) = \min_{\rho} \int_0^1 \sqrt{\langle \dot{\rho}(t), \dot{\rho}(t) \rangle_{\rho(t)}} dt,$$

Gradient flow of $S(\cdot)$

$$
\begin{aligned}
\frac{dS(\rho(t))}{dt} &= -\mathrm{trace}((\log(\rho) + I)\dot{\rho}) \\
&= -\mathrm{trace}((\log(\rho) + I)\nabla^* \int_0^1 \rho^s v \rho^{1-s} ds) \\
&= -\mathrm{trace}((\nabla_L \log \rho)^* \int_0^1 \rho^s v \rho^{1-s} ds) \\
&= -\langle \nabla_L \log \rho, v \rangle_\rho
\end{aligned}
$$

$\Rightarrow$ greatest ascent direction $v = -\nabla_L \log \rho$.

Gradient flow of $S(\cdot)$ is Lindblad's eqn.

$$\dot\rho = \nabla_L^*(\rho \circ \underbrace{(\nabla_L \log \rho)}_{v})$$
$$= -\nabla_L^* \int_0^1 \rho^s (\nabla_L \log \rho) \rho^{1-s} ds$$

Gradient flow

$$\dot\rho = -\nabla_L^* \nabla_L \rho = \Delta_L \rho$$

– miracle identity $\nabla_L \rho = \int_0^1 \rho^s (\nabla_L \log \rho)\rho^{1-s} ds$
(cf. $\partial_x \rho = \rho\, \partial_x (\log \rho)$)

– Cf. with JKO: "gradient flow of Shanon entropy = heat equation"

Recap & discussion

Quantum Schrödinger Bridges: interpolation of density matrices to restore considency

Two-state formalism - pre/post selection: Time-symmetric, Sinkhorn-applies, large-deviations interpretation, mostly classical

General quantum channel: convergence of non-commutative Sinkhorn is open

Questions:
– Is there a natural distance between quantum channels $\mathcal{E}^\dagger, \mathcal{F}^\dagger$?
– Is there a connection with non-commutative OT?
– non-commutative change of measure (?)
and, is there a non-commutative large-deviations interpretation?

$$\mathcal{F}^\dagger_{0:T}(\cdot) := \chi_T \left(\mathcal{E}^\dagger_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$