Using Statistical State Dynamics to Study the Mechanism of Wall-Turbulence

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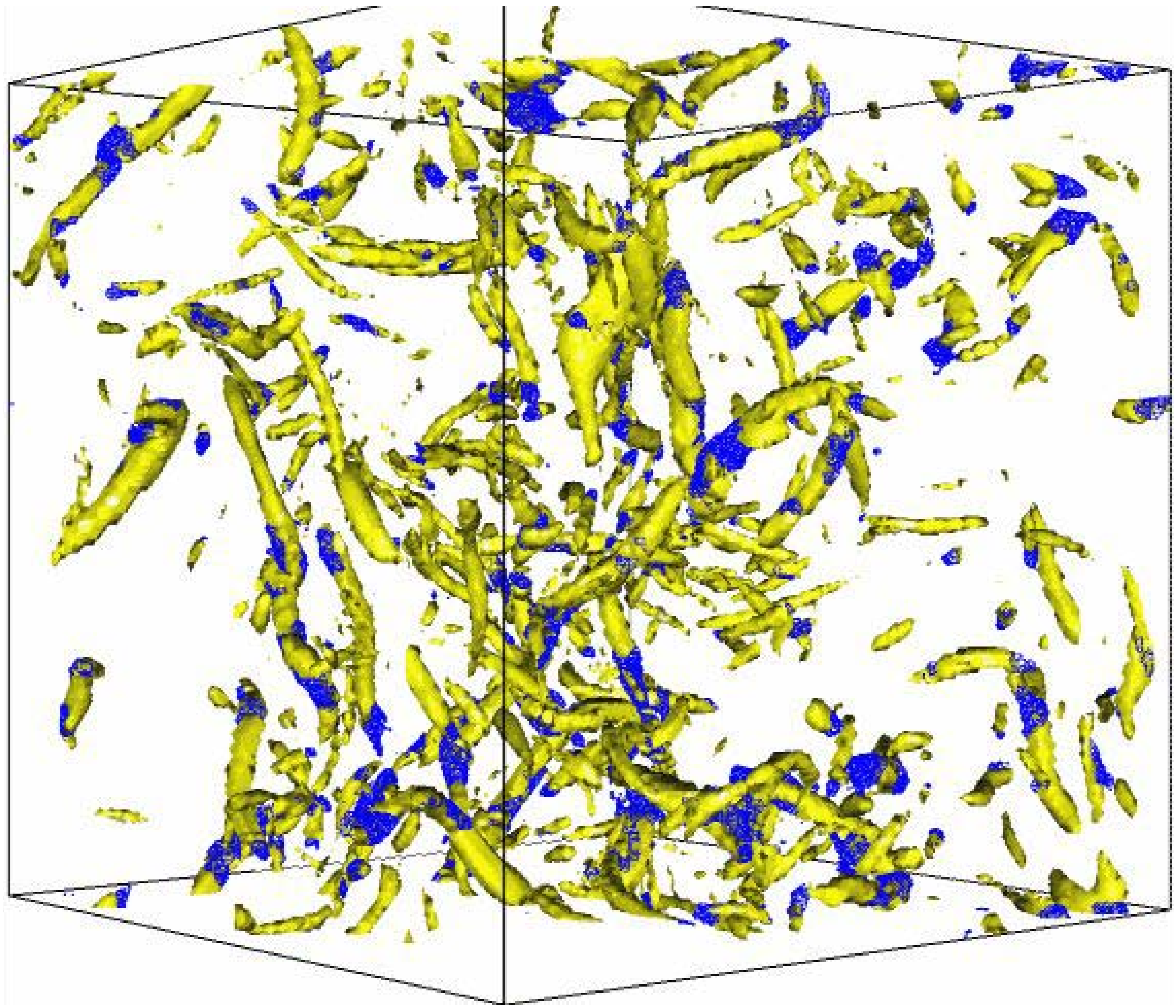
I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.

Horace Lamb (1932)

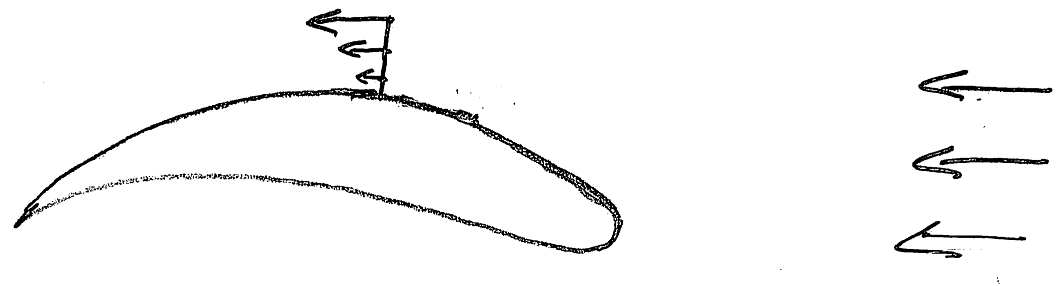
Isotropic Homogeneous Turbulence



Isosurfaces of the velocity gradient tensor

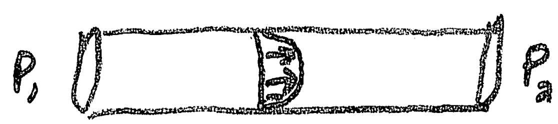
simulation by Andrew Ooi

Turbulence in Wall-Bounded Shear Flow

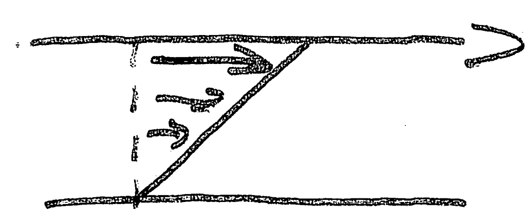


wings

boundary layers



pipes



channels

- Turbulence manifests in the dynamics of individual realizations but a comprehensive theory of turbulence has analytical expression only in statistical state dynamics (SSD).
- Chaos in the trajectory of a realization of a turbulent state is a familiar concept but the dynamics of turbulence is fundamentally associated with chaos not of the realization trajectory but of the trajectory of the underlying statistical state.

Fundamental Problems in Wall-Turbulence

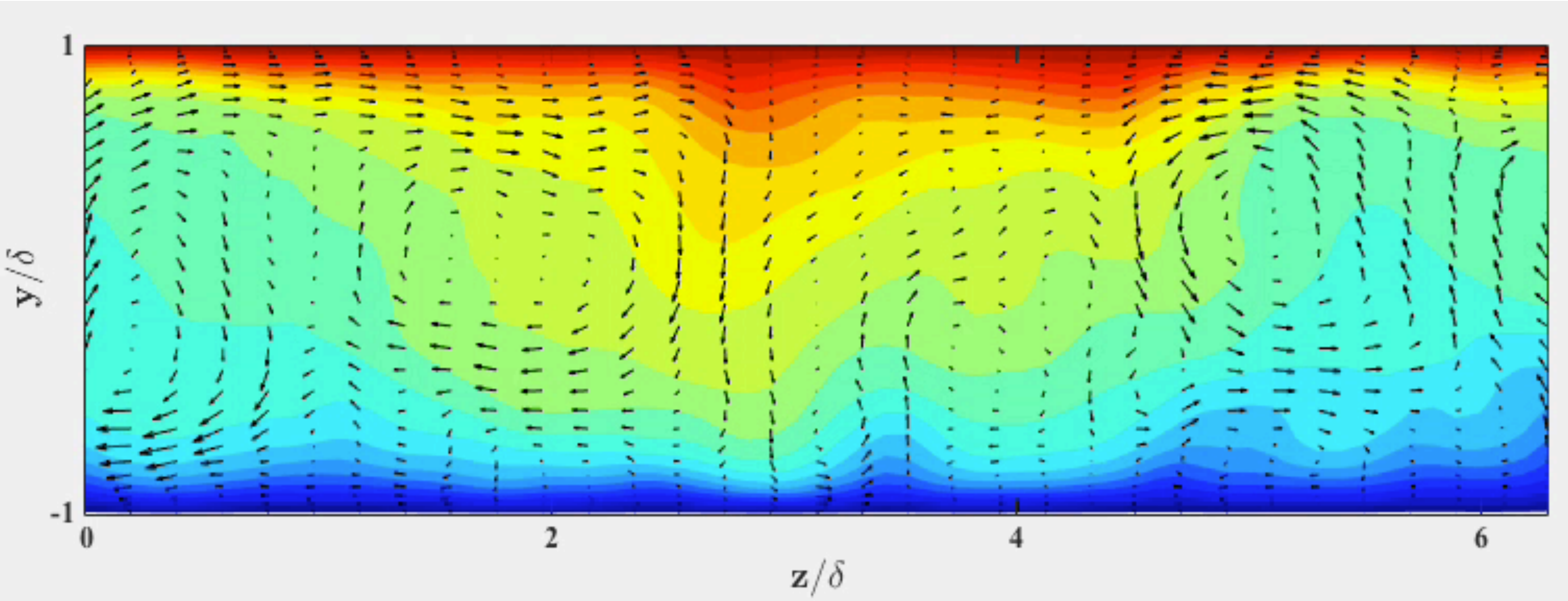
- Identify the mechanism producing transition to the turbulent state (transient growth..).
- Identify the state promoted to - that is identify the dynamical process producing a systematic transfer of energy from the forced shear flow to the turbulent perturbation field (SSP..). [Hamilton, Kim, Waleffe 95; Waleffe 97]
- Identify the mechanism regulating the SSP to produce the observed statistical steady turbulent state.

- Fast inflectional instabilities are not supported by wall-turbulence because the velocity profile has one sign of vorticity.

- However, fast transient growth of the roll/streak structure is supported.

- Evidence suggests that the turbulence is maintained by these optimally growing roll/streak structures.

- The maintenance question can be posed as identifying how the roll/streak structure is destabilized.



kx=0 component Lx= 4π, R=1000:

Simulations based on 'channelflow' code
[Peyret 2002, Gibson 2007]

- Given the large amplification available the Reynolds matrix can be destabilized by small feedbacks (Coriolis, centrifugal, Reynolds stresses).

- One example is destabilization by free stream turbulence.

- To understand the fundamental basis of this destabilization mechanism we will use a stochastic turbulence model which is the perturbation component of the RNL model.

