Towards low-order models of turbulence

Turbulence in Engineering Applications
Long Programme in Mathematics of Turbulence

IPAM, UCLA

Ati Sharma[*] Beverley McKeon[+]

Rashad Moarref[+] Paco Gomez[x] Ashley Willis[#]
Hugh Blackburn[x] Murray Rudman[x]

[*]University of Southampton
[+]California Institute of Technology
[x]Monash
[#]University of Sheffield

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Thanks to EOARD, AFOSR, ARC

Aim

Provide a set of **basis functions**, **derived from equations**, that economically represent wall turbulence and can be used to **make predictions** about it.

Fidelity is not required, but we would like graceful degradation.

Are low order models possible?

- Navier-Stokes equations are infinite-dimensional
- but turbulence has ‘structure’
- exact solutions / coherent structures should also be captured, if they continue into turbulent *Re*
- So turbulence should admit more formal (rather than phenomenological) reduced order models that describe structure

Overview

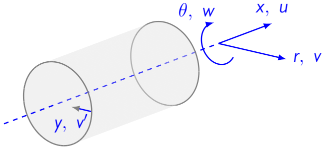
- Model formulation
- Sparsity of frequencies in turbulent DNS
- Scaling of resolvent modes, triads
- Representation of TW solutions

NSE in Fourier domain

Fourier modes for velocity in all (three) homogeneous directions

$$\mathbf{u}(r; x, \theta, t) = \sum_{k,n,\omega} \mathbf{u}_{\mathbf{k}}(r) e^{i(kx+\theta n-\omega t)}$$

$$\mathbf{u}_{\mathbf{k}}(r) := \hat{\mathbf{u}}(r; k, n, \omega)$$



subtract out steady-state / $\mathbf{u}_0$ / mean / $(k, n, \omega) = (0, 0, 0)$ and substitute for nonlinear term

$$\mathbf{f}(r; x, \theta, t) := -\left(\mathbf{u} - \mathbf{u}_0(r)\right) \cdot \nabla \left(\mathbf{u} - \mathbf{u}_0(r)\right)$$

$$\mathbf{f}(r; x, \theta, t) := \sum_{k,n,\omega} \mathbf{f}_{\mathbf{k}}(r) e^{i(kx+\theta n-\omega t)}$$

Steady-state equation

The equation for $(k, n, \omega) = (0, 0, 0)$ gives the spatio-temporal mean

Re stress gradients supporting mean

$$0 = \boxed{\mathbf{f}_0(r)} - \mathbf{u}_0(r) \cdot \nabla \mathbf{u}_0(r) + Re^{-1} \nabla^2 \boxed{\mathbf{u}_0(r)}$$

time-space ave velocity

$\mathbf{u_k}$ requires $\mathbf{u}_0$ requires $\mathbf{u_k}$... just assume $\mathbf{u}_0$

Equation for fluctuations at (k, n, ω)

$$
\begin{aligned}
-i\omega \mathbf{u}_{\mathbf{k}}(r) = & -\mathbf{u}_0(r) \cdot \nabla \mathbf{u}_{\mathbf{k}}(r) \\
& -\mathbf{u}_{\mathbf{k}}(r) \cdot \nabla \mathbf{u}_0(r) \\
& + Re^{-1} \nabla^2 \mathbf{u}_{\mathbf{k}}(r) - \nabla p_{\mathbf{k}}(r) \\
& + \mathbf{f}_{\mathbf{k}}(r) \\
\\
= & \, \mathrm{L}_{\mathbf{k}}(\mathbf{u}_0)\mathbf{u}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}}(r)
\end{aligned}
$$

Equation for fluctuations at (k, n, ω)

