Quantum OT: quantum channels and qubits
Tutorial 2

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Plan

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Motivation

- In the classical OT theory, transport maps T pushing a probability $\rho(x)$ into $\sigma(y)$, are relaxed into Kantorovich couplings $\pi(x, y)$.
- The intepretation of $\pi(x, y)$ as a generalized transport map (plan) is probabilistic via conditioning

$$\pi(y|x) = \frac{\pi(x, y)}{\rho(x)},$$

  which defines a Markov transition kernel.
- We propose a quantum OT where the transport plan is a quantum channel.
- These plans are in correspondence with suitable couplings, akin to Golse-Mouhot-Paul-Caglioti (GMPC) OT, but with some differences.

Quantum coupling and no-cloning

- Recall that a quantum channel Φ on a system H a linear, completely positive, trace preserving map between operators on H, in particular it preserves states:

$$\rho \in \mathcal{S}(H) \quad \mapsto \quad \Phi(\rho) = \sigma \in \mathcal{S}(H).$$

- Definition: given quantum states $\sigma, \rho \in \mathcal{S}(H)$, a quantum transport plan $\Phi \in \mathcal{M}(\rho, \sigma)$ is a quantum channel such that $\Phi(\rho) = \sigma$.

- Problem: How to compute a transport cost along the plan?

- We need a joint system where both $(\rho, \Phi(\rho))$ are defined but also strongly correlated (why $\rho \otimes \Phi(\rho)$ is bad?).

  $\Rightarrow$ we need to make a copy ρ and then act with Φ on one copy.

Transpose map

Consider H and its dual H^*. Being Hilbert spaces, these are naturally identified

$$|\varphi\rangle \in H \mapsto \langle\varphi| \in H^*.$$

Given a linear operator $A : H \to H$, we define

- its adjoint $A^\dagger : H \to H$, $|\varphi\rangle \mapsto |A^\dagger\varphi\rangle$ such that

$$\langle A^\dagger\varphi|\varphi\rangle = \langle\varphi|A|\varphi\rangle.$$

- its transpose $A^T : H^* \to H^*$, $\langle\varphi| \mapsto A^T\langle\varphi|$ such that

$$A^T\langle\varphi| = \langle\varphi|A.$$

Exercise: pick an orthonormal basis $(|i\rangle)_i \subseteq H$ with dual basis $(\langle i|)_i$ and show that in coordinates $(A^\dagger)_{ij} = \overline{A_{ji}}$, while $(A^T)_{ij} = A_{ji}$.

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Purification

1. Given $\rho \in \mathcal{S}(H)$, a purification of ρ is a pure state on a larger system $|\psi_\rho\rangle \in H \otimes K$ such that
$$\mathrm{Tr}_K[|\psi_\rho\rangle\langle\psi_\rho|] = \rho.$$
2. Example: on a qubit system $H = \mathbb{C}^2$, if
$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|,$$
then the Bell state
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \in (\mathbb{C}^2)^{\otimes 2}$$
is a purification of ρ.
3. Notice that a purification cannot exist within classical probability, since pure states correspond to Dirac δ measures (hence also their marginals are δ's).

Given a purification $|\psi_\rho\rangle \in H \otimes K$, also

- $I \otimes U_K |\psi_\rho\rangle$ is a purification (U_K is any unitary)
- $|\psi_\rho\rangle \otimes |\phi\rangle \in H \otimes K \otimes K'$ for any additional $|\phi\rangle \in K'$.

To construct a canonical purification we identify $H \otimes H^*$ with $L^2(H)$ (Hilbert-Schmidt operators) via the Hilbert isometry

$$|\phi\rangle \otimes \langle\psi| \mapsto |\phi\rangle\langle\psi|$$

(extend by linearity). We use the notation for the inverse:

$$X \mapsto ||X\rangle\rangle \in H \otimes H^*$$

Exercises:

1. prove that it is indeed an isometry. Recall that the scalar products are respectively

$$(|\phi\rangle \otimes \langle\psi|, |\phi'\rangle \otimes \langle\psi'|\rangle)_{H\otimes H^*} = \langle\phi|\phi'\rangle\langle\psi'|\psi\rangle, \qquad (A, B)_{L^2(H)} = \mathrm{Tr}[A^\dagger B].$$