The lift-up mechanism dynamics for shear S and dissipation rate λ is:

$$\begin{pmatrix} \dot{u} \\ v \end{pmatrix} = \begin{pmatrix} -\lambda & S \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

This stable solution exhibits large transient growth for S large and λ small:

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} e^{-\lambda t} & Ste^{-\lambda t} \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

One idea is to destabilize the Reynolds matrix with feedback to v from u:

$$\begin{pmatrix} \dot{u} \\ v \end{pmatrix} = \begin{pmatrix} -\lambda & S \\ \epsilon & -\lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \quad eigenvalues = -\lambda \pm (\epsilon S)^{1/2}$$

Navier-Stokes Equations in mean/perturbation form

$$\bar{\mathbf{u}}_{tot} = \mathbf{U} + \mathbf{u} = (U, V, W) + (u, v, w)$$

2δ : channel height

$\mathbf{U}$: Streamwise mean velocity

$\pm U_w$: wall velocities

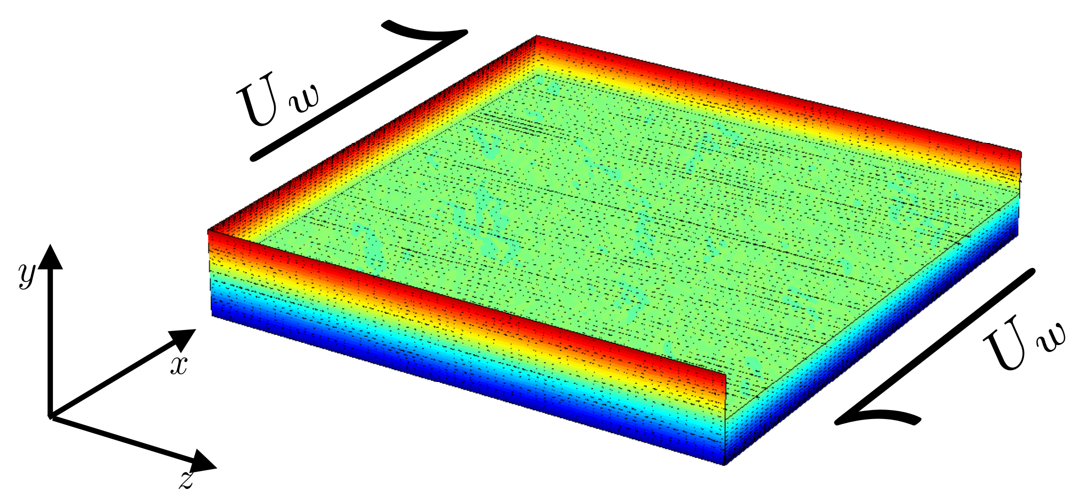
$\mathbf{u}$: Perturbation velocity

$$\mathrm{R} = \frac{U_w \delta}{\nu}$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U} = - < \mathbf{u} \cdot \nabla \mathbf{u} >$$

$$\partial_t \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \nabla \mathbf{p} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{u} = -(\mathbf{u} \cdot \nabla \mathbf{u} - < \mathbf{u} \cdot \nabla \mathbf{u} >)$$

$$\nabla \cdot \mathbf{U} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = \mathbf{0}$$



Restricted Nonlinear (RNL) Equations

- RNL is a SSD model closed at second order.
- It retains the full streamwise mean dynamics (first cumulant) and obtains the perturbation Reynolds stress from the perturbation covariance (second cumulant) $\mathbf{C}$:

$$\mathbf{U_t} + \mathbf{U}\cdot\nabla\mathbf{U} + \nabla\mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}}\boldsymbol{\Delta}\mathbf{U} = -<\mathbf{u}\cdot\nabla\mathbf{u}>= \mathbf{L}(\mathbf{C})$$

- The perturbation covariance may be approximated from a finite or infinite ensemble of perturbation equations sharing the same mean flow.
- The third cumulant (perturbation-perturbation nonlinearity in the perturbation equation) is parameterized.
- This parameterization may set the perturbation-perturbation nonlinearity to zero or use a temporally white but spatially correlated stochastic process (e.g. Leith (1996)).
- Here we use an infinite ensemble so the covariance solves the time dependent Lyapunov equation:

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}(\mathbf{A}(\mathbf{U}))^{\dagger} + \mathbf{Q}$$

In which $\mathbf{A}(\mathbf{U})$ is the matrix of the linearized perturbation dynamics and $\mathbf{Q}$ is the spatial correlation of the stochastic process parameterizing the perturbation nonlinearity.

$$\mathbf{U_t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \mathbf{\Delta U} = \mathbf{L}(\mathbf{C})$$

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}(\mathbf{A}(\mathbf{U}))^\dagger + \mathbf{Q}$$

- The RNL system is a SSD for the co-evolution of the state variables (U(t), C(t)).

- RNL system is deterministic, autonomous and nonlinear.

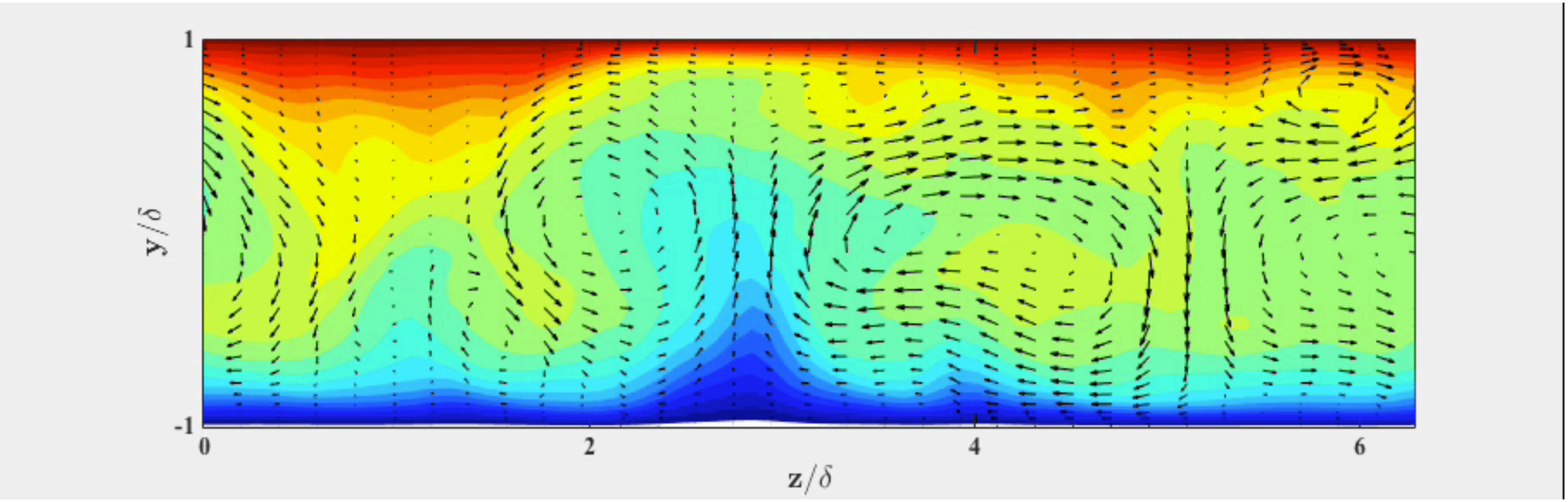
- Trajectories of (U(t), C(t)) may converge to a fixed point, a limit cycle or a chaotic attractor.

- Bifurcations can be explored by linear perturbation analysis of the fixed points in this system.

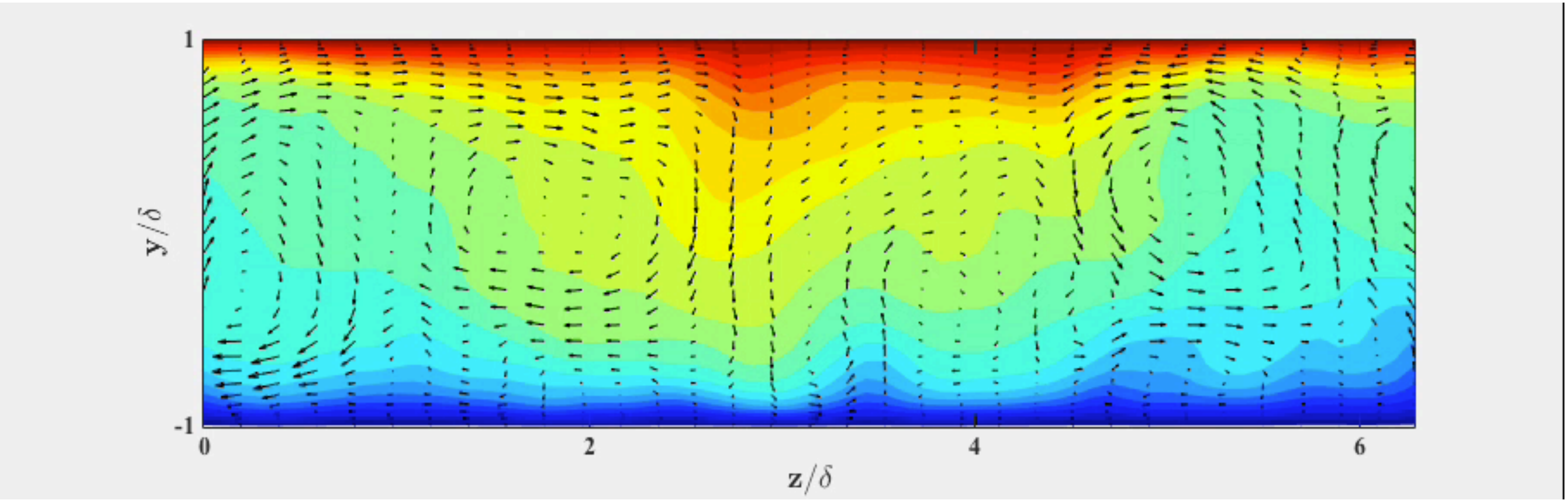
- Chaos in RNL corresponds to chaos not of a realization of turbulence but rather to chaos of a statistical state trajectory.

RNL supports turbulence similar to DNS

RNL



DNS



shown is kx=0 component Lx= 4π, R=1000:

Simulations based on 'channelflow' code

[Peyret 2002, Gibson 2007]

- Whereas DNS turbulence is complex and not well understood in contrast RNL turbulence is completely characterized.
- RNL turbulence is simple (rank1) while DNS turbulence is of high rank.
- This reduction in complexity is spontaneous and understood.
- The spontaneous reduction in complexity of the turbulence is accompanied by a natural reduction in the number of streamwise modes supporting the turbulence.