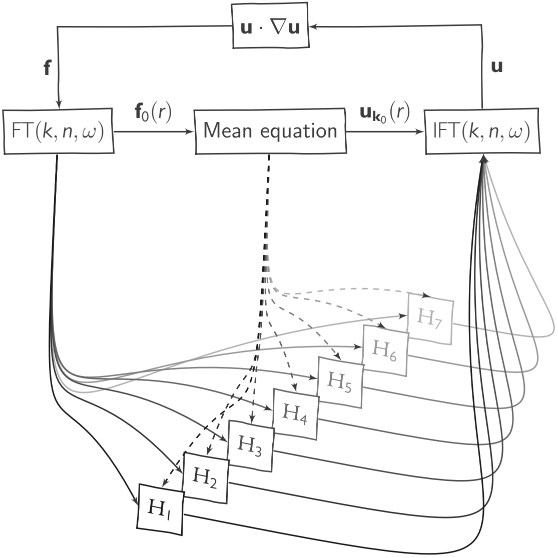
interaction between scales

$$\mathbf{u}_{\mathbf{k}}(r) = \boxed{\mathrm{H}_{\mathbf{k}}}\ \boxed{\mathbf{f}_{\mathbf{k}}(r)}$$

linear resolvent

The resolvent $\mathrm{H}_{\mathbf{k}} = (i\omega - \mathrm{L}_{\mathbf{k}})^{-1}$ is the *frequency-domain* (so travelling waves) transfer function from nonlinear interaction between scales to velocity field

NSE as a network of resolvents



$\mathbf{f_k}(r) \longrightarrow \boxed{\mathrm{H}_\mathbf{k}} \longrightarrow \mathbf{u_k}(r)$

Resolvent

UNIVERSITY OF Southampton

$$\mathrm{H}_\mathbf{k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

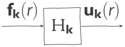
$D = \partial_r^2 + r^{-1}\partial_r - r^{-2}(n^2+1) - k^2$, states are (u_r, u_θ, u_x)

- loss of translational symmetry in r causes non-normality / source of energy
- as $c = \omega/k \to \mathbf{u}_0$ and $Re \to \infty$, $\quad \|\mathrm{H}_\mathbf{k}\| \to \infty$
- Energetic response at critical layer becomes very important at high Re.
- response becomes more localised around critical layer

McKeon & Sharma JFM 2010

Some points on interpretation / FAQ

- Valid for large fluctuations
- Clear interpretation for linear operators formed using the *mean* profile (not perturbation analysis around $\dot{\mathbf{u}} = 0$)
- Eddy viscosity not needed (Re stress model $\sim \mathbf{f_k}$)
- Haven't used scale separation arguments as in RDT
- Focuses discussion on forcing model/interpretation (c.f. earlier frequency-domain works by Jovanovic; Bamieh)
- Time (ω) on equal footing with homogeneous spatial directions (k, n)
- Importance of critical layer is clear

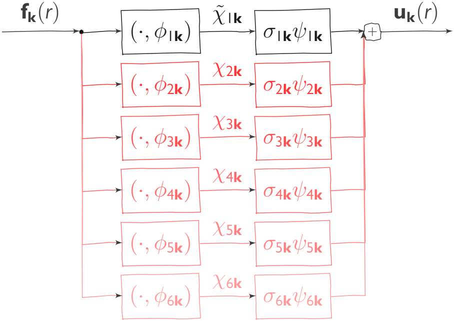


Approximating a single resolvent

SVD approximates an operator by directions of principal gain

$$\mathrm{H}_\mathbf{k} = \sum_{m=1}^{\infty} \psi_{m\mathbf{k}}(r)\sigma_{m\mathbf{k}}\phi^*_{m\mathbf{k}}(r)$$

Each σ_m is a (real) gain, σ_1 is the maximum gain. Velocity field response is $\psi_{m\mathbf{k}}(r)$.



$\mathbf{f}_{\mathbf{k}}(r) \rightarrow \boxed{\mathrm{H}_{\mathbf{k}}} \rightarrow \mathbf{u}_{\mathbf{k}}(r)$

Approximation of $\mathrm{H}_{\mathbf{k}}$ (by gain)

- Since often $\sigma_1 \gg \sigma_2$, often reasonable to approximate $\mathbf{u}_{\mathbf{k}}$ by leading $\psi_{1\mathbf{k}}(r)$
- This gives *radial form (or structure)* of velocity field at $\mathbf{k}$
- Reduces NSE solution to a weighted sum of response modes

$$\mathbf{u}(x, \theta, r, t) \approx \sum_{\mathbf{k},m} \chi_{m\mathbf{k}} \sigma_{m\mathbf{k}} \psi_{m\mathbf{k}}(r) e^{i(kx+n\theta-\omega t)}$$

