Encoded Universality – adapting quantum processing to physical interactions

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Overview

- Universal quantum computation - some history
- Change of paradigm
- Example: exchange-only qc
- General: Lie algebra formalism for encoded universal computation
- Solid state: exchange-based qc
  - Heisenberg, symmetric “XY”, asymmetric “XY”, with crossterms, ...
- Gas phase: scalable ion trap qc
Quantum circuits

Barenco et al. ’95:

Single-qubit gates and CNOT → every unitary transformation
"Easy" and "hard" interactions (system-dependent)

"Easy": intrinsic interactions “natural” to the system, easy to tune, rapid
"Hard": slower, require higher device complexity, high decoherence

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Can we avoid “hard” interactions?
“Almost” every interaction is universal

Deutsch et al. (’95), Lloyd (‘95):

Almost any interaction on two qubits is universal

In the generic sense but ... 

Nature is not generic!
Traditionally:
*manipulate* the physical system

 Universal encoded computation:
interactions *given* by the physical system

find a way to **make these universal**
language of Hamiltonians

\[ U(t) = \exp(iHt) \quad \text{which } \textit{interactions} \text{ are universal?} \]

given \( H = \{H_1, H_2, \ldots, H_n\} \) can one generate any unitary transformation? (exact or approximate)

\[ e^{it_1 H_1} \cdot e^{it_2 H_2} \cdot e^{it_3 H_3} \ldots \approx U \]

\( H \) has to generate the Lie algebra \( su(N) \) of the unitary group \( SU(N) \)

1) \[ U(a) = e^{iaH} \quad \text{\textit{scalar multiple}} \]

2) \[ e^{i(aH_1 + bH_2)} = \lim_{p \to \infty} \left\{ e^{iaH_1/p} e^{ibH_2/p} \right\}^p \quad \text{\textit{linear combination}} \]

3) \[ e^{i[H_1, H_2]} = \lim_{p \to \infty} \left\{ e^{-iH_1/\sqrt{p}} e^{iH_2/\sqrt{p}} e^{iH_1/\sqrt{p}} e^{-iH_2/\sqrt{p}} \right\}^p \quad \text{\textit{Lie bracket}} \]
e.g. Heisenberg exchange, $E$

$$E_{ij} = \sum_{\alpha=x,y,z} \sigma^i_{\alpha} \cdot \sigma^j_{\alpha}$$

(Pauli matrices)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$|0\rangle|1\rangle \leftrightarrow E \leftrightarrow |1\rangle|0\rangle$

- omnipresent in solid state physics (« easy »)
- is not universal: preserves the total spin of the qubits

**Lie algebra of $E$**:

On three qubits:

$$H_0 := E_{12} + E_{13} + E_{23}; \quad H_1 := \frac{1}{4\sqrt{3}} (E_{13} - E_{23})$$

$$H_3 := \frac{1}{12} (-2E_{12} + E_{23} + E_{13}); \quad H_2 := i[H_3, H_1]$$

$$[H_0, H_i] = 0 \quad [H_i, H_j] = i\epsilon_{ijk} H_k$$

$$\{H_1, H_2, H_3\} \quad \text{su}(2)$$
the algebra $L(E)$ of $E$ (on 3 qubits)

The algebra $L^3(E)$ splits as:

$$L^3(E) \cong S_1 \otimes I_4 \oplus S_2 \otimes I_2$$

$\text{su}(2) \subset S_2$

Encoded qubit?

$$|0_L\rangle = \frac{1}{\sqrt{2}} (|010\rangle - |100\rangle)$$

$$|1_L\rangle = \frac{1}{\sqrt{6}} (2|001\rangle - |010\rangle - |100\rangle)$$

Simulation of all operations of one qubit ($\text{su}(2)$) with $L^3(E)$ on the encoded qubit!
the algebra $L^n(E)$ of E (on $n$ qubits)

the algebra $L^n(E)$ splits as:

$$L^n(E) \cong \bigoplus_{j=0}^{n/2} S_n \otimes I_{2j+1}$$

commutant $L'$ of $L^n(E)$:

$$L' = \{ A : [A, L] = 0 \quad \forall L \in L^n(E) \}$$

$L'$ is generated by

$$S_\alpha = \sum_{i=1}^{n/2} \sigma_\alpha^n$$

as a Lie algebra, $L'$ splits into irreducible representations of $\text{su}(2)$
Let $S$ be a $\dagger$-closed algebra closed under multiplication and linear combination. Then the underlying space $H$ is isomorphic to

$$H \cong \bigoplus_{j \in \mathcal{J}} C^{n_j} \otimes C^{d_j}$$

such that $S$ and its commutant $S'$ split as:

$$S \cong \bigoplus_{j \in \mathcal{J}} M(C^{n_j}) \otimes I_{d_j} \quad \quad S' \cong \bigoplus_{j \in \mathcal{J}} I_{n_j} \otimes M(C^{d_j})$$

where $M(C^d)$ (or $M(C^n)$) is the algebra of all matrices on $C^n$.