2. Show that if $X \in L^2(H)$, $A, B \in L^2(H)$, $B \in L^2(H^*)$, then

$$A \otimes B^T ||X\rangle\rangle = ||AXB\rangle\rangle.$$

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Given $\rho \in \mathcal{S}(H)$, its canonical purification is defined as

$$||\sqrt{\rho}\rangle\rangle \in H \otimes H^*,$$

i.e., the auxiliary system is $K = H^*$.To see it is a purification:

1. Use spectral theorem to diagonalize $\rho = \sum_i p_i |i\rangle\langle i|$ with orthonormal basis $(|i\rangle)_i$.
2. Then $\sqrt{\rho} = \sum_i \sqrt{p_i} |i\rangle\langle i|$, hence

$$||\sqrt{\rho}\rangle\rangle = \sum_i \sqrt{p_i} |i\rangle \otimes \langle i|.$$

3. Taking the partial trace:

$$\mathrm{Tr}_{H^*}[||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||] = \sum_i p_i |i\rangle\langle i| = \rho.$$

4. We also notice that

$$\mathrm{Tr}_H[||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||] = \sum_i p_i \langle i| \otimes |i\rangle = \rho^T.$$

- Notice that we can extend the construction to a larger systems, e.g. letting

$$|\psi_\rho\rangle \in H_1 \otimes H_2^* \otimes H_3 \otimes H_4^*$$

(with $H_1 = H_2 = H_3 = H_4$) given by

$$|\psi_\rho\rangle = \sum_i \sqrt{p_i}|i\rangle \otimes \langle i| \otimes |i\rangle \otimes \langle i|.$$

- This way,

$$\mathrm{Tr}_{12}[|\psi_\rho\rangle\langle\psi_\rho|] = \mathrm{Tr}_{34}[|\psi_\rho\rangle\langle\psi_\rho|] = \mathrm{Tr}_{23}[|\psi_\rho\rangle\langle\psi_\rho|] = ||\sqrt{\rho}\rangle\rangle\langle\sqrt{\rho}||.$$

and also $\mathrm{Tr}_{14}[|\psi_\rho\rangle\langle\psi_\rho|]$ up to a swap transpose.

Couplings

We define a quantum couplings $\mathcal{C}(\rho, \sigma)$ as a state $\Pi \in \mathcal{S}(H \otimes H^*)$ such that

$$\mathrm{Tr}_H[\Pi] = \rho^T, \qquad \mathrm{Tr}_{H^*}[\Pi] = \sigma.$$

Examples:

1. (identity coupling) $\Pi = ||\sqrt{\rho}\rangle\rangle\langle\sqrt{\rho}|| \in \mathcal{C}(\rho, \rho)$.
2. (product coupling) $\Pi = \sigma \otimes \rho^T$.

Remarks:

- if $\sigma = |\phi\rangle\langle\phi|$ is pure, then $\mathcal{C}(\sigma, \rho) = \{\sigma \otimes \rho^T\}$.
- $\mathcal{C}(\rho, \sigma)$ is in natural correspondence with $\mathcal{C}(\sigma, \rho)$, via the swap transpose:

$$|\phi\rangle \otimes \langle\psi| \mapsto |\psi\rangle \otimes \langle\phi|.$$

- Acting with a partial transpose on $\Pi \in \mathcal{C}(\rho, \sigma)$ we obtain $\Pi' \in L^1(H \otimes H)$ such that

$$\mathrm{Tr}_2[\Pi'] = \sigma, \quad \mathrm{Tr}_1[\Pi'] = \rho.$$

However it is not a coupling in the sense of GMPC (except special cases) since partial transpose is not completely positive.

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Correspondence between couplings and plans

Given a quantum transport plan $\Phi \in \mathcal{M}(\rho, \sigma)$, ($\Phi(\rho) = \sigma$, we induce the coupling

$$\Pi_\Phi = \left(\Phi \otimes \mathbb{I}_{L(H^*)}\right) ||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}|| \in \mathcal{C}(\rho, \sigma).$$

Indeed,

- Π_Φ is positive by complete positivity of Φ,
- we have

$$\mathrm{Tr}_{H^*}[\Pi_\Phi] = \Phi\, \mathrm{Tr}_{H^*}[||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||] = \Phi\rho = \sigma,$$

- and

$$\mathrm{Tr}_H[\Pi_\Phi] = \mathrm{Tr}_H[||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||] = \rho^T.$$

The correspondence is a bijection.

