Asymptotic Solutions to Many-body Dynamics II: Incompressible Navier-Stokes equations, viscosity and Green-Kubo formula

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Conservation law: Total number of the particles.

$$d\eta_0 + \left\{ w_{0,1} - w_{-1,0} \right\} = \text{ martingale}$$

$$\text{micro current: } w_{0,1} = \eta_0[1 - \eta_1]$$

Hydro equation in the Euler Limit : $(x, t) = (X/\varepsilon, T/\varepsilon)$:

Burgers eq.: $\partial_T m + [m(1-m)]_X = 0$

Next order correction

$$\partial_T m + [m(1-m)]_X = \varepsilon \nu \Delta m$$

To see ν to the leading order, we need to consider

$$\text{incompressible limit} \quad m = 1/2 + \varepsilon u, \qquad T \to \varepsilon^{-1} T$$

$$\partial_T u + u u_X = \nu \Delta u$$

Time dependent correlation (or structure) function

$$S_\rho(x,t) = \langle \eta_0(0); \eta_x(t) \rangle_\rho = E_\rho[\eta_0 \, e^{t\mathcal{L}} \eta_x] - E_\rho[\eta_0] E_\rho[\eta_x]$$

Consider $\rho = 1/2$ so that there is no global drift. Suppose

$$S(x,t) \sim (2\pi t)^{-d/2} \exp\left\{ -\frac{x^2}{2tD(t)} \right\}$$

$$D(t) \; = \; t^{-1} \sum_{x \in \mathbb{Z}^d} x^2 S(x,t)$$

Green-Kubo formula: Simple consequence of the Ito's calculus.

$$D := \text{ static term} + \int_0^\infty dt \sum_x \underbrace{\langle\!\langle \tilde{w}_{0,1} \, ; \, e^{t\mathcal{L}} \; \tilde{w}_{0,1} \rangle\!\rangle_\rho}_{\text{current-current correl. funct.}}$$

$$\langle\!\langle g, h \rangle\!\rangle_\rho = \sum_{x \in \mathbb{Z}^d} \langle \tau_x h \, ; \, g \rangle_\rho \, , \qquad \underbrace{\tilde{\omega}_{0,1} = (\eta_0 - \tfrac{1}{2})(\eta_1 - \tfrac{1}{2})}_{\text{renormalized current}}$$

Theorem (Esposito, Marra, Landim, Olla, Y): For $d > 2$ we have

1. Green-Kubo formula converges.

2. The convergence to the Burgers equation holds in the incompressible limit.

3. The interpretation of the next order correction holds before the shock occurs.

$$\text{incompressible limit} \quad m = 1/2 + \varepsilon u,$$

$$\frac{\eta_1 + \cdots \eta_N}{N} = \text{ mean } + N^{-1/2}, \qquad N = \varepsilon^{-d}$$

The accuracy needed to make sense of u is ε.

$$\varepsilon^{-1}N^{-1/2} = \varepsilon^{d/2-1} \to 0 \text{ iff } d > 2.$$

Large deviation of lattice gas (Quastel-Y):

$$P_\varepsilon(u_\varepsilon \sim u) \approx \exp\left(-\varepsilon^{-d+2} I(u)\right),\ d = 3$$

$$I(u(\cdot,\cdot)) := \int_0^T \left\| \frac{\partial u}{\partial T'} + \pi\left(u \cdot \nabla u\right) - \nu \Delta u \right\|_{-1}^2 dT'$$

π projection onto the divergence free part.

Beijeren-Kutner-Spohn conjectured

$$D(t) \sim (\log t)^{2/3}, \quad d = 2;$$

$$D(t) \sim t^{1/3}, \quad d = 1$$

Mode-Coupling Theory.

Kardar-Parisi-Zhang in $d = 1$. KPZ equation.

Green-Kubo formula (nonstandard form):

$$\lambda^2 \int_0^\infty e^{-\lambda t}\, t D_{11}(t) dt \sim \frac{1}{2} + \sum_{x \in \mathbb{Z}^d} \langle\!\langle w_1 \,,\, (\lambda - \mathcal{L})^{-1} w_1 \rangle\!\rangle.$$

$$w_1 = \tilde{w}_{0,1} = (\eta_0 - \tfrac{1}{2})(\eta_1 - \tfrac{1}{2})$$

Landim-Quastel-Salmhofer-Y

$$\langle\langle w_1, (\lambda - \mathcal{L})^{-1} w_1 \rangle\rangle \geq |\log \lambda|^{1/2} \quad d = 2.$$

$$\langle\langle w_1, (\lambda - \mathcal{L})^{-1} w_1 \rangle\rangle \geq \lambda^{-1/4} \quad d = 1.$$

Theorem (Y) $d = 2$. There exists a constant $\gamma > 0$ such that for sufficiently small $\lambda > 0$,

$$|\log \lambda|^{2/3} e^{-\gamma |\log\log\log \lambda|^2}$$

$$\leq \langle\langle w_1, (\lambda - \mathcal{L})^{-1} w_1 \rangle\rangle$$

$$\leq |\log \lambda|^{2/3} e^{\gamma |\log\log\log \lambda|^2}$$

Fundamental Question: How to solve the equation

$$\Big(\lambda - \mathcal{L}\Big)u = w$$

with $\mathcal{L}$ an operator in infinite dimension?