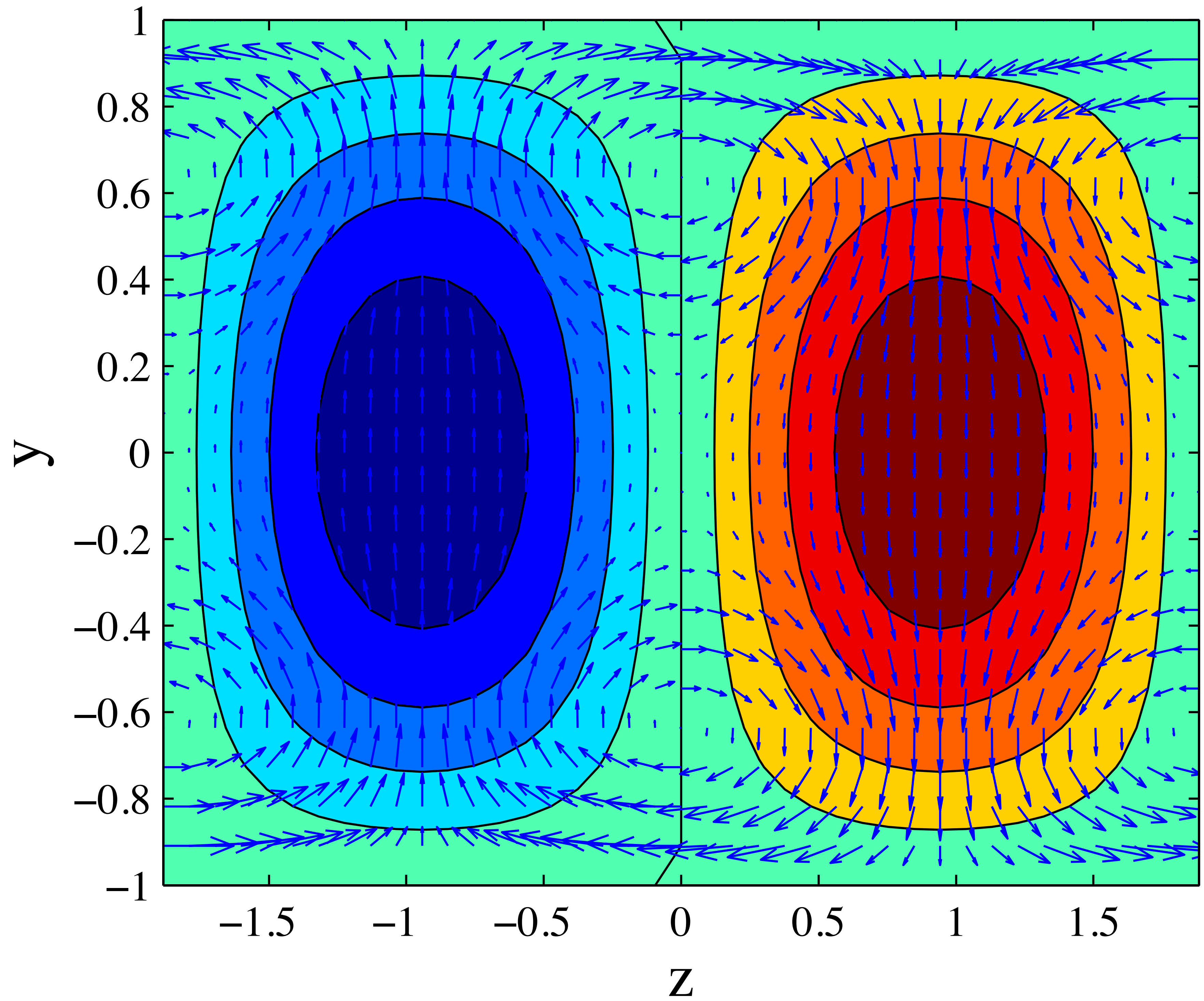
Restricted Nonlinear (RNL) Equations

$$\mathbf{U_t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla \mathbf{P} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U} = \mathbf{L}(\mathbf{C})$$

$$\mathbf{C_t} = \mathbf{A}(\mathbf{U})\mathbf{C} + \mathbf{C}(\mathbf{A}(\mathbf{U}))^{\dagger} + \mathbf{Q}$$

- By itself the second of these equations constitutes a stochastic turbulence model (STM) for the perturbations.
- We can exploit this STM to understand a fundamental mechanism of wall-turbulence dynamics.

Consider perturbing a stochastically maintained turbulence in Couette flow with a small streak and solving the STM for the forcing of the roll (V,W) that results.

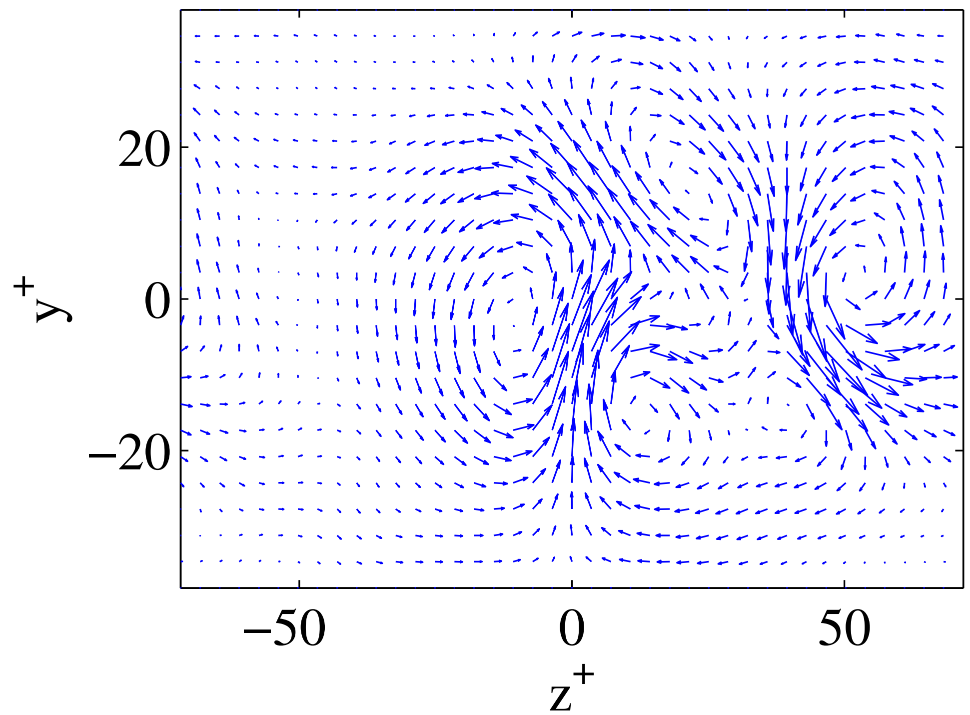


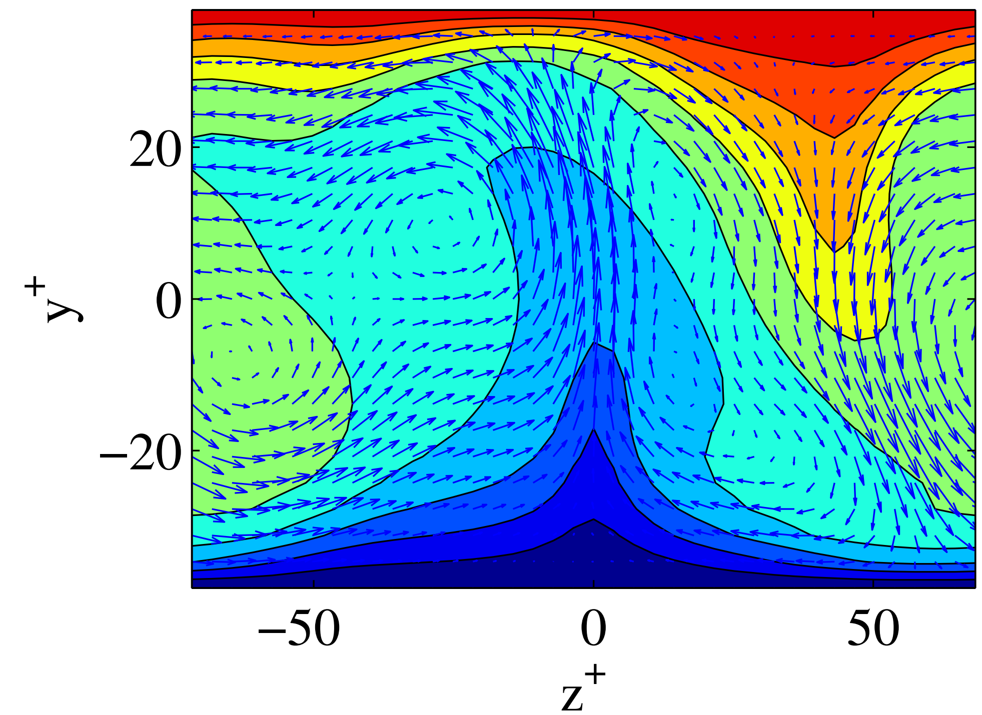
R=600

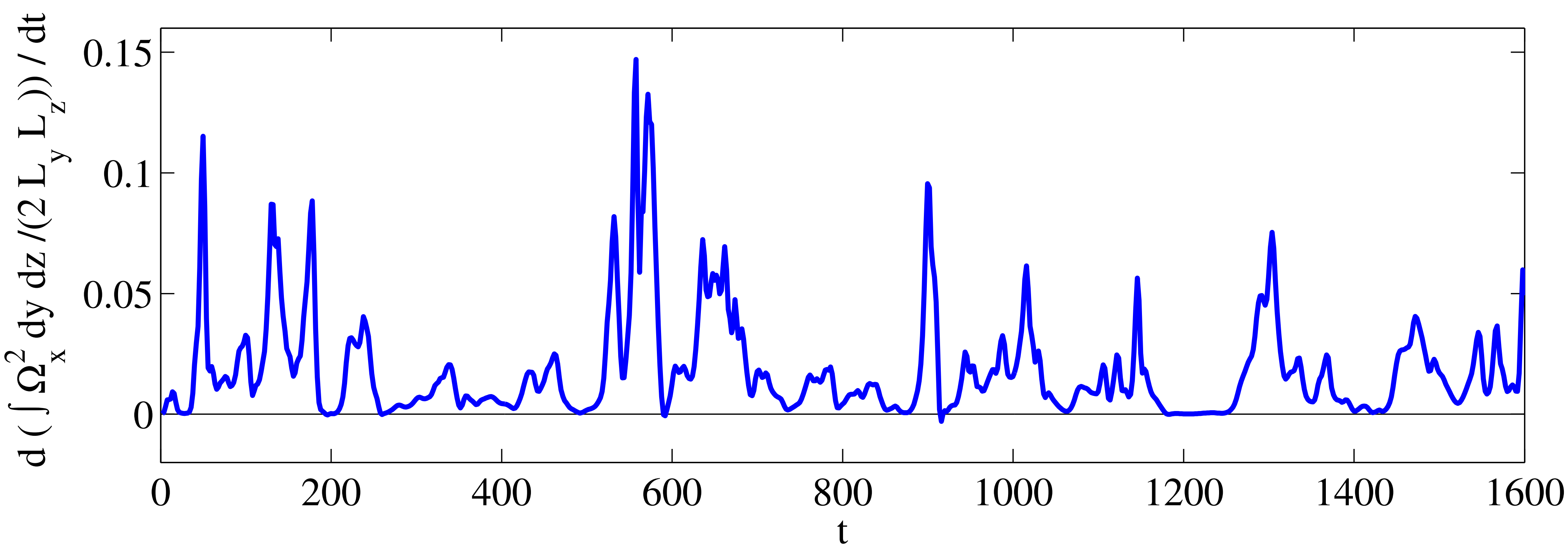
Reynolds stresses are organized by the imposed streak to produce lift up configured to amplify the imposed streak.

