Wall Turbulence as an Open Dynamical System

The Input-Output View

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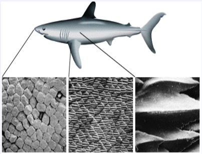
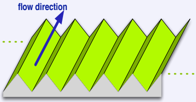
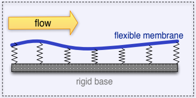
Control of Boundary Layer Turbulence

“passive” control

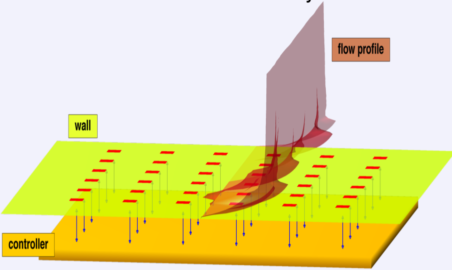






corrugated skin   compliant skin

active control with
sensor/actuator arrays



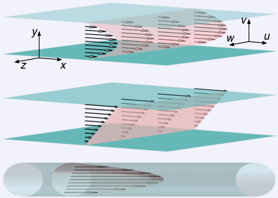
- Other “open loop” schemes:
  - Oscillating walls
  - Body force traveling waves

Caveat: *Plant’s dynamics are not well understood*

obstacles { not only device technology
also: *dynamical modeling and control design*

Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:



$$
\begin{aligned}
\partial_t \mathbf{u} &= -\nabla_{\mathbf{u}}\mathbf{u} - \text{grad}\, p + \tfrac{1}{R}\Delta \mathbf{u} \\
0 &= \text{div}\, \mathbf{u}
\end{aligned}
$$

- **Hydrodynamic Stability**: view NS as a dynamical system
- *laminar flow* $\bar{\mathbf{u}} :=$ a stationary solution of the NS equations (an *equilibrium*)

Mathematical Modeling of Transition: Hydrodynamic Stability

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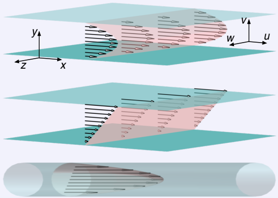

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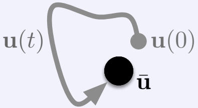
- **Hydrodynamic Stability**: view NS as a dynamical system
- *laminar flow* $\bar{\mathbf{u}}$:= a stationary solution of the NS equations (an *equilibrium*)

laminar flow $\bar{\mathbf{u}}$ stable $\longleftrightarrow$ i.c. $\mathbf{u}(0) \neq \bar{\mathbf{u}}$, $\mathbf{u}(t) \xrightarrow{t\to\infty} \bar{\mathbf{u}}$

- typically done with dynamics linearized about $\bar{\mathbf{u}}$
- various methods to track further "non-linear behavior"



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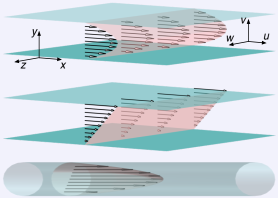

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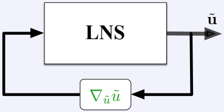
- **Hydrodynamic Stability**: view NS as a dynamical system
- A very successful (*phenomenologically predictive*) approach for many decades
- However: *it fails badly in the special (but important) case of streamlined flows*

Mathematical Modeling of Transition: Incorporating Uncertainty

- Decompose the fields as $\mathbf{u} = \underset{\text{laminar}}{\bar{\mathbf{u}}} + \underset{\text{fluctuations}}{\tilde{\mathbf{u}}}$

- Add a time-varying *exogenous disturbance* field $\mathbf{d}$ (e.g. random body forces)

$$\begin{aligned} \partial_t \tilde{\mathbf{u}} &= -\nabla_{\bar{\mathbf{u}}} \tilde{\mathbf{u}} - \nabla_{\tilde{\mathbf{u}}} \bar{\mathbf{u}} - \text{grad}\, \tilde{p} + \frac{1}{R} \Delta \tilde{\mathbf{u}} - \nabla_{\tilde{u}} \tilde{u} + \mathbf{d} \\ 0 &= \text{div}\, \tilde{\mathbf{u}} \end{aligned}$$