- Using $m = 1$ or $m < M$ is great simplification, but remains to find coefficients $\chi_{m\mathbf{k}}$

Finding coefficients

- Coefficients are fixed by $\mathbf{f_k}$ in a quadratic equation

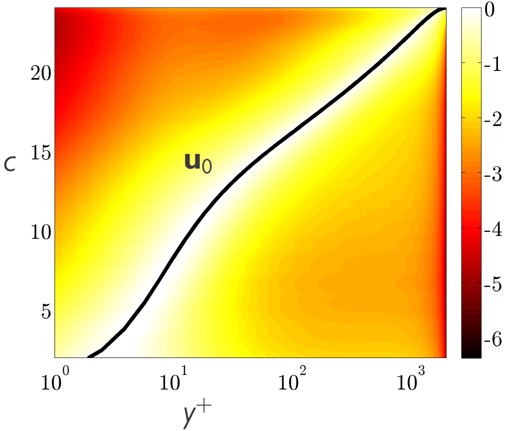
$$\chi_j = \sum_{ab} \sigma_j N_{jab} \chi_a \chi_b, \quad \text{where } N_{jab} = (-\psi_a \cdot \nabla \psi_b,\ \phi_j)$$

- Natural truncation where $\sigma_j N_{jab} < \varepsilon$ (low-order / sparsity)

Ways to proceed

1. assume forcing model over $\mathbf{k}$ — spectra
2. symmetries of $\phi_i,\ \psi_i;\ \{\chi_i\};\ N_{jab}$ — scaling
3. pick representative combinations of $\chi_{\mathbf{k}}$ — structure
4. solve truncated $\chi_j = \sum_{ab} \sigma_j N_{jab} \chi_a \chi_b$
   — "approximate exact" solutions

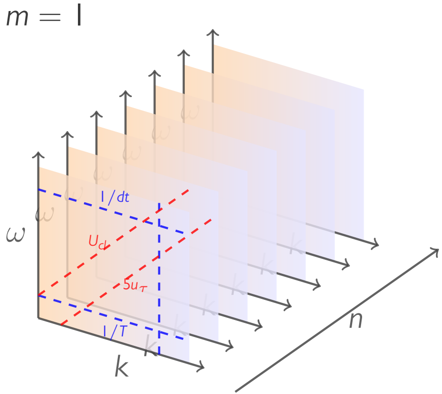
Localisation of response at critical layer



$\log(Re_\tau^{-2} E_{uu}(y,c))$; normalised at each y^+; $Re_\tau = 2003$

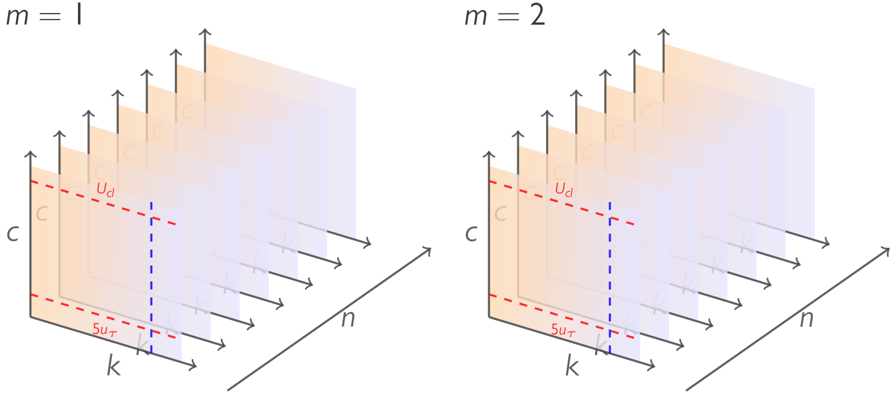
Moarref et al JFM 2013

Turbulence as sheets of coefficients



- energy quite localised around critical layer
- large areas over-resolved in k or ω resulting in stiffness of equations

Turbulence as sheets of coefficients



- Truncate outside $5u_\tau < c < U_{cl}$; above high k, n, ω; $m \lesssim 2$
- need for 4D data (DNS / experiment)