- Can linearize the RNL system about the Couette flow equilibrium and find the unstable eigenfunctions which are roll/streak structures that grow exponentially in free stream turbulence. These are intrinsically SSD instabilities.
- However, the interesting result for our purposes is not these eigenfunctions but rather the existence of a universal fast mechanism supporting the roll/streak structure.

Forcing of the streak by its organization of perturbation Reynolds stresses occurs on the advective time scale and underlies maintenance of the roll/streak structure in turbulent flows.







Review of Parametric Instability

The undamped harmonic oscillator in energy coordinates with restoring force perturbation ω' has dynamics:

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1+\omega') & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

Just as in the NS equations for turbulent flow the instantaneous growth rates and directions are eigenvalues and eigenvectors of the linearized dynamics $\frac{A+A'}{2}$ which in this case are $\pm\frac{\omega'}{2}$ and $(-1,1);(1,1)$ respectively.

As the solution vector rotates it grows when aligned along (-1,1) and decays when aligned along (1,1) averaging to zero net growth.

If the restoring force perturbation is applied as the solution vector passes (-1, 1) and removed as it passes (1, 1) the solution vector does not lose passing (1,1) its gain on passing (-1,1) and will be exponentially destabilized by this time dependent restoring force despite being stable at each instant.

Conceptually this is the mechanism maintaining turbulence (and the reason turbulence is necessarily time-dependent).

- This is the familiar mechanism of the Mathieu equation by which the time dependent harmonic oscillator is destabilized.

- This mechanism requires resonant forcing and it is not the mechanism producing parametric growth in turbulent boundary layers.

- The parametric growth mechanism in wall-turbulence is that of Oseledets (1968): it is the stochastic parametric mechanism that produces the unstable Lyapunov spectrum in random matrix dynamics.

- This mechanism depends on the convexity of the exponential propagator and can be understood by considering the stretching of a material line in a turbulent nondivergent fluid.

The stochastic parametric mechanism destabilizes (almost) any time-dependent dynamics. To see this consider the growth of a line segment in a nondivergent fluid with an imposed stochastically time-dependent velocity of hyperbolic form $\psi = -\alpha x y$ (locally at an instant of time)

Line segments stretch along the x-axis and contract along the y-axis:

$$\delta x(t) = \delta x(0) e^{\alpha t} = (1 + \alpha t + ..)$$
$$\delta y(t) = \delta y(0) e^{-\alpha t} = (1 - \alpha t + ..)$$

If the fluid velocity is delta correlated in time there is no growth but with finite correlation time the growth is exponential.

- The compelling similarity of RNL and NS turbulence and the great simplicity of RNL dynamics motivates closer study of the mechanisms underlying RNL turbulence.

- The method we adopt is to synchronize two RNL systems so that the perturbation dynamics can be studied in isolation.

Consider an RNL turbulence self-sustaining without stochastic forcing (Q=0):

$$\partial_t \mathbf{U_a} + \mathbf{U_a} \cdot \nabla \mathbf{U_a} + \nabla \mathbf{P_a} - \frac{\mathbf{1}}{\mathbf{R}} \boldsymbol{\Delta} \mathbf{U_a} = \mathbf{L}(\mathbf{C_a})$$

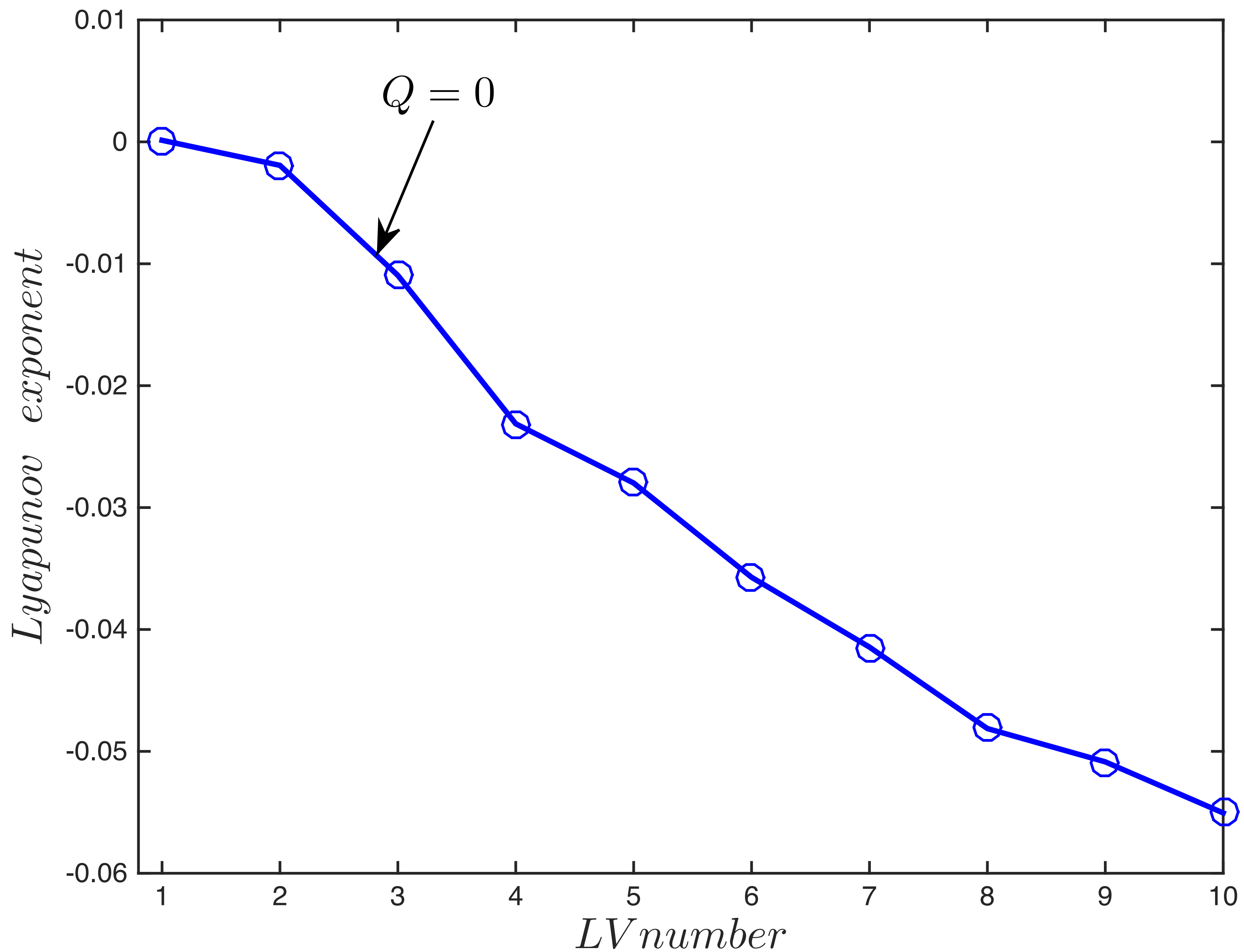
$$\partial_t \mathbf{C_a} = \mathbf{A}(\mathbf{U_a})\mathbf{C_a} + \mathbf{C_a}(\mathbf{A}(\mathbf{U_a}))^\dagger$$

Now impose the streak alone from this turbulence on the perturbations dynamics of a second RNL system initialized with a random full rank covariance:

$$\partial_t \mathbf{C_b} = \mathbf{A}(\mathbf{U_a})\mathbf{C_b} + \mathbf{C_b}(\mathbf{A}(\mathbf{U_a}))^\dagger$$

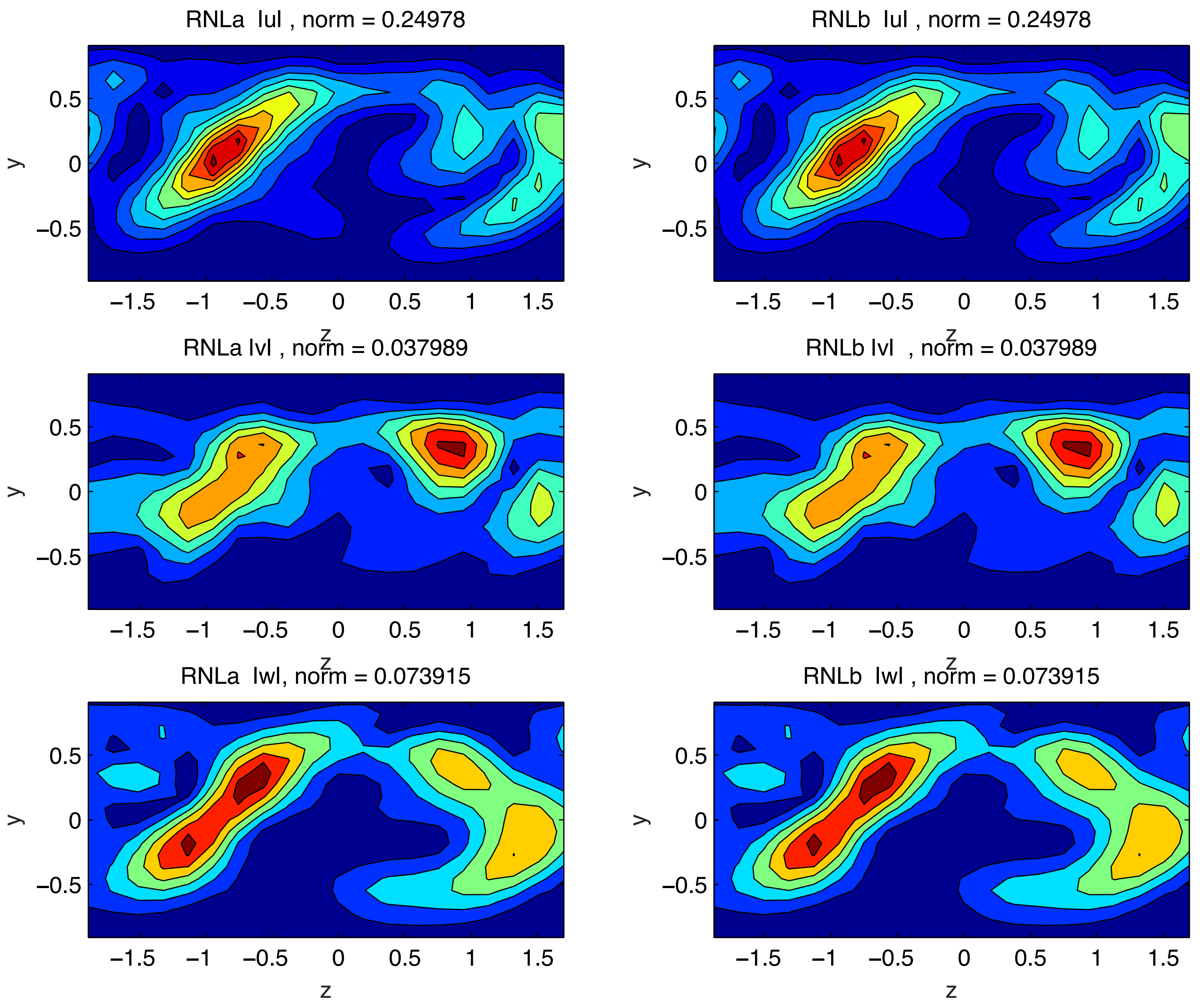
- Examination of the perturbation dynamics reveals a linear essentially stochastic time-dependent system so the asymptotic structure of the perturbation field is the first Lyapunov vector.
- Given that the Lyapunov vector is a component of the state trajectory the associated Lyapunov exponent is necessarily zero.

