Asymptotic Solutions to Many-body Dynamics I: The Euler Equations

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Asymmetric Simple Exclusion

1. **configuration**: $\eta \in \{0,1\}^{\mathbb{Z}}$

2. jump to right with prob p, to left prob q.

Totally asymmetric to the right $d = 1$, $\quad p = 1$, $q = 0$.

generator: $$(\mathcal{L}f)(\eta) = \sum_{x\in\mathbb{Z}} \eta_x \left[1 - \eta_{x+1}\right] \left[f(\eta^{x,x+1}) - f(\eta)\right] .$$

$$(\eta^{x,y})_z = \begin{cases} \eta_z & \text{if } z \neq x,\ y, \\ \eta_x & \text{if } z = y \text{ and} \\ \eta_y & \text{if } z = x. \end{cases}$$

Invariant measures: Bernoulli (product) measure with density ρ, μ_ρ. $\langle \quad \rangle_\rho = E^{\mu_\rho}$.

The formal adjoint w.r.t. any μ_ρ is

$$(\mathcal{L}^* f)(\eta) \ := \ \sum_{x \in \mathbb{Z}} \eta_x \left[1 - \eta_{x+1}\right] \left[f(\eta^{x,x-1}) - f(\eta)\right] \ .$$

i.e., the generator to jump to the left.

Denote the density of the system w.r.t. μ by $f(t,\eta)$. Then the forward equation is given by

$$\partial_t f(t,\eta) = (\mathcal{L}^* f)(t,\eta)$$

Question: How to solve this equation?

Conservation law: Total number of the particles.

Ito's formula: $\frac{d}{dt}\eta_0 = \mathcal{L}\eta_0 +$ martingale

$$\frac{d}{dt}\eta_0 + \left\{ w_{0,1} - w_{-1,0} \right\} = \text{ martingale}$$

micro current: $w_{0,e} = \eta_0[1 - \eta_1]$

$$\frac{d}{dt} \textbf{ conserved quantities} + \mathrm{div} \ (\textbf{ micro current}) = 0$$

Euler Limit : $(x, t) = (X/\varepsilon, T/\varepsilon)$ large-scale long-time regime.

Symmetric case: $p = q = 1/2$.

$$\frac{d}{dt}\eta_0 = \frac{1}{2}[\eta_{-1} - 2\eta_0 + \eta_1] + \text{ martingale}$$

Heat equation in the diffusive scaling limit.

Empirical density

$$\mu_\eta^\varepsilon(dX, T) = \left\{ \varepsilon \sum_y \delta(X - \varepsilon y)\, \eta_y(\varepsilon^{-1} T) \right\} dX$$

an one parameter family of random measures (depends on the law of η) indexed by T.

Goal: Hydrodynamical limit

$$\mu_\eta^\varepsilon(dX, T) \to \delta\Big(m(X, T) dX\Big)$$

and $m(X, T)$ satisfies the hydrodynamical equation.

Need a closed equation for the local density.

Boltzmann-Gibbs principle: System locally is in equilibrium.

$$\mathop{\mathrm{Av}}_{x:\varepsilon x \sim X} \eta_x(1-\eta_{x+1}) - m(X)[1-m(X)] \quad \text{in the limit } \varepsilon \to 0$$

micro current $\to$ **macro current**

1. System is locally time invariant.

2. Classify the time invariant measures.

3. Show that "micro current $\to$ macro current" holds for all time invariant measures.

$$\lim_{\varepsilon,\ell} \sup_{\rho} P^{\mu_\rho}\left[\operatorname*{Av}_{|x|\le\ell} \eta_x(1-\eta_{x+1}) - m[1-m] \to 0 \right] = 1, \quad m := \operatorname*{Av}_{|x|\le\ell} \eta_x$$

Burgers eq.: $\partial_T m + [m(1-m)]_X = 0$ (entropy cond., Rezakhanlou)

Questions:

1. In what scale ℓ (relative to ε) the system is locally equilibrium? Mesoscale fluctuations should not occur if the hydro limits hold.

2. Where does the entropy condition come from?

3. Where does the viscosity come from?

$$\partial_T u + u u_X = \nu u_{XX}$$

Need diffusive re-scaling and Green-Kubo formula.

Set the reference measure to $\mu_{1/2}$, i.e., the uniform measure.