$R^+ = 314$
sparsity in ω introduced by finite-length pipe

$\max_{\mathbf{k}}(\sigma_{\mathbf{k},n,\omega,1})$

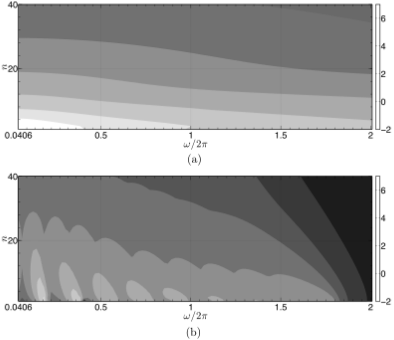


FIG. 1. Distribution of resolvent amplification $\log_{10}(\sigma_{n,\omega,1})$ in the energetically active subset of azimuthal wavenumbers n and frequencies $\omega/2\pi$ for the first singular vector $m = 1$ at $R^+ = 314$. (a) One-dimensional resolvent model. (b) Two-dimensional resolvent model.

1D resolvent (upper); 2D resolvent (lower)

Gomez et al *PoF 2014*

DNS $R^+ = 314$ projection, mode amplitudes fixed by nonlinearity

$\max_{\mathbf{k}}(\sigma_{\mathbf{k},n,\omega,1})$

F. Gómez, H. M. Blackburn, M. Rudman, A. S. Sharma and B. J. McKeon

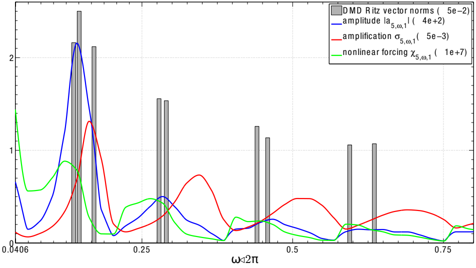
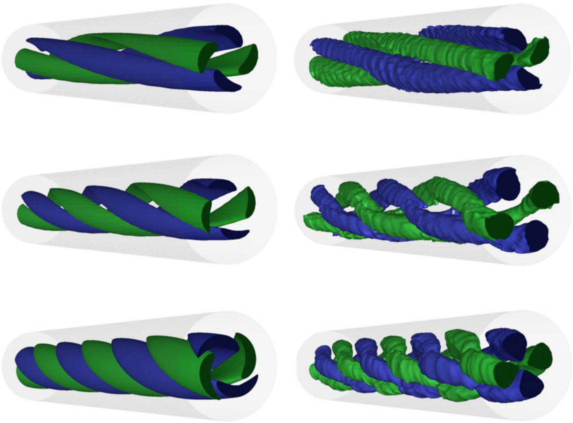


Fig. 1 Distribution of amplitude $a_{5,\omega,1}$, amplification $\sigma_{5,\omega,1}$ and nonlinear forcing $\chi_{5,\omega,1}$ in frequency of the first singular value m = 1 corresponding to the azimuthal wavenumber n = 5. Bars indicate most energetic frequencies computed via DMD of DNS data.

- 2D resolvent; sparsity in ω of $\mathbf{f_k}$, σ, $\mathbf{u_k}$;
- amplitude peaks not exactly at σ peaks (need to understand $\mathbf{f_k}$)

Gomez et al *iTi 2014*

Comparison with DMD structure in DNS

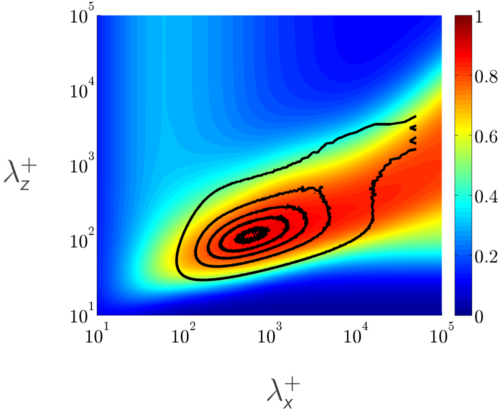


Comparisons between resolvent modes (*left*) and DMD modes (*right*) at same frequencies $\omega/2\pi = 0.1826$ (*top*) $\omega/2\pi = 0.3652$ (*middle*) $\omega/2\pi = 0.5479$ (*bottom*) at $n = 2$. Colored isosurfaces indicate $\pm 1/3$ of maximum streamwise f uctuating velocity.