Synchronized system perturbation field converges to the first Lyapunov vector of the primary system.



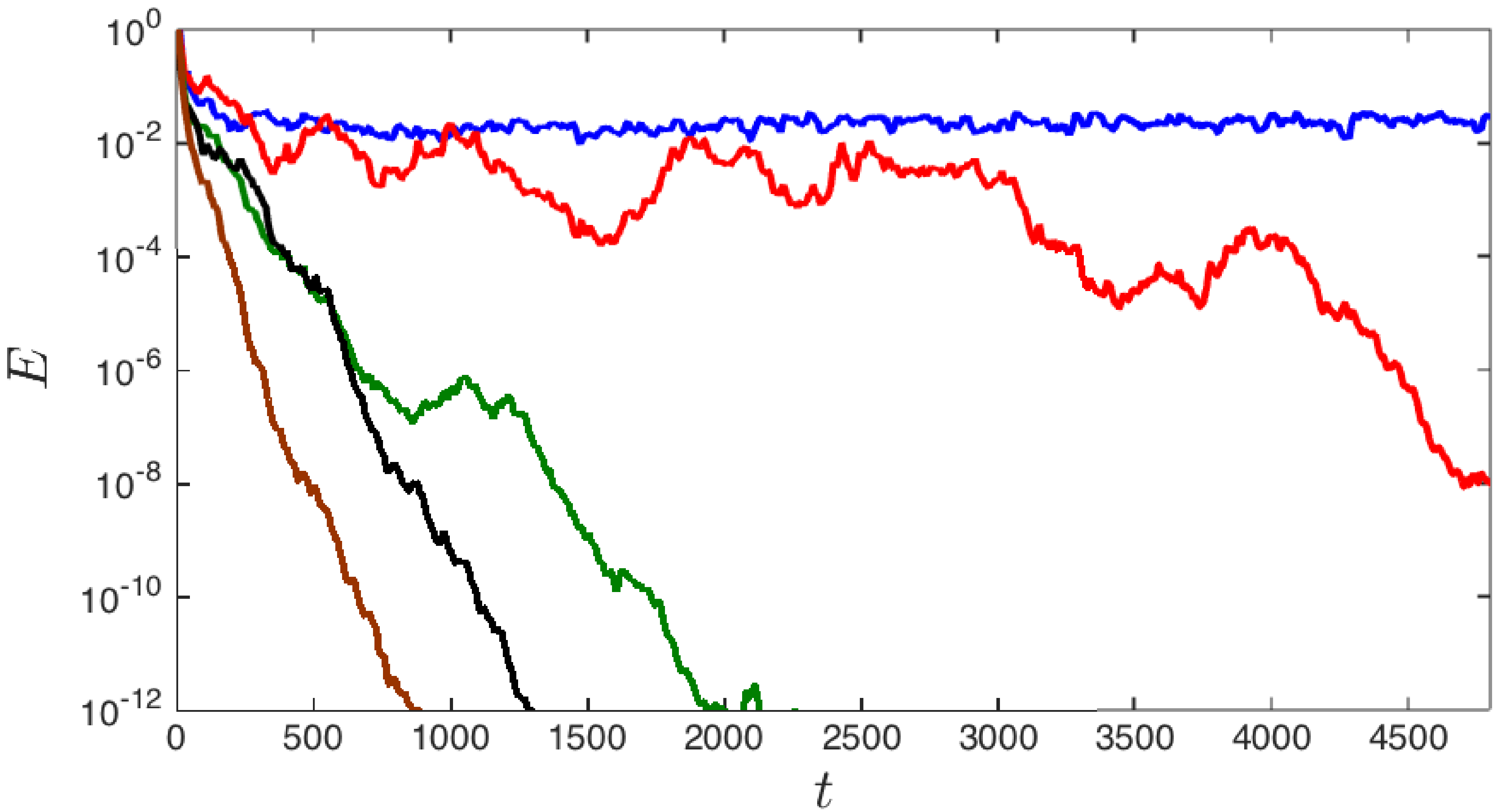
The channel has length Lx = 1.75, width Lz = 1.2 and R = 600; the single streamwise wavenumber k=2 pi/Lx is retained by the dynamics of LV1

Synchronized system perturbation field converges to the first Lyapunov vector of the primary system which supports only a single streamwise mode greatly reducing the complexity.



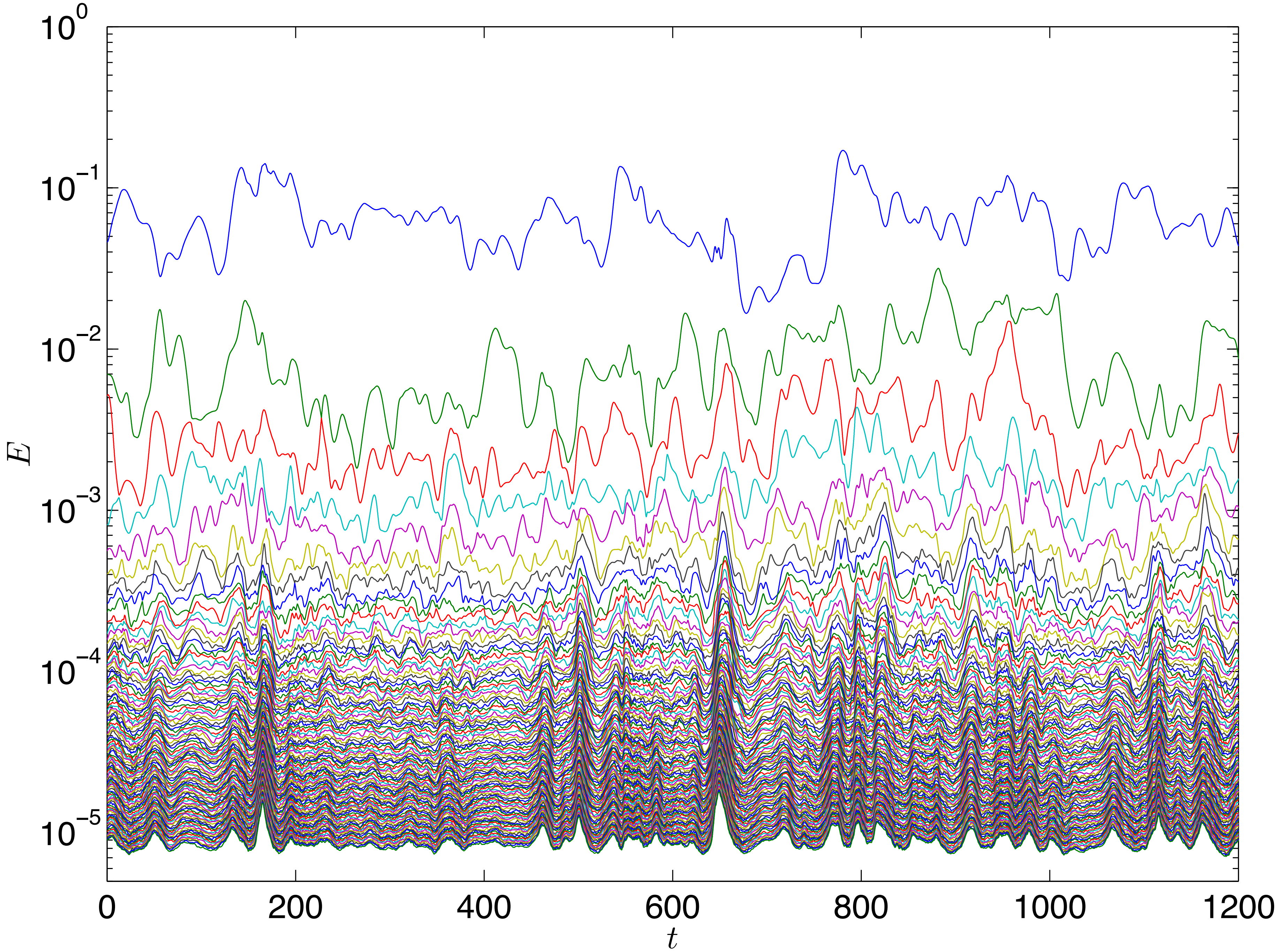
The channel has length Lx = 1.75, width Lz = 1.2 and R = 600; the single streamwise wavenumber k=2 pi/Lx is retained by the dynamics.

Stochastic forcing of the synchronized dynamics reveals the existence of both the trajectory Lyapunov vector (active subspace) as well as the other Lyapunov vectors with negative exponents (passive subspace).