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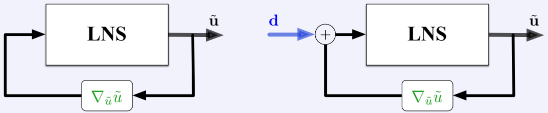

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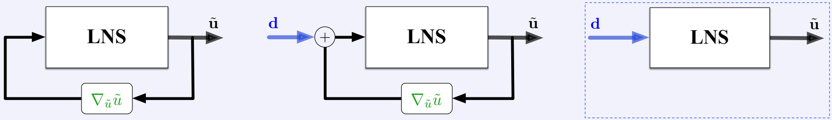


Input-Output view of the Linearized NS Equations

Farrell, Ioannou, '93 PoF

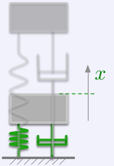
BB, Dahleh, '01 PoF

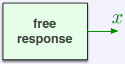
Jovanovic, BB, '05 JFM

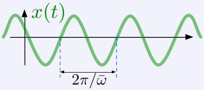
Internal Modes vs. External Resonances

A Detour

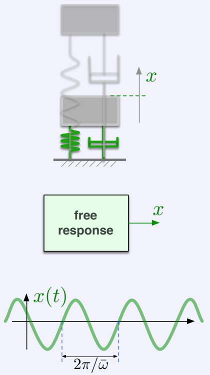
Internal Modes vs. External Resonances

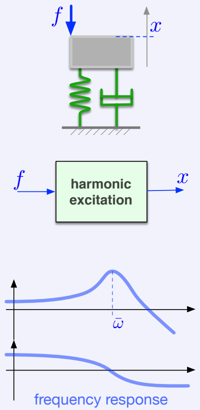






Internal Modes vs. External Resonances





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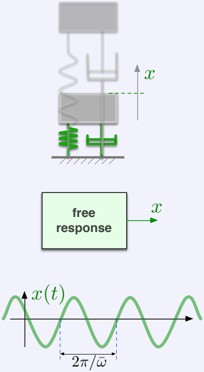
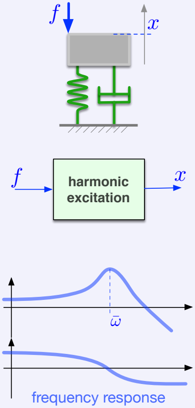
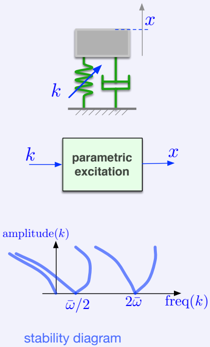
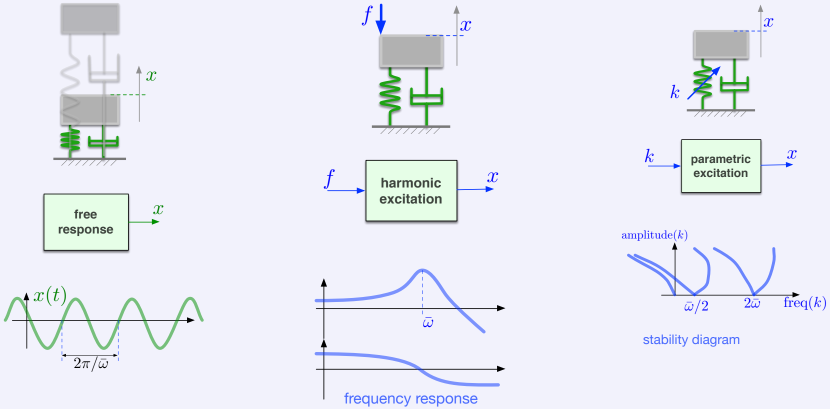