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- Let $\Pi \in \mathcal{C}(\rho, \sigma) \subseteq \mathcal{S}(H \otimes H^*)$, and use spectral theorem:

$$\Pi = \sum_i p_i ||A_i\rangle\rangle\langle\langle A_i||$$

with $A_i \in L(H)$ orthonormal basis, $\sum_i p_i = 1$.

- For simplicity assume that $\rho > 0$ is invertible. Define

$$\Phi(X) = \sum_i \sqrt{p_i}\, A_i\, \rho^{-\frac{1}{2}}\, X\, \rho^{-\frac{1}{2}}\, A_i^\dagger \sqrt{p_i} = \sum_i B_i X B_i^\dagger$$

which a Kraus representation with $B_i = \sqrt{p_i} A_i \rho^{-1/2}$:

$$\sum_i B_i^\dagger B_i = \sum_i p_i \rho^{-1/2} A_i^\dagger A_i \rho^{-1/2} = \rho^{-1/2} \left(\sum_i p_i A_i^\dagger A_i \right) \rho^{-1/2}$$
$$= \rho^{-1/2} \rho \rho^{-1/2} = \mathbb{I}_H.$$

- Then, $\Phi(\rho) = \sum_i p_i A_i A_i^\dagger = (\sigma^T)^T = \sigma$ using the exercise below.

Exercise: Show that

$$\mathrm{Tr}_H[||X\rangle\rangle\langle\langle X||] = (XX^\dagger)^T, \qquad \mathrm{Tr}_{H^*}[||X\rangle\rangle\langle\langle X||] = X^\dagger X.$$

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Plan

Quantum transport cost

We now follow closely the GMPC, but with our notion of couplings: given a cost observable C, we minimize $\mathrm{Tr}[C\Pi]$

- Fix $R_1, \ldots, R_N$, self-adjoint operators on H. Define the quadratic cost operator:

$$C = \sum_{i=1}^{N} (R_i \otimes \mathbb{I}_{H^*} - \mathbb{I}_H \otimes R_i^T)^2$$

- Given states $\rho, \sigma \in \mathcal{S}(H)$ and $\Pi \in \mathcal{C}(\rho, \sigma)$, the quantum transport cost is

$$C(\Pi) = \mathrm{Tr}[C\Pi] \geq 0.$$

- Minimizing yields the square quantum Wasserstein (pseudo-)distance:

$$D(\rho, \sigma)^2 = \inf_{\Pi \in \mathcal{C}(\rho,\sigma)} C(\Pi).$$

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Example/ Exercise

- By compactness, the $\inf$ in in fact a $\min$, i.e., there exists an optimal coupling, which also yields an optimal plan $\Phi \in \mathcal{M}(\rho, \sigma)$.
- If $\Pi = \sigma \otimes \rho^T$, then

$$C(\Pi) = \operatorname{Tr}[\sigma \otimes \rho^T C]$$
$$= \sum_{i=1}^{N} \operatorname{Tr}[\sigma R_i^2] + \operatorname{Tr}[\rho R_i^2] - 2 \operatorname{Tr}[\sigma R_i] \operatorname{Tr}[\rho R_i]$$

- If either σ (or ρ) is a pure state, then

$$D(\sigma, \rho) = \sqrt{\sum_{i=1}^{N} \operatorname{Tr}[\sigma R_i^2] + \operatorname{Tr}[\rho R_i^2] - 2 \operatorname{Tr}[\sigma R_i] \operatorname{Tr}[\rho R_i]}.$$

Can we write explicitly $C(\Pi)$ in terms of quantum transport plans Φ?