Need a setup so that the solution depends essentially on finite number of variables.

Two basic setups:

1. Duality for probabilistic problems (or Foch space representation).

2. BBGKY hierarchy for classical or quantum problems.

Polynomial: $h = \sum h^{(2)}(x,y)\xi_x\xi_y + \sum h^{(3)}(x,y,z)\xi_x\xi_y\xi_z + \cdots$

$$\xi_x = \eta_x - \frac{1}{2}, \qquad \langle \xi_x, \xi_y \rangle = 0$$

Degree one functions are of the form $\sum C(x,y)[\xi_x - \xi_y]$ and vanishing in our norm. The coefficients

$$(h^{(2)}, h^{(3)}, \cdots) \in \otimes_{j=1}^{\infty} \mathcal{M}_j$$

$\mathcal{M}_n$: space of symmetric functions of n variables on $\mathbb{Z}^2$.

$$f(x,x) = 0$$

$$w_1 = \xi_0 \xi_{e_1}, \quad w_1(0,e_1) = w_1(e_1,0) = 1, \quad = 0 \text{ otherwise}$$

$\mathcal{L} = \mathcal{S} + \mathcal{A}$, $\mathcal{S}$: symmetric part, $\mathcal{A}$ the asymmetric part.

$$\mathcal{S} : \mathcal{M}_n \to \mathcal{M}_n \qquad (\mathcal{S}g) = \Delta g$$

with   Neumann boundary condition on the hard core.

$$\mathcal{A}_+ : \mathcal{M}_n \to \mathcal{M}_{n+1}, \quad \mathcal{A}_- = -\mathcal{A}_+^* : \mathcal{M}_n \to \mathcal{M}_{n-1}$$

$$\Big(\mathcal{A}_+ g\Big) = \text{Symm}\Big\{\Big[g(x_1 + e_1, x_2) - g(x_1, x_2)\Big]\delta(x_3 - x_1 - e_1)1(x_2 \neq x_3)\Big\}$$

$\mathcal{S} + \mathcal{A}$ is not a generator in prob. sense in the graded space.

In terms of creation and annihilation operators and drop the hard core conditions $\Longrightarrow$

3rd non-selfadjoint Eucliden field "Hamiltonian":

$$\mathcal{L} = \sum_x \Big[\nabla a^+ x \nabla a_x + a_x^+ n_{x+1} - n_x \nabla a_{x+1}^+ + \nabla a_{x+1} n_x \Big]$$

where $n_x = a_x^+ a_x$ is the number operator

Question: What is the infrared (large distance) limit? Infrared fix points of the renormalization group flow?

Gaussian for $d > 2$, non-Gaussian for $d \leq 2$.

Typical problems with RG method: complicated combinatoric estimates on Feynman diagrams .

Non-selfadjointness. Few low dimensional fixed points are identified even for selfadjoint field theory.

$$(\lambda - L)u = w, \qquad w = w_1$$

$$\lambda - L = D - A; \quad L = \Delta + A, \quad D = \lambda - \Delta$$

- S is second order operator and A is first order. For large distance, $A \gg S$.

- A has a big kernel since it is first order! In the kernel of A, $S \gg A$.

- Since A_+ creates a particle

$$A_+^{n,n+1} \leq nS_n, \qquad d > 2 \Longrightarrow \text{GK converges}$$

$$V(x_1 - x_2) \leq C[-\Delta_{x_1} - \Delta_{x_2}], \quad V \text{ is short-range}, d > 2$$

Monotonicity inequality:

$$\langle\!\langle w, \left[D_2 + A_+^* D_3^{-1} A_+\right]^{-1} w\rangle\!\rangle \leq \cdots \leq \langle\!\langle w, (\lambda - L_N)^{-1} w\rangle\!\rangle$$

$$\cdots \leq \langle\!\langle w, \left[D_2 + A_+^* \left\{D_3 + A_+^* D_4^{-1} A_+\right\}^{-1} A_+\right]^{-1} w\rangle\!\rangle$$

$$D = \lambda - \Delta, \qquad \cfrac{1}{D + A^* \cfrac{1}{D + A^* \cfrac{1}{D + A^* \cfrac{1}{D + \cdots} A} A} A}$$

Denote the Fourier variables

$$p = (r, s), \qquad \mathbf{p}_n = (p_1, \cdots, p_n)$$

dispersion law in $d = 2$:

$$r^2 + s^2 \Rightarrow r^2 |\log(\lambda + r^2 + s^2)|^{2/3} + s^2$$

is a fixed point.

In the x direction, the "correction" from the first order part dominates the symmetric part

$\Rightarrow$ Far from perturbation of free theory.