$\text{su}(n_j) \subseteq M(C^{n_j})$

universal computation “for free”?
BUT...

For S the interaction algebra, e.g.,

\[ H = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}, \quad \{S_{\alpha}\} \rightarrow S \]

the full multiplicative algebra \( S \) is not at our disposition, only the Lie algebra \( L \)

however the Lie algebra splits into irreducible components in the same basis:

\[ \mathcal{L} \cong \bigoplus_{j \in J} S_{n_j} \otimes I_{d_j} \]
central problem of “Encoded Universality”

Given an ensemble of generators $H$ with Lie algebra $L(H)$ which splits as

$$\mathcal{L} \cong \bigoplus_{j \in \mathcal{J}} S_{n_j} \otimes I_{d_j}$$

can one find a component $S_{n_j} \otimes I_{d_j}$ s.t. $S_{n_j}$ contains $su(n_j)$?

$su(n_j) \in S_{n_j}$ ?

Yes

encode the quantum information into the corresponding sub-space of dimension: $n_j$

Conjoining – a useful tensor structure

Introduce a cutoff that defines a single “qudit”.

In principle:

For larger $n$, find larger component $S_{n_k}$ with better encoding ratio $r$?

$$r = \log_2(\text{dim } S_{n_k}) / n$$

Need to guarantee uniformity of quantum circuits! ("form" of circuit should not depend on size of problem)
Introduce cutoff -> tensor product structure

Conjoining subsystems:

Kempe et al., Qu. Inf. & Comp. 1, 33 (2001)
Heisenberg interaction (isotropic exchange)

\[ E_{i,j} = \vec{\sigma}_i \cdot \vec{\sigma}_j \]

- E is universal with encoding*
- introduce tensor structure, eg. blocks with 3 qubits**

![Diagram showing encoded qubits and operations](image)

efficient implementation of encoded gates found by **numerical search****

serial - 19 operations for CNOT, 4 operations for 1-qubit
parallel - 7 operations for CNOT, 3 operations for 1-qubit

**DiVincenzo, Bacon, Kempe, Whaley, NATURE 408, 339 (2000)
exchange-only CNOT

nearest neighbor exchange coupling

exchange gate

\[ e^{iJE_{i,j}t} \]


**Tradeoffs**

factor of 3 in space (encoding)

factor of \(\sim 10\) in time
Spintronic Quantum Computer

use **exchange interaction** between qubits for two-qubit gates
tune by raising/lowering barrier between dots to control overlap
coupling strength: on \( \approx 0.1 \text{ meV} \), off \( \approx 0 \)

single qubit gates needed to supplement exchange \( \Rightarrow \) high
demands on g-factor engineering, strong inhomogeneous
magnetic fields, slow microwave manipulations, ...
Physical Systems

Solid state systems with spin-spin interaction

isotropic exchange
$H_I(t) = J(t) \, S_1 \cdot S_2$
Loss, DiVincenzo, PRA \textbf{57}, 120 (1998)

spin-orbit coupling anisotropy
$H_{SO}(t) = b(t) \, (S_1 \times S_2) + S_1 \cdot B(t) \cdot S_2$
Moriya, Phys.Rev. \textbf{120}, 91 (1960)
Bonesteel et al., PRL \textbf{87}, 207901 (2001)

dipole-dipole interaction
$H_D(t, R) = d(t, R) \, (S_1 \cdot S_2 - 3 \, S_1 \cdot \Gamma(t) \cdot S_2)$
Burkard, Loss, PRL \textbf{88}, 047903