- Recall that
$$\Pi = \left(\Phi \otimes \mathbb{I}_{L^1(H^*)}\right) ||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||.$$
- we have (letting $\Phi^\dagger$ the adjoint of the channel)
$$\begin{aligned}\mathrm{Tr}[\Pi C] &= \mathrm{Tr}[||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}|| \left(\Phi^\dagger \otimes \mathbb{I}_{L(H^*)}\right) C] \\ &= \langle\langle\sqrt{\rho}|| \left(\Phi^\dagger \otimes \mathbb{I}_{L^1(H^*)}\right) C||\sqrt{\rho}\rangle\rangle.\end{aligned}$$
- With slight abuse, write $R_i = R_i \otimes \mathbb{I}_{H^*}$ and $R_i^T = \mathbb{I}_H \otimes R_i^T$, and notice that they commute so that
$$(R_i - R_i^T)^2 = R_i^2 + (R_i^2)^T - 2R_i \otimes R_i^T.$$
- We get
$$\left(\Phi^\dagger \otimes \mathbb{I}_{L(H^*)}\right)(R_i - R_i^T)^2 = \Phi^\dagger(R_i^2) + (R_i^2)^T - 2\Phi^\dagger(R_i) \otimes R_i^T.$$
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We want to to rewrite

$$\langle\langle\sqrt{\rho}||\Phi^\dagger(R_i^2)+(R_i^2)^T-2\Phi^\dagger(R_i)\otimes R_i^T||\sqrt{\rho}\rangle\rangle.$$

Recall that $A\otimes B^T||X\rangle\rangle=||AXB\rangle\rangle$, so

1. For the first term:

$$\begin{aligned}\langle\langle\sqrt{\rho}||\Phi^\dagger(R_i^2)||\sqrt{\rho}\rangle\rangle &= \langle\langle\sqrt{\rho}||\Phi^\dagger(R_i^2)\sqrt{\rho}\rangle\rangle\\ &= \mathrm{Tr}[\sqrt{\rho}\Phi^\dagger(R_i^2)\sqrt{\rho}]=\mathrm{Tr}[\rho\Phi^\dagger(R_i^2)]\\ &= \mathrm{Tr}[\Phi(\rho)R_i^2]=\mathrm{Tr}[\sigma R_i^2].\end{aligned}$$

2. Similarly, for the second term:

$$\begin{aligned}\langle\langle\sqrt{\rho}||(R_i^2)^T||\sqrt{\rho}\rangle\rangle &= \langle\langle\sqrt{\rho}||\sqrt{\rho}R_i^2\rangle\rangle\\ &= \mathrm{Tr}[\sqrt{\rho}\sqrt{\rho}R_i^2]=\mathrm{Tr}[\rho R_i^2].\end{aligned}$$

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For the third term:

$$\begin{aligned}\langle\langle\sqrt{\rho}||\Phi^\dagger(R_i)\otimes R_i^T||\sqrt{\rho}\rangle\rangle &= \langle\langle\sqrt{\rho}||\Phi^\dagger(R_i)\sqrt{\rho}R_i\rangle\rangle \\ &= \mathrm{Tr}[\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}R_i] = \mathrm{Tr}[R_i\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}] \\ &= \langle\langle\sqrt{\rho}R_i||\Phi^\dagger(R_i)\sqrt{\rho}\rangle\rangle.\end{aligned}$$

Summing upon $i = 1, \ldots, n$, we obtain the equivalent expressions

$$\begin{aligned}C(\Pi) &= \sum_{i=1}^N \mathrm{Tr}[\sigma R_i^2] + \mathrm{Tr}[\rho R_i^2] - 2\,\mathrm{Tr}[R_i\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}] \\ &= \sum_{i=1}^N \langle\langle\sqrt{\sigma}R_i||\sqrt{\sigma}R_i\rangle\rangle + \langle\langle\sqrt{\rho}R_i||\sqrt{\rho}R_i\rangle\rangle - 2\langle\langle\sqrt{\rho}R_i||\Phi^\dagger(R_i)\sqrt{\rho}\rangle\rangle.\end{aligned}$$

For the third term:

$$\begin{aligned}\langle\langle\sqrt{\rho}||\Phi^\dagger(R_i)\otimes R_i^T||\sqrt{\rho}\rangle\rangle &= \langle\langle\sqrt{\rho}||\Phi^\dagger(R_i)\sqrt{\rho}R_i\rangle\rangle \\ &= \mathrm{Tr}[\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}R_i] = \mathrm{Tr}[R_i\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}] \\ &= \langle\langle\sqrt{\rho}R_i||\Phi^\dagger(R_i)\sqrt{\rho}\rangle\rangle.\end{aligned}$$