Local Gibbs state (local equilibrium states) with chemical potential λ.

$$\widetilde{\psi}_\lambda^\varepsilon(\eta) = Z_\varepsilon^{-1} \exp\left\{\sum_x \lambda(\varepsilon x)\eta_x\right\}$$

where Z_ε is the normalization. Define the density

$$m(x) = E^{\widetilde{\psi}_\lambda^\varepsilon}[\eta_x]$$

Then λ and m are related by

$$\lambda(x) = \log \frac{m(x)}{1 - m(x)}, \qquad m(t,\theta) = \frac{1}{1 + e^{-\lambda(t,\theta)}}.$$

Define the local Gibbs state with density m by choosing the chemical to have the density m, i.e.,

$$\psi_m^\varepsilon = \widetilde{\psi}_\lambda^\varepsilon$$

Definition. Let f, g be two probability density. The relative entropy is defined by

$$S(f|g) = \int f \log \frac{f}{g} \, d\mu$$

Theorem 1: Suppose that m_T is a smooth solution to the Burgers equation up to T_0. Let f_t solve the froward equation. If

$$s(f_0|\psi^{\varepsilon}_{m_0}) := \varepsilon^d S(f_0|\psi^{\varepsilon}_{m_0}) \to 0 \text{ as } \varepsilon \to 0$$

Then for $T \leq T_0$ we have $s(f_{\varepsilon^{-1}T}|\psi^{\varepsilon}_{m_T}) \to 0$.

$$
\begin{array}{ccc}
m_0(X) & \xrightarrow{\text{Burgers}} & m(X,T) \\
\text{local equilibrium} \Big\downarrow & & \Big\downarrow \text{local equilibrium} \\
\psi^{\varepsilon}_{m_0} & \longrightarrow & \psi^{\varepsilon}_{m_T} \\
s(f_0|\psi^{\varepsilon}_{m_0}) \to 0 \Big\downarrow & & \Big\downarrow s(f_{\varepsilon^{-1}T}|\psi^{\varepsilon}_{m_T}) \to 0 \\
f_0 & \xrightarrow{\text{ASEP}} & f_{\varepsilon^{-1}T}
\end{array}
$$

Construct approximating solution to the forward equation.

Relative entropy inequality: Suppose $\partial_t f_t = \mathcal{L}^* f_t$, $\mathcal{L}^*$ is w.r.t. μ and ψ_t is any function. Then

$$\frac{d}{dt} S(f_t|\psi_t) \le \int f_t \frac{(\mathcal{L}^* - \partial_t)\psi_t}{\psi_t} \, d\mu$$

Proof:

$$\begin{aligned}\frac{d}{dt}\int f_t \log(f_t/\psi_t)\, d\mu &= \int (\mathcal{L}^* f_t) \log(f_t/\psi_t)\, d\mu + \int [\dot{f}_t - \tfrac{\dot{\psi}_t}{\psi_t}]\, d\mu \\ &= \int f_t \mathcal{L} \log(f_t/\psi_t)\, d\mu - \int f_t \tfrac{\dot{\psi}_t}{\psi_t}\, d\mu\end{aligned}$$

Φ concave $\Longrightarrow \mathcal{L}\Phi(g) \le \Phi'(g)\mathcal{L}g$. Thus

$$\int f_t \mathcal{L} \log(f_t/\psi_t)\, d\mu \le \int f_t \frac{\mathcal{L}(f_t/\psi_t)}{f_t/\psi_t}\, d\mu$$
$$= \int \psi_t \mathcal{L}(f_t/\psi_t)\, d\mu = \int f_t \psi_t^{-1} \mathcal{L}^* \psi_t \, d\mu$$

dynamical variational approach

$$\partial_t S(f_t|\psi_t) \le \int f_t \left\{ \psi_t^{-1} \left[\mathcal{L}^* - \partial_t\right] \psi_t \right\} d\omega$$

Jensen inequality

$$\int f W \, d\omega \le S(f|\psi) + \log \int \psi \exp\left(W\right) \, d\omega$$

$$\partial_t S(f_t|\psi_t) \le S(f_t|\psi_t) + \underbrace{\log \int \psi_t \exp\left\{ \psi_t^{-1} \left[\mathcal{L}^* - \partial_t\right] \psi_t \right\} d\omega}_{\text{indep of } f_t \text{ and } \to 0}$$

- Need classification of invariant measures.
- Use large deviation argument.