Qubits coupled via cavity modes

$H(t) = J(t) \, (\sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y})$
Imamoglu et al., PRL \textbf{83}, 4204 (1999)
Anisotropic Exchange \((XY)\)

\[
(H_{XY})_{ij} = \frac{J_{ij}}{2} (\sigma^i_x \sigma^j_x + \sigma^i_y \sigma^j_y) \equiv J_{ij} A_{ij}
\]

“\(XY\)”-interaction

- **Encode into qutrit:**
  \[
  |0_L\rangle = |100\rangle \quad \rightarrow \text{operators } A_{12}^L, A_{23}^L, A_{31}^L
  \]
  \[
  |1_L\rangle = |010\rangle
  \]
  \[
  |2_L\rangle = |001\rangle
  \]
  \[
  e.g., \quad A_{12}^L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
  \]

- **Construct** \([A^L_{\alpha\beta}, A^L_{\gamma\delta}] \rightarrow su(3) \)** 1-qutrit operations

- **Conjoin qutrits:**
  \[
  |0_L\rangle \otimes |1_L\rangle \rightarrow \text{9D subspace via commutant of } H_{XY}
  \]

- **\(H_{XY}\) generates \(su(9)\) on this subspace**

- **“Truncated qubit”**: use \(|0_L\rangle = |100\rangle\) and \(|1_L\rangle = |010\rangle\) only

- **Effectively**
  \[
  |0_L\rangle = |10\rangle \quad \text{with an ancillary qubit } |0\rangle \text{ for 1-qubit gates}
  \]

---

**Exact Gates for XY**

Gate sequences:
- 7 operations for single qubit operations (serial)
- 5 operations for $\text{Sqrt (-ZZ)}$ ($\equiv$ controlled phase)

"P3"-gate:

\[
P3(\gamma) = \begin{array}{c}
\uparrow \frac{\pi}{4} \\
\uparrow \frac{\pi}{2} \\
\downarrow \frac{\pi}{2} \\
\downarrow \frac{\pi}{2} \\
\uparrow \gamma/2 \\
\uparrow -\frac{\pi}{4}
\end{array}
\]

Truncated qubit:
- $|0_L\rangle = |10\rangle$
- $|1_L\rangle = |01\rangle$

Single qubit operations:
- $e^{i\phi\sigma_x}$
- $e^{i\phi\sigma_z}$

Two-qubit operation:
- $\sqrt{-\sigma_z\sigma_z}$

**U** = $e^{i\phi_1\sigma_x} \circ e^{i\phi_2\sigma_z} \circ e^{i\phi_3\sigma_x}$ (Euler angles)

Layout – Anisotropic Exchange

a) triangular array (qutrit)

b) “truncated qubit”

or
Physical Systems

Solid state systems with spin-spin interaction

isotropic exchange
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Loss, DiVincenzo, PRA 57, 120 (1998)

spin-orbit coupling anisotropy
\[ H_{SO}(t) = b(t) (\mathbf{S}_1 \times \mathbf{S}_2) + \mathbf{S}_1 \cdot \mathbf{B}(t) \cdot \mathbf{S}_2 \]
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dipole-dipole interaction
\[ H_{D}(t,R) = d(t,R) (\mathbf{S}_1 \cdot \mathbf{S}_2 - 3 \mathbf{S}_1 \cdot \Gamma(t) \cdot \mathbf{S}_2) \]
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\[ H(t) = J(t) (\sigma_{1x} \sigma_{2x} + \sigma_{1y} \sigma_{2y}) \]
Imamoglu et al., PRL 83, 4204 (1999)
Generalized Anisotropic Exchange

Two contributions to \( H \) (in 2D):

1) symmetric term, which couples the physical qubit states \(|01\rangle\) and \(|10\rangle\)

\[
H_{ij} = J \left( \sigma_{x,i} \sigma_{x,j} + \sigma_{y,i} \sigma_{y,j} \right) + K \left( \sigma_{x,i} \sigma_{y,j} - \sigma_{y,i} \sigma_{x,j} \right)
\]

2) antisymmetric term, coupling physical qubit states \(|00\rangle\) and \(|11\rangle\)

\[
h_{ij} = j \left( \sigma_{x,i} \sigma_{x,j} - \sigma_{y,i} \sigma_{y,j} \right) + k \left( \sigma_{x,i} \sigma_{y,j} + \sigma_{y,i} \sigma_{x,j} \right)
\]

Matrix representation

in the basis \( \mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \)

\[
\begin{pmatrix}
0 & 0 & 0 & j+ik \\
0 & 0 & J+iK & 0 \\
0 & J-iK & 0 & 0 \\
j-ik & 0 & 0 & 0
\end{pmatrix}
\]