Summing upon $i = 1, \ldots, n$, we obtain the equivalent expressions

$$\begin{aligned}C(\Pi) &= \sum_{i=1}^N \mathrm{Tr}[\sigma R_i^2] + \mathrm{Tr}[\rho R_i^2] - 2\,\mathrm{Tr}[R_i\sqrt{\rho}\Phi^\dagger(R_i)\sqrt{\rho}] \\ &= \sum_{i=1}^N \langle\langle\sqrt{\sigma}R_i||\sqrt{\sigma}R_i\rangle\rangle + \langle\langle\sqrt{\rho}R_i||\sqrt{\rho}R_i\rangle\rangle - 2\langle\langle\sqrt{\rho}R_i||\Phi^\dagger(R_i)\sqrt{\rho}\rangle\rangle.\end{aligned}$$

Distance between a state and itself

- If $\sigma = \rho$ and $\Phi = \mathbb{I}_{L^1(H)}$, then $\Pi = ||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||$ and

$$C(\Pi) = 2\sum_{i=1}^{N} \mathrm{Tr}[\rho R_i^2] - \mathrm{Tr}[\sqrt{\rho} R_i \sqrt{\rho} R_i].$$

- One has the general inequality

$$D^2(\rho, \sigma) \geq \frac{1}{2} C(||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||) + \frac{1}{2} C(||\sqrt{\sigma}\rangle\rangle\langle\langle\sqrt{\sigma}||).$$

- For $\sigma = \rho$ it yields that the identity channel is an optimal plan

$$D^2(\rho, \rho) = C(||\sqrt{\rho}\rangle\rangle\langle\langle\sqrt{\rho}||).$$

- We can also connect D^2 with the Wigner-Yanase square norm

$$D^2(\rho, \rho) = \frac{1}{4} \sum_{i=1}^{N} g_\rho\left(\mathbf{i}\left[\mathbf{R_i},\, \rho\right],\, \mathbf{i}\left[\mathbf{R_i},\, \rho\right]\right).$$

We define the Quantum W_2 divergence as

$$W_2(\sigma, \rho) = \sqrt{D^2(\sigma, \rho) - \frac{1}{2}\left(D^2(\sigma, \sigma) + D^2(\rho, \rho)\right)}. \quad (1)$$

We conjecture that W_2 is an actual distance under minimal assumptions.

Proposition (De Palma, Virosztek, Titkos, T.)

If $\dim(H) < \infty$ and

- the observables $\{R_i\}_{i=1}^d$ generate algebraically all operators,
- ρ and σ are invertible,

then

$$W_2(\rho, \sigma) = 0 \quad \Rightarrow \quad \rho = \sigma$$

and the identity channel is the only optimizer.

quantum Wasserstein isometries

- In Gehér et al., state transformations (called QW-isometries)

$$J : \mathcal{S}(H) \to \mathcal{S}(H) \qquad D^2(J(\sigma), J(\rho)) = D^2(\sigma, \rho)$$

for a given quantum system H and a set $\{R_i\}_{i=1}^d$ are introduced.

- For $H = \mathbb{C}^2$ and $\{X, Z\}$, the situation is already non-trivial:
  1. Isometries of the Bloch ball that fix the X, Z plane (rotations or symmetries) induce (trivial) QW-isometries
  2. There are QW-isometries that are neither injective nor surjective (on pure states)
  3. Inside the ball (i.e., for non pure states) there is rigidity: up to conjugation with trivial isometries, J is either the identity or

$$\rho = \rho(b_x, b_y, b_z) \quad \mapsto \quad J(\rho) = \rho(b_x, -b_y, b_z).$$

A modified triangle inequality

- Given ρ, σ, $\tau \in \mathcal{S}(H)$, we have the following modified triangle inequality:

$$D(\rho, \sigma) \leq D(\rho, \tau) + D(\tau, \tau) + D(\tau, \sigma).$$

- Interpretation: we pay some price to stay in τ and "prepare" to move from τ to σ.