Newton (Liouville) equation

$$H(\boldsymbol{x},\boldsymbol{v}) = \sum_i \frac{v_i^2}{2} + \sum_{i<j} W(x_i - x_j)$$

The Liouville operator is given by

$$\mathcal{L} = \sum_i \left[\frac{\partial H}{\partial v_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial v_i}\right]$$

Boltzmann hypothesis: $\mathcal{L}^*\omega = 0 \Longrightarrow \omega$ is Gibbs ?

Olla-Varadhan-Y.

$$\text{stationarity } + [s(\omega) \text{ finite}] + \text{vel. uncorrel} \Longrightarrow \text{Gibbs}$$

Theorem (Olla-Varadhan-Y.): Suppose the Boltzmann hypothesis holds. Then Theorem 1 holds for classical dynamics with the hydrodynamical equation given by the Euler equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial T} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) &= 0 \\
\frac{\partial (\rho\, \mathbf{u})}{\partial T} + \boldsymbol{\nabla} \cdot [\rho \mathbf{u} \otimes \mathbf{u} + P(e, \rho)] &= 0 \\
\frac{\partial (\rho e)}{\partial T} + \boldsymbol{\nabla} \cdot [\rho e \mathbf{u} - P(e, \rho) \mathbf{u}] &= 0
\end{aligned}
$$

$P(e, \rho)$ is the pressure computed from statistical mechanics.

$\psi^{\varepsilon}_{\mathbf{q}}$: local Gibbs state with conservative quantities $\mathbf{q}$

$$\begin{array}{ccc} \psi^{\varepsilon}_{\mathbf{q}_0} & \xrightarrow{\text{Euler equation}} & \psi^{\varepsilon}_{\mathbf{q}_T} \\ \big\downarrow & & \big\downarrow \; s(f_{\varepsilon^{-1}T} | \psi^{\varepsilon}_{\mathbf{q}_T}) \to 0 \\ f_0 & \xrightarrow{\text{Liouville equation}} & f_{\varepsilon^{-1}T} \end{array}$$

Quantum dynamics

The dynamics is given by the Schrödinger equation:

$$i\partial_t \psi_t = H_N \psi_t$$

where H_N is the Hamiltonian

$$H_N = -\frac{1}{2}\sum_{j=1}^{N} \Delta_j + \sum_{1 \le i < j \le N} W(x_i - x_j)$$

In statistical mechanics, the state is described by a density matrix (quantum equivalent of a probability measure), e.g.:

$$\gamma_t = \sum_n \rho_n \mathrm{Proj}_{\psi_{n,t}}$$

which satisfies the Schrödinger equation in the form

$$i\partial_t\gamma_t = [H, \gamma_t] := H\gamma_t - \gamma_t H$$

and the expectation of an observable X is given by

$$\gamma_t(X) = \mathrm{Tr}\gamma_t X, \qquad \langle\psi_t, X\psi_t\rangle$$

$\omega^{\varepsilon}_{\mathbf{q}}$: local Gibbs state with conservative quantities $\mathbf{q}$.

Theorem: (Nachtergaele-Y) Assuming the quantum version of the Boltzmann hypothesis and some additional cutoff assumptions. Then the Euler equations holds provided that the pressure function is computed via quantum statistical physics.

$$\begin{array}{ccc} \omega^{\varepsilon}_{\mathbf{q}_0} & \xrightarrow{\text{Euler equation}} & \omega^{\varepsilon}_{\mathbf{q}_T} \\ \downarrow & & \downarrow s(\gamma_{\varepsilon^{-1}T} | \omega^{\varepsilon}_{\mathbf{q}_T}) \to 0 \\ \gamma_0 & \xrightarrow{\text{Schrödinger equation}} & \gamma_{\varepsilon^{-1}T} \end{array}$$

The energy h_x operator is given by:

$$h_x = \frac{1}{2}\nabla a_x^+ \nabla a_x + \frac{1}{2}\int dy W(x-y) a_x^+ a_y^+ a_y a_x$$

Key Difficulty: **Quantum mechanics are noncommutative and involves noncommutative operators**.

1. $f(A,B)$ in general is not defined unless A, B commute.

2. Operator inequalities are subtle:

$$|A+B| \leq |A| + |B|, \qquad |AB| \leq |A||B|$$

are both false.

Outline

1. Cutoff of high momentum.

2. Constructing a commuting version of local energy, momentum and density.

3. Cutoff of phase transition region.

4. Ergodicity.

5. Virial theorem to compute the pressure function.

6. Large deviation for commuting variables.