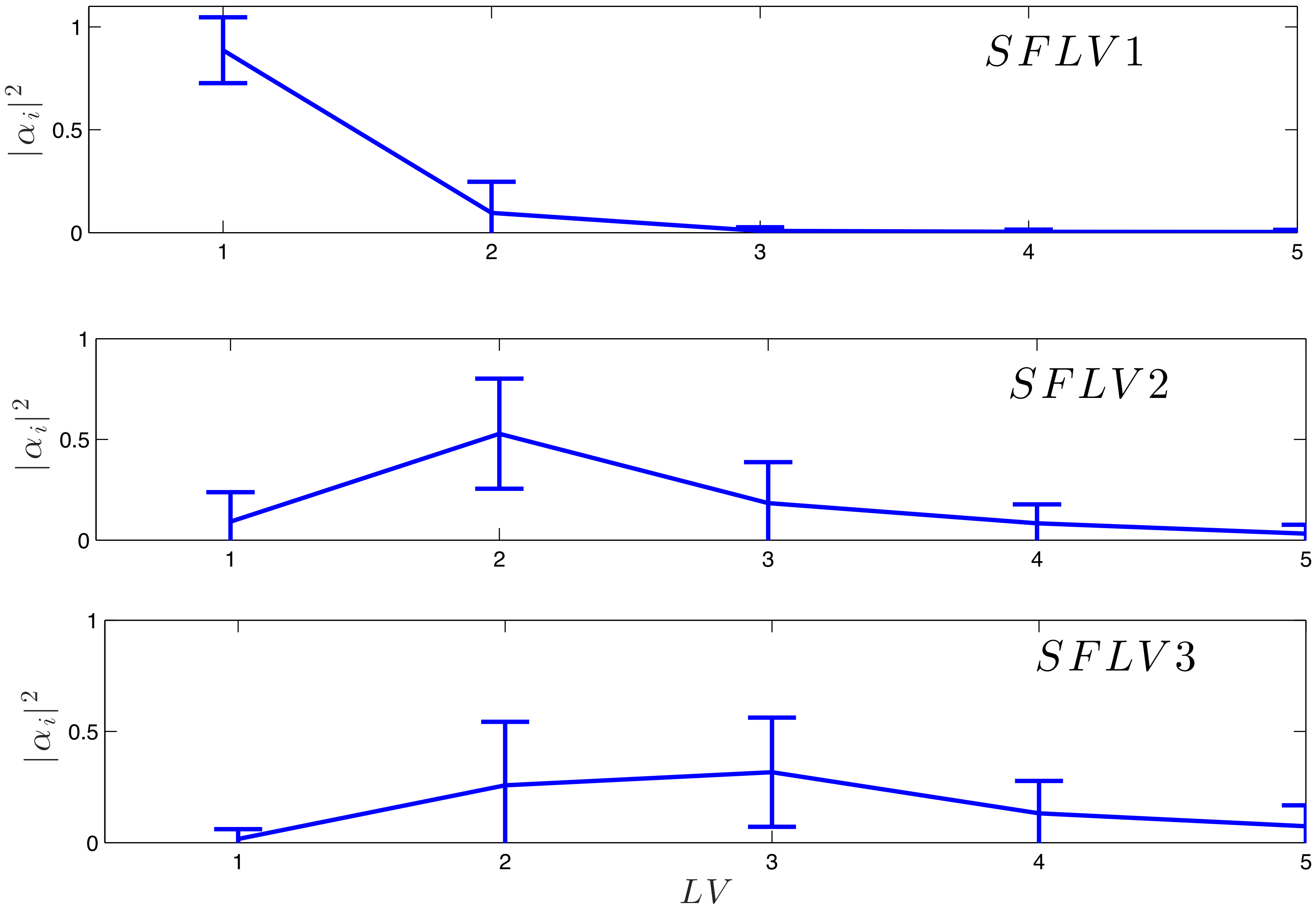


- Inserting the stochastic parameterization for the perturbation-perturbation nonlinearity reveals the entire set of Lyapunov vectors of the passive subspace.

- These structures are maintained parametrically by their interaction with the fluctuating mean flow.

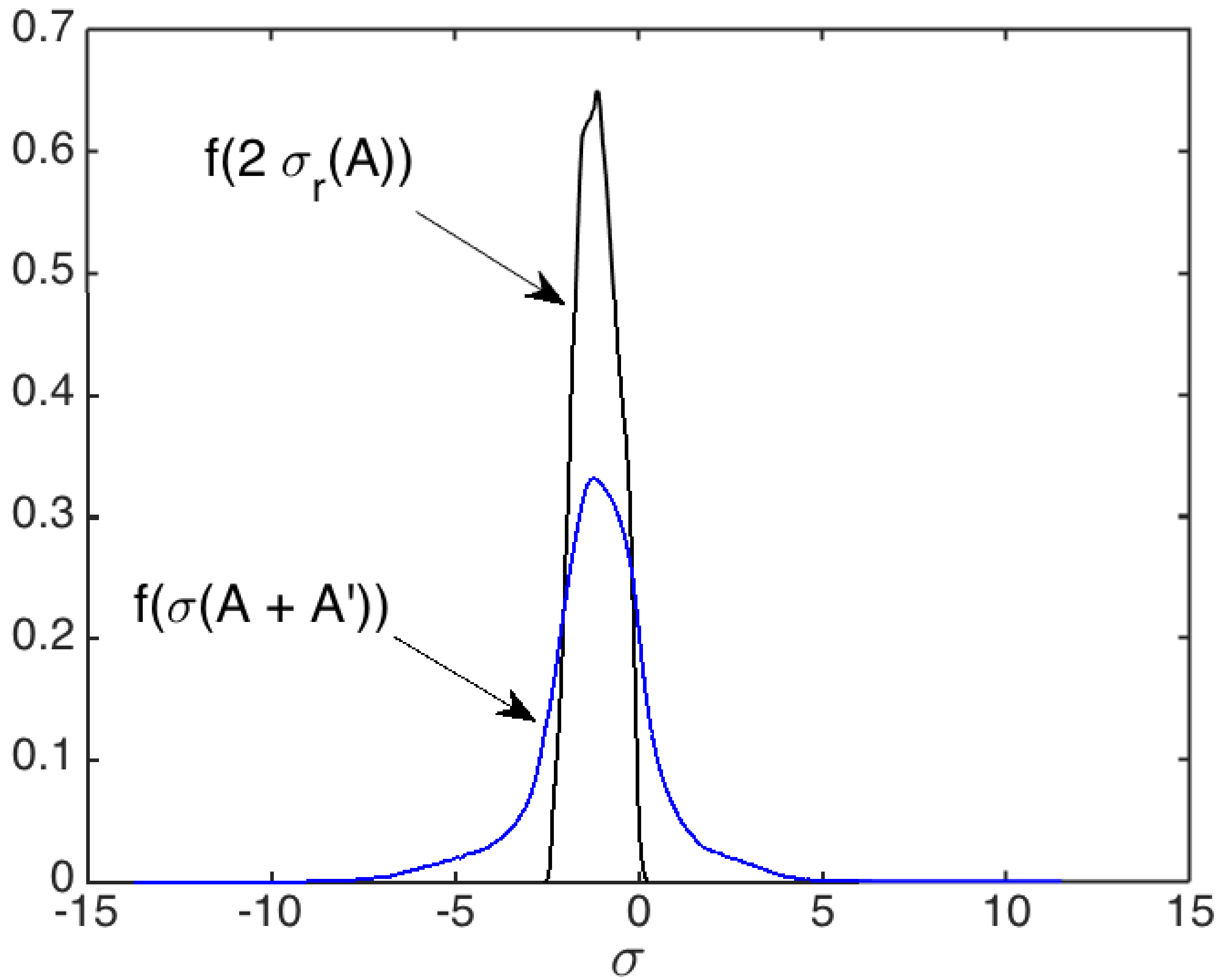


Projection on the Lyapunov vectors of the Q=0 flow shows strong support of the perturbation variance by the LV's

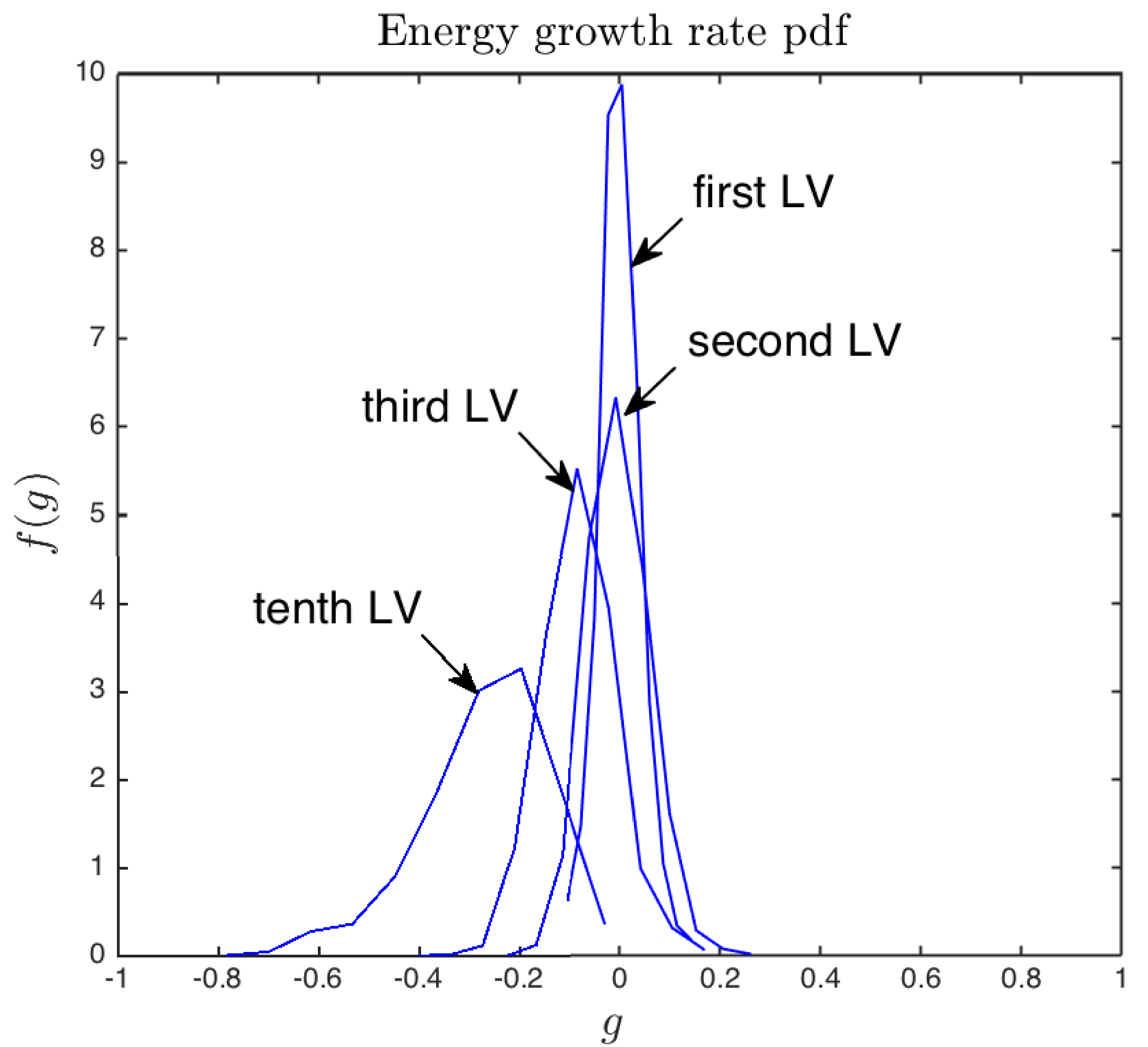


- We wish to examine the mechanism by which LV1 is maintained and regulated.
- Method is to diagnose the synchronized system dynamics.