Vala & Whaley, PRA 66, 022304 (2002)
3-qubit encoding

Algebraic approach: commutant of $H$ is spanned by

$$Z = \bigotimes_k s_{z,k} \quad \text{and} \quad X = \bigotimes_k s_{x,k}, \quad \text{for} \quad k = 1, 2, 3$$

2-by-2 block diagonal structure in the basis set

$B = \{|000>, |111>, |110>, |001>, |101>, |010>, |011>, |100>\}$

the Lie algebra of $H$ and Hilbert space split as

$$L = L(4) \otimes I_2 \quad \text{and} \quad H = \mathbb{C}^4 \otimes \mathbb{C}^4$$

and code spaces are defined as

(I) $\{|000>, |110>, |101>, |011>\}$

(II) $\{|111>, |001>, |010>, |100>\}$
Action of the Hamiltonian

Hamiltonian can be expressed in either basis

\{ |000>, |110>, |101>, |011> \} or \{ |111>, |001>, |010>, |100> \}:

```
\begin{pmatrix}
0 & h & 0 & 0 \\
0 & 0 & h & 0 \\
0 & 0 & 0 & H \\
0 & 0 & H & 0
\end{pmatrix}
```

```
\begin{pmatrix}
0 & 0 & 0 & h \\
0 & 0 & H & 0 \\
0 & H & 0 & 0 \\
h & 0 & 0 & 0
\end{pmatrix}
```
encoded logical qubit

the complete su(2) can be generated over subsets

\{ |110>, |101>, |011> \} and \{ |001>, |010>, |100> \}

of code spaces

\{ |000>, |110>, |101>, |011> \} and \{ |111>, |001>, |010>, |100> \}

→ the qubit can be represented by any pair of states from

one subset, e.g.,

\[ |0_L> = |110> \quad |1_L> = |011> \quad ... \]

\[ |0_L> = |001> \quad |1_L> = |100> \quad ... \]

TRIANGULAR LAYOUT ...
the full su(2) algebra over a single logical qubit is generated via the commutation relations between exchange interactions over physical qubits: e.g.

\[ [H_{13}, H_{23}] = i (J^2 - j^2) \mathbf{s}_{y,12} \]

and

\[ [H_{12}, \mathbf{s}_{y,12}] = i 2 J \mathbf{s}_{z,12} \]
Entangling two-qubit operation $C(Z)$ results from application of the encoded $s_z$ operation onto the physical qubits 2-3-4 in the triangular architecture, plus single-qubit operations.
Gate Sequences

Vala & Whaley, PRA 66, 022304 (2002)

For the Hamiltonian

$$H_{ij} = J (\sigma_{x,i} \sigma_{x,j} + \sigma_{y,i} \sigma_{y,j}) + j (\sigma_{x,i} \sigma_{x,j} - \sigma_{y,i} \sigma_{y,j})$$

commutation relations are applied through conjugation

$$U(\sigma_{y,12}, \phi) = \exp(-i \sigma_{y,12} \phi) = \exp(iH^{13}\Theta/2) \exp(iH^{23}\phi/2J) \exp(-iH^{13}\Theta/2)$$

with condition on the duration of conjugation:

$$\Theta = 0 \ (mod \ \pi)/j = (\pi/2) \ (mod \ \pi) / J$$

conjugation turns off the antisymmetric terms ($\sim j$)

a similar construction is valid for the general anisotropic interaction with cross-product terms
quantum circuit for a commutator

\[ U(\sigma^y_{12}, \phi) \]

\[ e^{-iH^{13} \Theta/2} \]

\[ e^{-iH^{23} \phi'/2} \]

\[ e^{iH^{13} \Theta/2} \]
Implementing generalized anisotropic exchange:

encode logical qubit into 3 physical qubits, e.g.  $|0_L\rangle = |110\rangle$  $|1_L\rangle = |011\rangle$

- one-qubit gates generated from application of exchange interactions and their commutators to physical qubits within a single logical qubit

- entangling two-qubit operation $C(Z)$ from application of encoded $s_z$ operation onto the physical qubits 2-3-4 in the triangular architecture, plus one-qubit operations
Scalable Array-Based Ion Trap
Quantum Computation - with protection