Gomez et al *PoF 2014*

Spectra: direct from resolvent gains

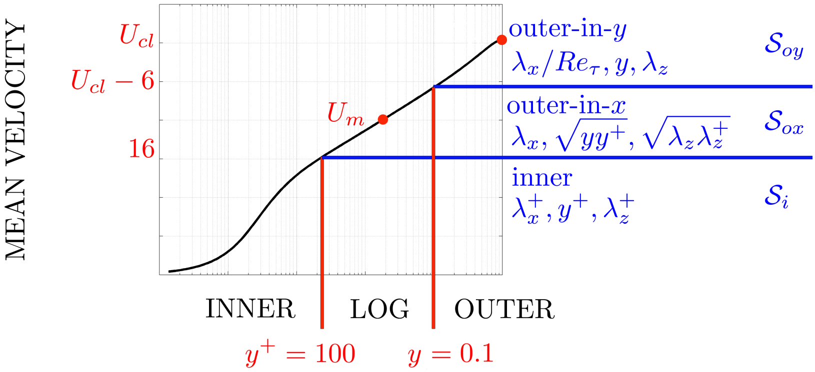
E_{uu} for unit white noise forcing on first two modes only at $y^+ = 15$ vs DNS



(in a channel) Moarref & al JFM 2013

DNS (lines): Hoyas & Jimenez PoF 2006

Scaling of modes from symmetries in the resolvent



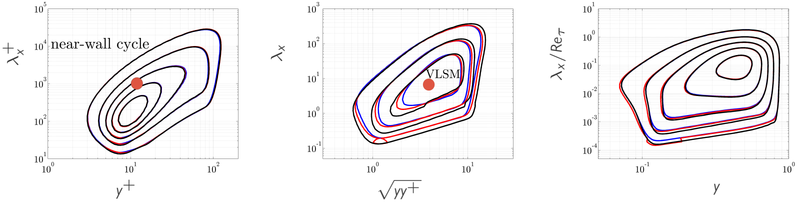
- mean profile scaling regions induce symmetries in resolvent
- reveals scalings of response modes

Moarref & al JFM 2013

Re-scaling of u, λ_x streamwise energy spectra

$$E_{uu}(y;k_x) = \overbrace{\boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{inner}} + \overbrace{\boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{self-similar (analytical)}} + \overbrace{\boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}}^{\text{outer}}$$

E_{uu}/Re_τ^2



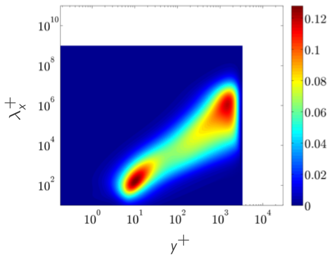
$Re_\tau = 3333,\ 10000,\ 30000$

Re-scaling of u, λ_x streamwise energy spectra

$$E_{uu}(y;k_x) = \underbrace{\boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c}}_{\text{inner}} + \underbrace{\boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c}}_{\text{self-similar (analytical)}} + \underbrace{\boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}}_{\text{outer}}$$

E_{uu}/Re_τ^2

$Re_\tau = 3333$



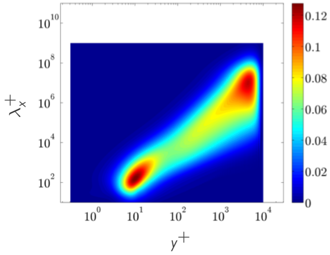
Re-scaling of u, λ_x streamwise energy spectra

inner | self-similar (analytical) | outer

$$E_{uu}(y;k_x) = \boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\mathrm{d}c} + \boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\mathrm{d}c} + \boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\mathrm{d}c}$$