- We do not know whether the term $D(\tau, \tau)$ can in general be removed.
- Bunth et al. showed that triangle inequality hold W_2 if (at least) one state is pure.

- A similar inequality is not known for the GMPC case.

Plan

Bosonic (Gaussian) systems

- Bosonic quantum particles (e.g. photons) with position Q and momentum P obey the canonical commutation relations

$$(CCR) \quad [Q, P] = \mathbf{i}\mathbb{I}_H.$$

- They provide a non-commutative analogue of $\mathbb{R}^2$:

$$|\psi(x)\rangle \in H = L^2(\mathbb{R}) : \qquad Q|\psi\rangle = |x\psi(x)\rangle, \qquad P|\psi\rangle = -\mathbf{i}|\frac{d}{dx}\psi(x)\rangle$$

  (with natural domains).
- Ladder (annihilation/creation) operators

$$a = \frac{Q + \mathbf{i}P}{\sqrt{2}} \quad a^\dagger = \frac{Q - \mathbf{i}P}{\sqrt{2}}.$$

- Number operator: $N = a^\dagger a$.

Exercise Write a, $a^\dagger$ and N as differential operators (on smooth functions)

- Vacuum state: $|0\rangle$ such that $N|0\rangle = 0$. Fock states:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

- Coherent states, for $\alpha = q + ip \in \mathbb{C}$ are defined as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

they are not an orthogonal systems – but are an overcomplete basis:

$$\frac{1}{\pi}\int_{\mathbb{C}}|\alpha\rangle\langle\alpha|d\alpha = \mathbb{I}_H.$$

- Gaussian states are $\rho = \exp(\mathrm{pol}(a, a^\dagger)) \in \mathcal{S}(H)$ with $\mathrm{pol}(\cdot,\cdot)$ second degree polynomial (not generic however!).

Exercise: Compute explicitly the first 3 Fock states $|0\rangle, |1\rangle, |2\rangle \in L^2(\mathbb{R})$. Show that they are orthonormal.

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- The setting extends to m modes, with self-adjoint (unbounded) operators $Q_1, \dots, Q_m$, $P_1, \dots, P_m$ on $H = L^2(\mathbb{R}^m)$ satisfying

$$[Q_i,\, P_j] = \mathbf{i}\,\delta_{ij}\,\mathbb{I}_H\,, \qquad [Q_i,\, Q_j] = [P_i,\, P_j] = 0\,, \qquad i,\, j = 1,\, \dots,\, m\,.$$

- Define the quadratures $\{R_1, \dots, R_{2m}\} = \{Q_1, \dots, Q_m, P_1, \dots, P_m\}$.
- A Gaussian state is an exponential of a quadratic polynomial in the R_i's:

$$\rho = \exp\left(-\frac{1}{2} \sum_{i,\, j=1}^{2m} (R_i - r_i\,\mathbb{I}_H)\, h_{ij}\, (R_j - r_j\,\mathbb{I}_H) + c \right)\,,$$

- We consider the quadratic cost operator, and the associated quantum Wasserstein distance

$$C = \sum_{i=1}^{2m} (R_i \otimes \mathbb{I}_{H^*} - \mathbb{I}_H \otimes R_i^T)^2, \quad D(\rho,\sigma)^2 = \inf_{\Pi \in \mathcal{C}(\rho,\sigma)} C(\Pi).$$

(careful with domains! compare with classical non compact case).

OT between Gaussian states

Theorem: let $\rho, \sigma \in \mathcal{S}(H)$ be Gaussian states. Then

$$D(\rho, \sigma)^2 = \mathrm{Tr}[C\Pi_{opt}]$$

for some

- Gaussian coupling $\Pi_{opt} \in \mathcal{S}(H \otimes H^*)$
- corresponding to a plan Φ_{opt} that is a Gaussian quantum channel (i.e., maps Gaussian states into Gaussian states).

Remarks:

1. Gaussian states are a finite dimensional sub-manifold $\to$ possible numerical schemes.
2. In the case of "isotropic" Gaussians (thermal states) $\omega(\nu)$ with covariance matrix νI_{2m} we have an explicit formula.
3. Ask Fanch Coudreuse for more on this!

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Plan

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