- ions stored in pairs (the encoded qubit)
- Sorensen-Molmer gates provide all single qubit rotations
- pairs of encoded qubits are moved into an interaction ion trap in order to perform two qubit operations
- the encoded states are also a DFS that protects the qubits from collective dephasing
- collective dephasing errors result from ambient magnetic field fluctuations and moving ions

Brown, Vala & Whaley
quant-ph/0207155
Sorensen-Molmer Entangling Operation

- 2 lasers are applied to 2 ions and the ions entangled by a virtual interaction with a common motional mode
- 1 laser blue detuned, 1 laser red detuned, by common frequency, $\Delta$
- changing the phase of each laser → perform two body Hamiltonian:

\[
H = \omega R(\theta)R(\phi)
\]

\[
R(\theta) = \cos(\theta)X + i\sin(\theta)Y
\]


$\omega = \eta^2 \Omega^2 / \Delta$

$\eta$ = Lamb Dicke parameter

$\Omega$ = single ion Rabi frequency
encoded 2-qubit operations: I

Kielpinski, Monroe, Wineland

• 2-qubit encoding $|0_L> = |01>$, $|1_L> = |10>$

• encoded 2-qubit operations performed using Sorensen-Molmer single physical 4-qubit entangling operation ($RR=RRRR$)

• experimental challenges: 4-ion entangling gates are less robust and slower than 2-ion entangling gates.
encoded 2-qubit operations: II

Brown, Vala, Whaley quant-ph/0270115

• XY encoding $|0_L\rangle = |010\rangle$, $|1_L\rangle = |100\rangle$; rapid alternation of polarization creates effective $H_{XY}$; universal on this encoding

• encoded 2-qubit operations performed using 5 physical 2-qubit operations (conjugation)

• experimental challenges: difficult to address individual ions, need to alternate pulses
Decoherence Minimization

\[ \mathcal{H}_{\text{rest}} \]

Sorensen-Molmer

\[ \exp(-i \sigma_x^i \sigma_x^j T) \]

\[ \exp(-i \sigma_y^i \sigma_y^j T/\hbar) \]

\[ \exp(-i \sigma_x^i \sigma_x^j T/\hbar) \]

\[ \exp(-i \sigma_y^i \sigma_y^j T) \]

\[ \mathcal{H}_{\text{DFS}} \]

Decoherence

\[ \sim T^3 / 3 n^2 \]

\( (n \gg T) \)

Fidelity
Is encoded universality always possible?

NO!

- non-interacting fermions (Valiant, Terhal&DiVincenzo, Knill ’01)
- nearest-neighbor XY-interaction
- linear optics quantum computation

Criterion: Kempe, Bacon et al. Qu. Comp. & Inf. 1, 33 (2001)

If a set of Hamiltonians (over n qubits) allows for (encoded) universal computation then the Lie algebra $L(H)$ contains exponentially many linearly independent elements

$$\text{some component } S_{n_k} \text{ has to contain } su(n_k) \text{ where } \log_2(n_k) \text{ is a polynomial function of } n$$

e.g., $H = \{\sigma_z^i, \sigma_x^i \sigma_x^{i+1}\}$ is not universal with any encoding
Summary of results:

I. we can do quantum computing (all required logic operations, i.e. universal QC) with only the exchange operation, via encoded universality

- general algebraic framework (\(00, 01\))
- universality of various exchange interactions proven (\(00-02\))
- encodings found (\(00-02\))
- gate-sequences for encoded gates found (\(00, 02\))
- robustness and fidelity of these gate-sequences studied (\(02\))

II. we can map exchange-based encoded universality to other physical systems
Conclusions

• Encoding into sub-spaces allows to make certain interactions universal

• Representation theory of Lie groups - powerful tool

• E and XY (symm, asymm, generalized) *alone* are universal - important simplification of physical implementations
References


Earlier related work on universal QC on DFS:


- tensor product of encoded qubits
  **conjoined codes**  Bacon et al., *PRL* 85, 1758 (2000)
  Bacon et al., quant-ph/0102140

- find entangling operations
  **Lie algebraic analysis**  Kempe et al., *PRA* 63, 042307 (2001)
  Kempe et al., JQIC (2001)

- efficient implementation
  **numerical search**  e.g. Heisenberg exchange
  serial coupling - 19 operations for CNOT, 4 operations for 1-qubit
  parallel coupling - 7 operations for CNOT, 3 operations for 1-qubit