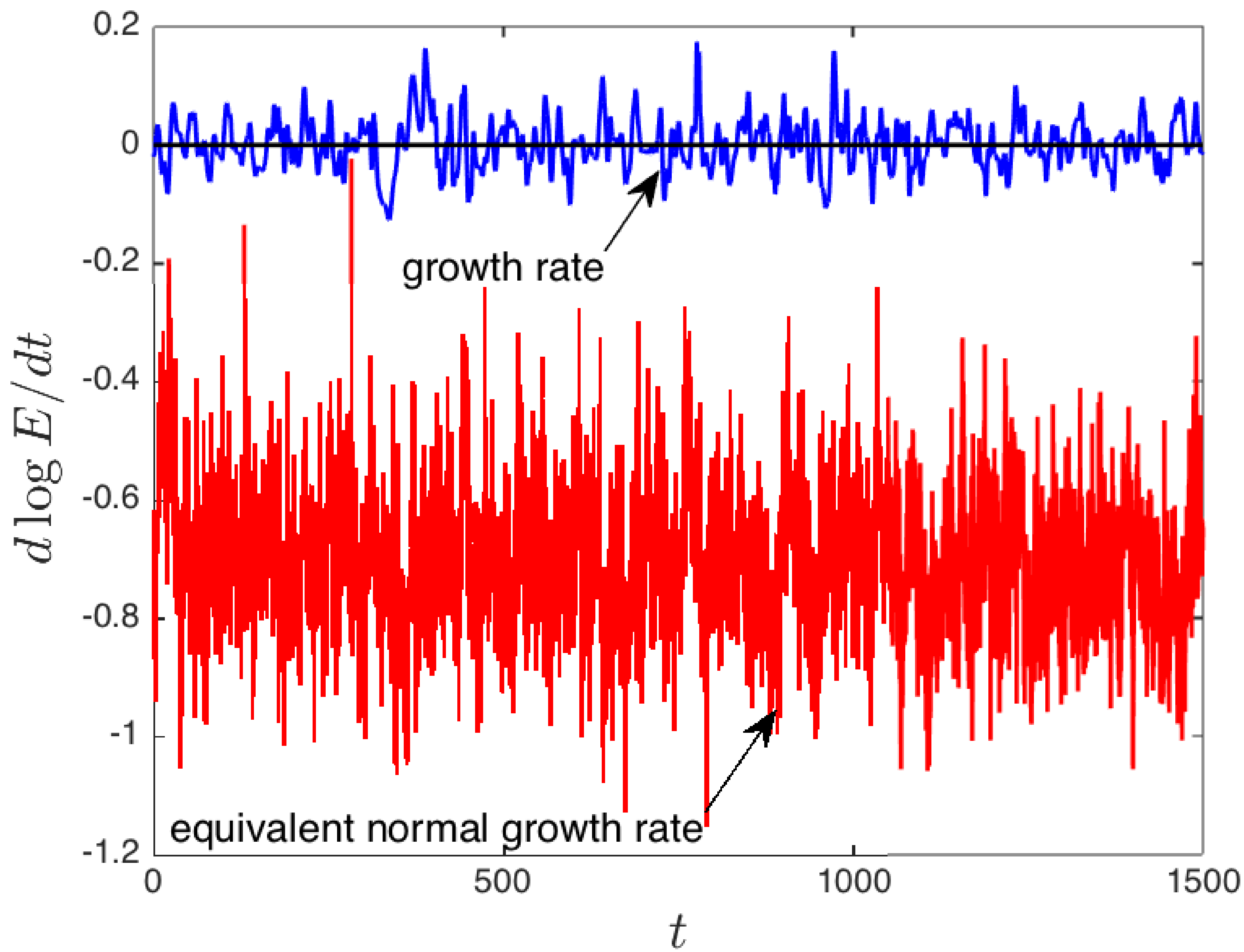
- Instantaneous growth rate possible for a perturbation (t=[0,5000]).
- Note instability boundary.



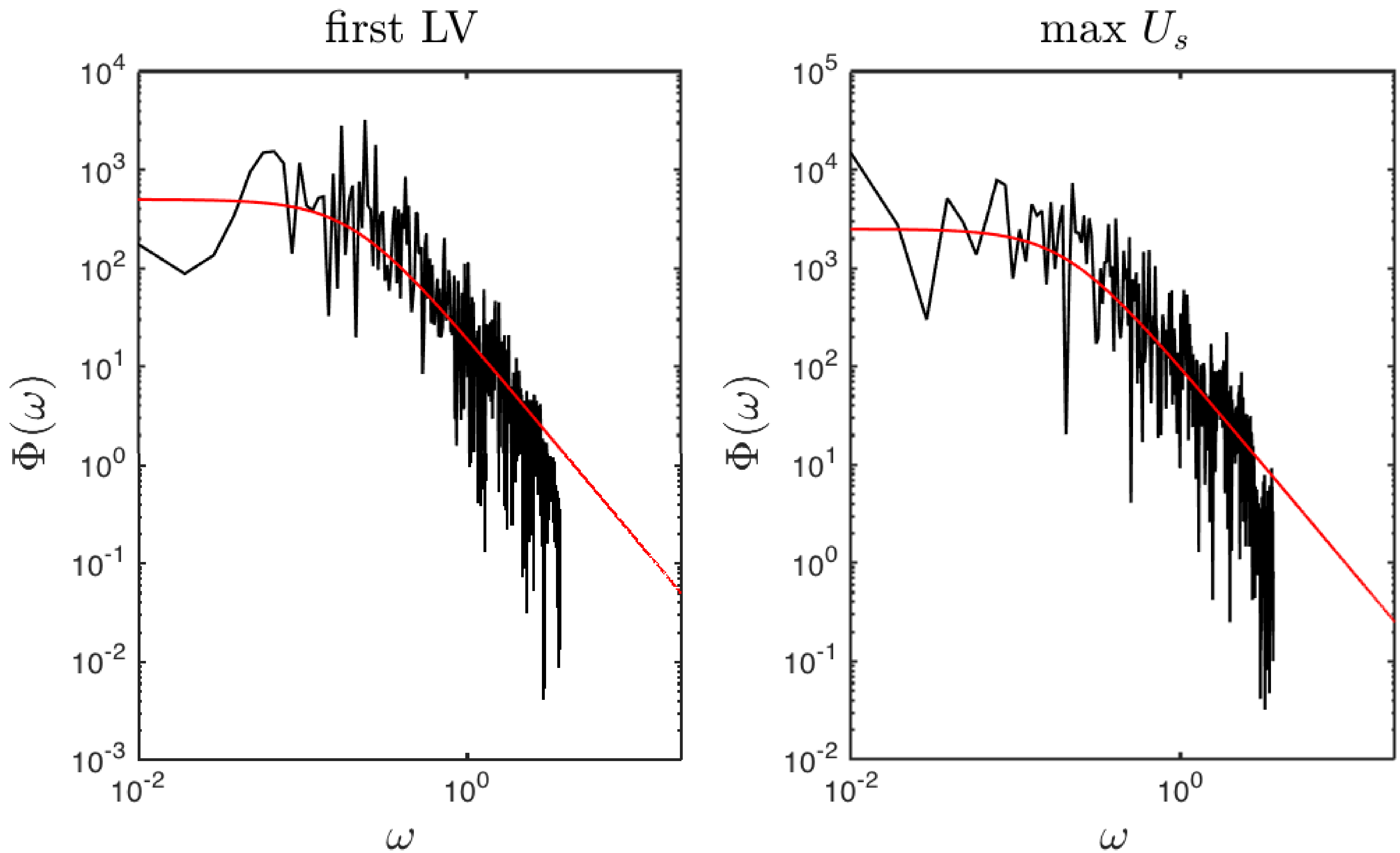
Instantaneous growth rate achieved by Lyapunov state vectors (t=[0,5000]).



Partition of the instantaneous growth rate of the LV1 into modal and non-modal sources.