E_{uu}/Re_τ^2

$Re_\tau = 10000$

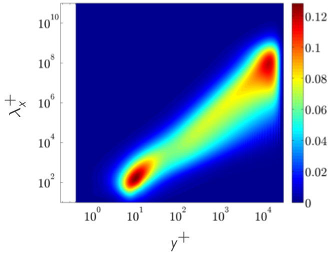


Re-scaling of u, λ_x streamwise energy spectra

$$E_{uu}(y;k_x) = \overbrace{\boxed{\int_0^{U^+=16} E_{uu}(y;k_x,c)\,\mathrm{d}c}}^{\text{inner}} + \overbrace{\boxed{\int_{U^+=16}^{U_{CL}-6} E_{uu}(y;k_x,c)\,\mathrm{d}c}}^{\text{self-similar (analytical)}} + \overbrace{\boxed{\int_{U_{CL}-6}^{U_{CL}} E_{uu}(y;k_x,c)\,\mathrm{d}c}}^{\text{outer}}$$

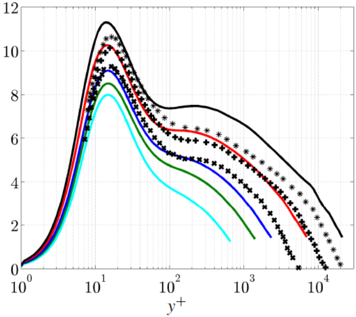
E_{uu}/Re_τ^2

$Re_\tau = 30000$



Fitting the $\{\chi_i\}$

Introducing $W(c)$ fitted at $Re_\tau = 2300$, using mode scalings, and assuming scalings for $W(c)$, generate SW energy intensity for high Re_τ, Note logarithmic scaling in overlap region[I]



$Re_\tau = 934$ (DNS) $Re_\tau = 2,300$ (DNS) $Re_\tau = 3,333$ $Re_\tau = 10,000$ $Re_\tau = 30,000$

Moarref et al JFM 2013; [I] *Marusic et al 2013*

Fitting the $\{\chi_i\}$

- Can fix $\{\chi_i\}$ by projection on DNS
- Norm choice is biased towards E_{uu} at these Re
- You can do better (E_{uv} etc) with more modes*

**Moarref et al, PoF 26 2014*

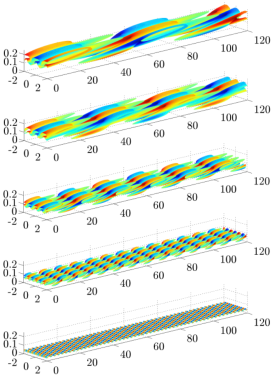
Geometric self-similarity; hierarchies of modes

Under assumptions:

1. modes local in y
2. critical layer term scales geometrically $U(y) - c = g(y/y_c)$
3. $ik_x(U(y) - c)$ term balances with $Re_\tau^{-1}\Delta$ term

log region is necessary to obtain invariant $\mathrm{H}_{\mathbf{k}}$ (and mode scalings).

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{bmatrix} y_c^+ y_c \overline{H}_{11} & (y_c^+)^2 y_c \overline{H}_{12} & (y_c^+)^2 y_c \overline{H}_{13} \\ y_c \overline{H}_{21} & y_c^+ y_c \overline{H}_{22} & y_c^+ y_c \overline{H}_{23} \\ y_c \overline{H}_{31} & y_c^+ y_c \overline{H}_{32} & y_c^+ y_c \overline{H}_{33} \end{bmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

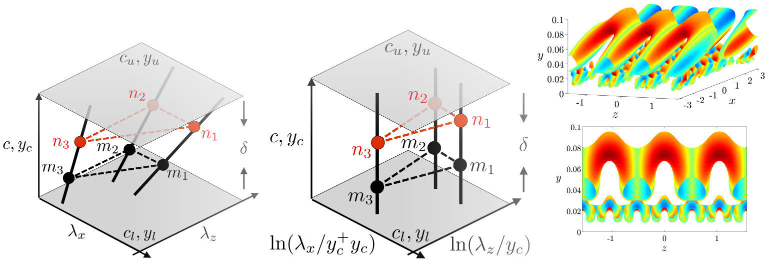


Moarref & al JFM 2013

Similar arguments derive classical inner and outer Re scalings for resolvent modes

Scaling of the interaction coefficient tensor

- Forcing is given by $\nabla \cdot (uu)$ of triadically-consistent modes, $\sigma_j N_{jab}$ in model.
- Scalings for self-similar modes (log region) have been found. For a given triad, decays exponentially with $c_u - c$.
- We get hierarchies of triads; an interacting triad at one y_c determines triads at all y in a region.