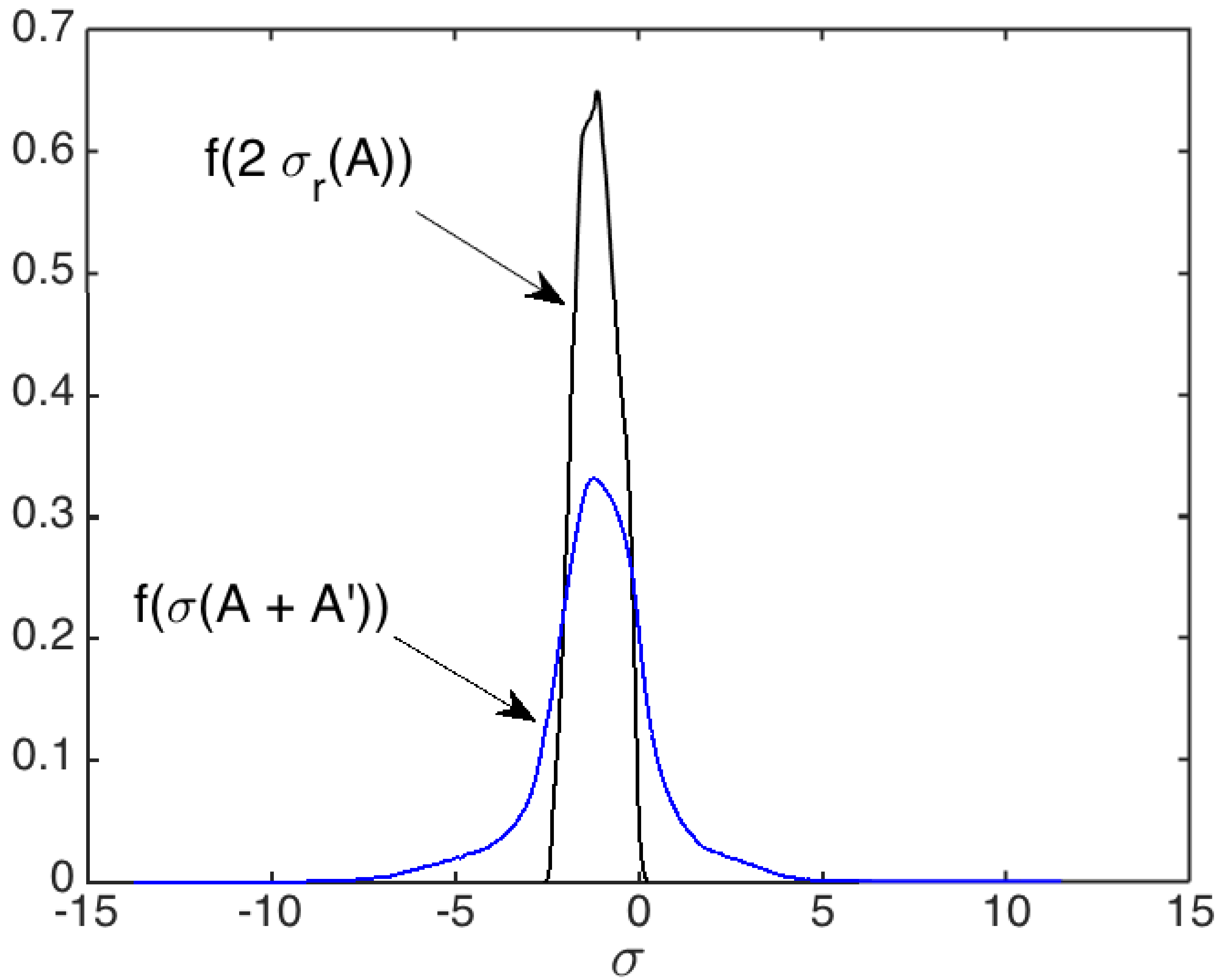


Spectra of the instantaneous growth rate of the perturbation state vector (LV1) and the mean streak reveal identical red noise processes.

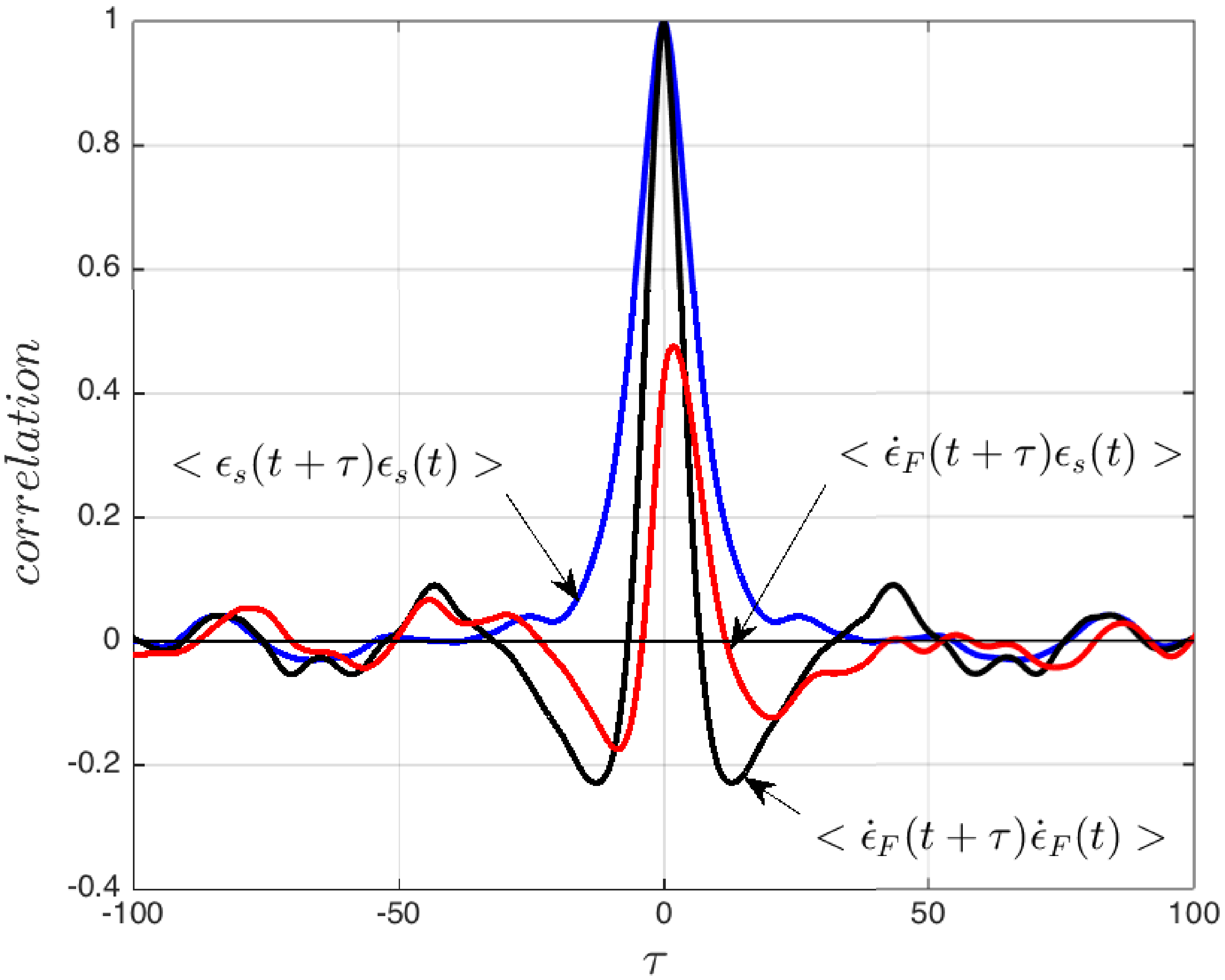


Turn now to the question of how the system is regulated to maintain a statistical steady turbulent state.

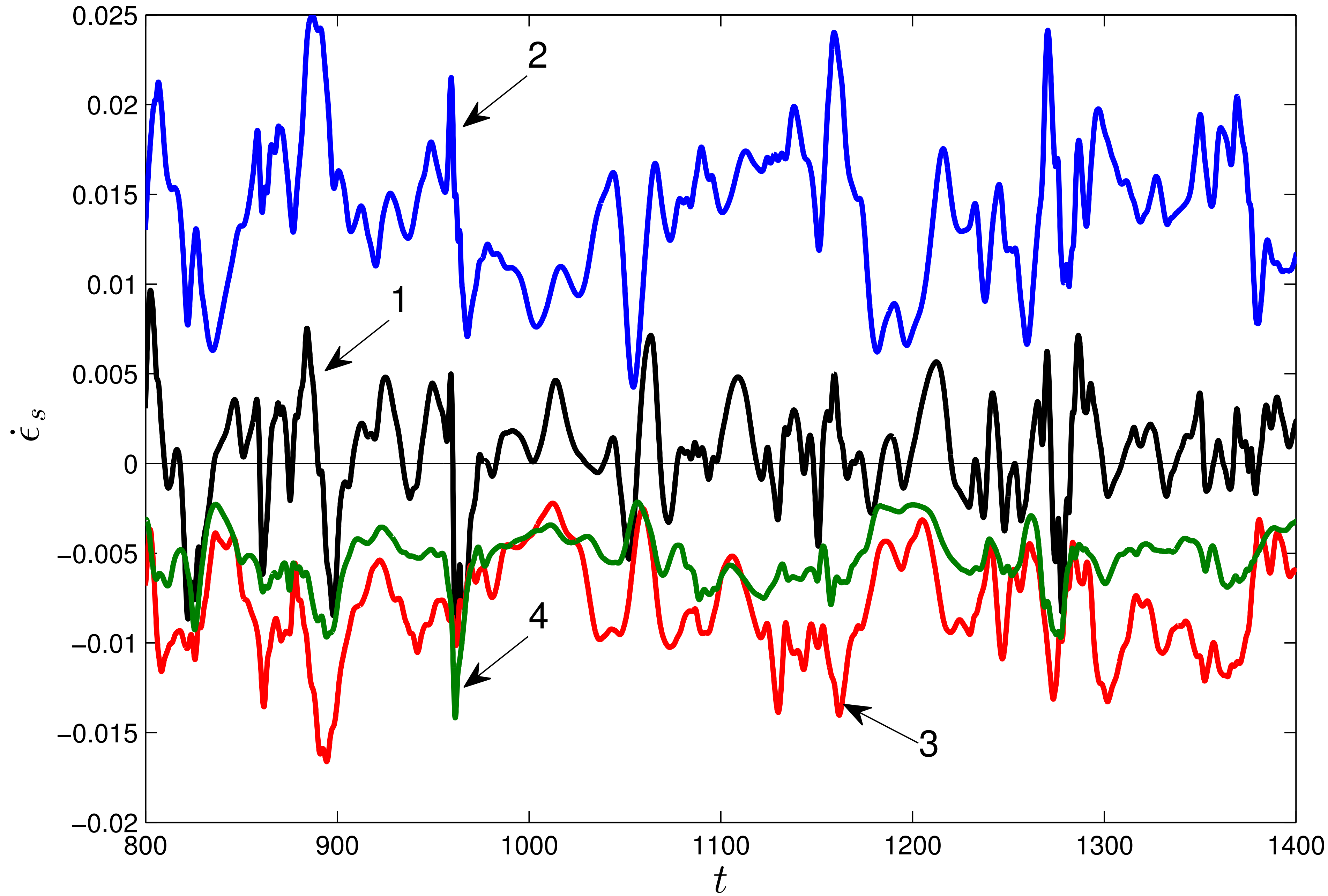
- Instantaneous growth rate possible for a perturbation (t=[5000]).
- Note instability boundary.



Auto and cross-correlation of streak amplitude and Reynolds stress damping reveals a time scale far shorter than that of the instability -> regulation of the streak occurs on the advective time scale.



The statistical state is regulated by a rapid perturbation Reynolds stress mediated feedback associated with streak inflection which is in balance with roll-induced lift-up. The mechanism of this regulation is adjoint mode growth which occurs on the advective time scale.



streak growth rate (1), lift-up (2), Reynolds stress (3), damping (4)

Conclusions

- SSD provides a powerful tool for studying the dynamics of turbulence.
- RNL model is a second order SSD model that maintains highly realistic turbulence.
- The dynamics of the RNL system are directly connected to NS dynamics.
- The RNL dynamics are naturally minimal.
- The RNL dynamics are completely characterized analytically.
- RNL turbulence is maintained by the stochastic parametric growth mechanism which is a universal property of time-dependent dynamical systems.
- RNL turbulence is regulated by adjoint mode growth on the advective time scale.