Moarref et al, in review; triad from Sharma & McKeon JFM 2013

Self-exciting processes in the model

Notice that for a mode combination to self-excite, we just require resulting nonlinear forcing terms not to be orthogonal to the resolvent forcing modes.

This is true for the example plotted.

We have still not yet solved for χ ...we need a toy problem.

The exact travelling wave solutions

- These provide a nice testbed, since single c
- In high Re turbulence, if model modes represent structure well, and if exact solutions also do, then modes pick out regions of state space more densely crossed due to nearby solutions with only low unstable dimension
- So, in neighbourhood of a solution, the modes should compactly describe that solution

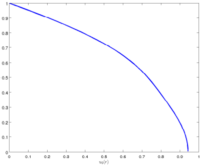
Projection of solutions onto model modes

- 15 pipe solutions provided by A Willis of Sheffield, generated by continuation to $Re_B = 5300$
- Also channel solutions provided by M Graham of Wisonsin-Madison (presented at APS)
- S and N solutions presented, upper and lower branch*
- modes generated using $\mathbf{u}_0$ of solution

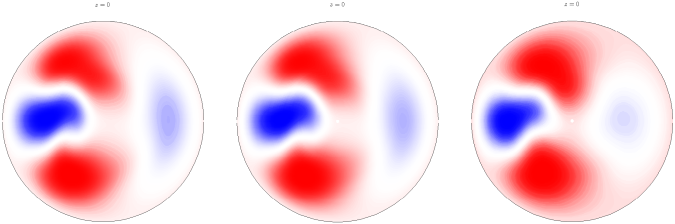
original solutions from Pringle et al, Phil. Trans. R. Soc. A, 2009

S1 solution 3403.0007

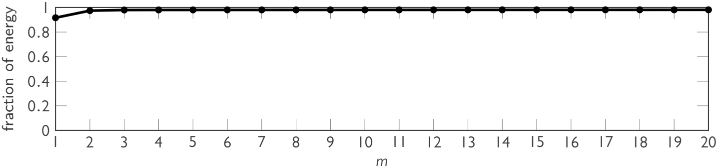
close to laminar; well represented with one mode per k



actual solution | $m = 1 \ldots 5$ | $m = 1$

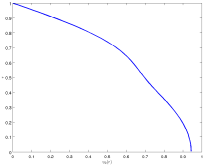


fraction of solution energy, keeping m singular values per Fourier mode

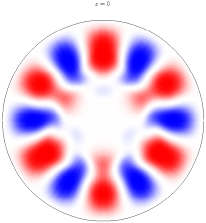


N3L solution 6507.1000

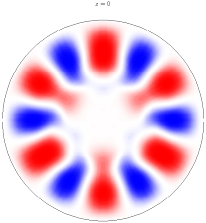
lower branch; close to laminar; well represented



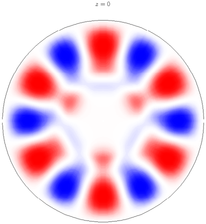
actual solution



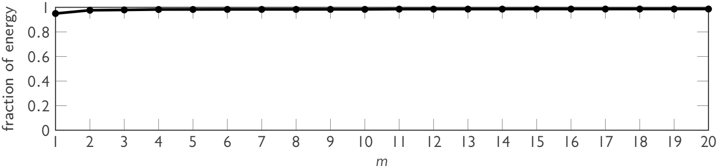
$m = 1 \ldots 5$



$m = 1$



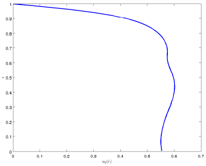
fraction of solution energy, keeping m singular values per Fourier mode

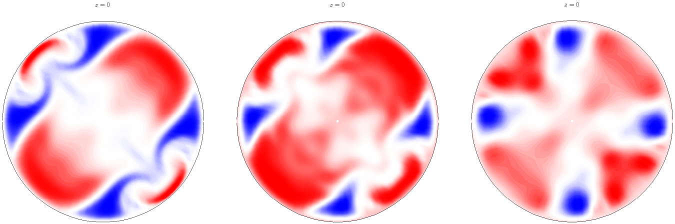


N2U solution 6502.0050

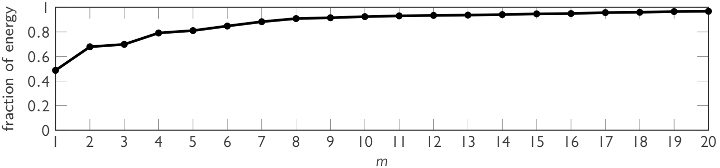
more ‘turbulent’; less well represented

actual solution $\qquad m = 1 \dots 5 \qquad m = 1$



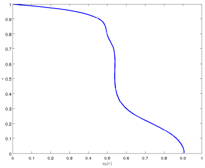


fraction of solution energy, keeping m singular values per Fourier mode

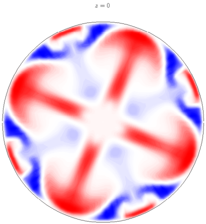


N4U solution 6512.1000

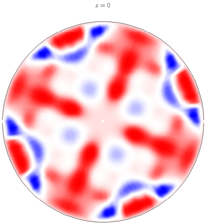
more ‘turbulent’; less well represented; not visited in turbulent DNS



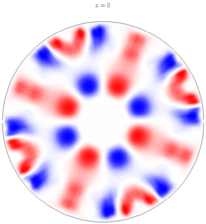
actual solution



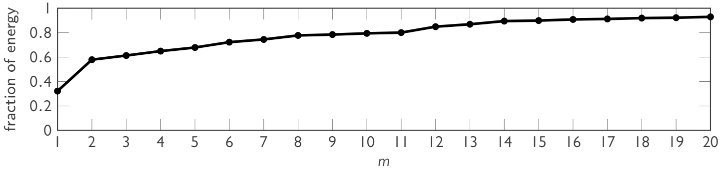
$m = 1 \ldots 5$



$m = 1$

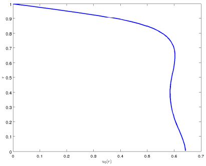


fraction of solution energy, keeping m singular values per Fourier mode

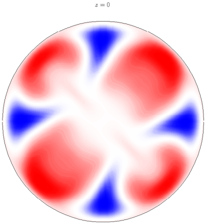


N2U solution 6502.0001

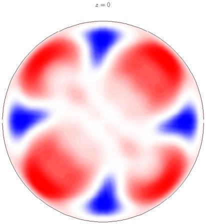
more ‘turbulent’; but well represented ?!



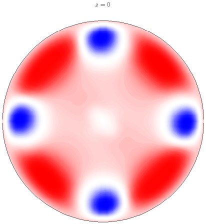
actual solution



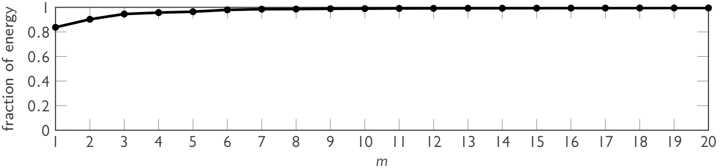
$m = 1 \ldots 5$



$m = 1$



fraction of solution energy, keeping m singular values per Fourier mode



What is missing in the model?

- Modes capture S-type and N-type (lower branch) extremely well
- Some upper branch solutions captured, others not; don't know why
- First step to demonstrating χ solution for exact solutions
- Methods may extend to finding new exact solutions (using modes as 'seeds')

Conclusions

- Derived a gain-optimal basis from NSE; sparse
- Captures turbulent structure and TW solns quite nicely
- Self-exciting mode combinations exist (amplitude dependent)
- Scalings of modes (*Re*; geometric); interaction coeffs & triads (log region) now known

For the future

- Investigating solutions for coefficients using TW solutions
- Use approximate solutions in χ-space to ‘seed’ exact solution solvers
- Truncated models may lead to reduced/toy DNS

Pipe mode code available at
http://github.com/mluhar/resolvent

Feel free to play.