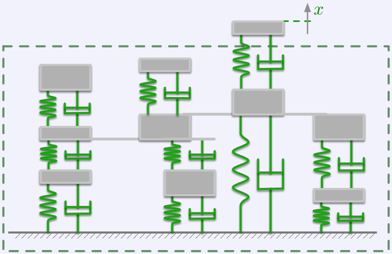
Internal Modes vs. External Resonances

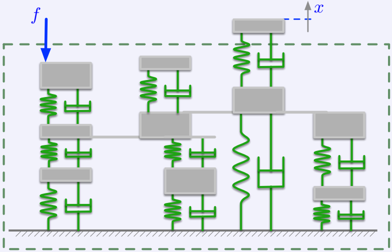


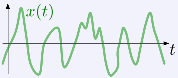
- Typically:
  internal modes frequencies $\longleftrightarrow$ externally excited response frequencies

Internal Modes vs. External Resonances

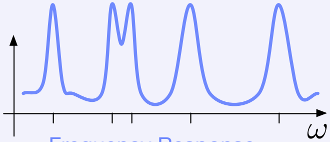
Does this correspondence hold for large-scale systems?







power spectral density



Frequency Response

Internal Modes vs. External Resonances

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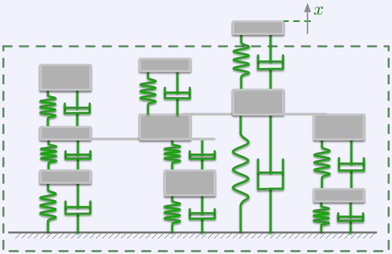

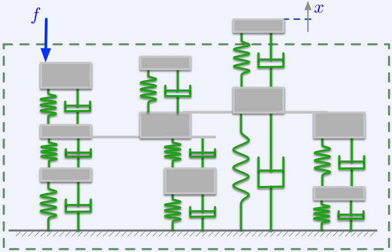

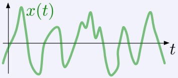

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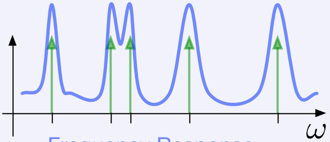

Frequency Response

typically:

internal modes $\longleftrightarrow$ external resonances

Internal Modes vs. External Resonances

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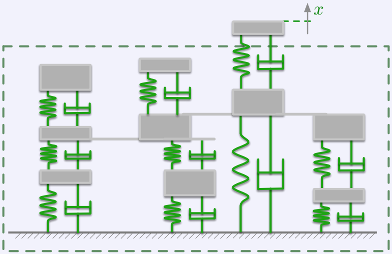
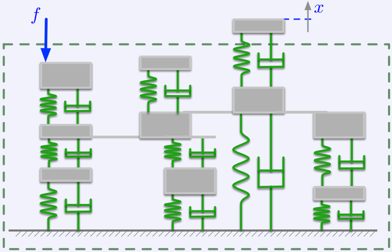
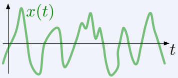
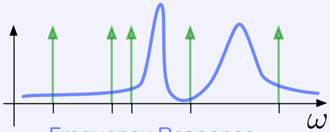








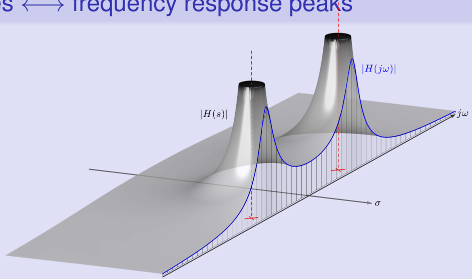
However: this may not hold in general

even in linear systems!

Modal vs. Input-Output Response

Typically: underdamped poles $\longleftrightarrow$ frequency response peaks

cf. The "rubber sheet analogy":



ODE (state space model)

$$\begin{aligned} \dot{\psi}(t) &= A\,\psi(t) + B\,d(t) \\ \tilde{\mathbf{u}}(t) &= C\,\psi(t) \end{aligned}$$

Transfer Function

$$H(s) = C\,(sI - A)^{-1}\,B$$

- eigs(A) $=$ poles$(H(s))$

Modal vs. Input-Output Response

However: Pole Locations $\not\leftrightarrow$ Frequency Response Peaks

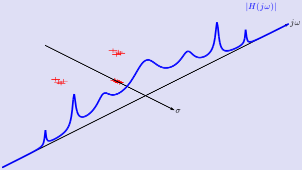
Theorem: *Given any desired pole locations*

$$z_1, \ \ldots, \ z_n \ \in \mathbb{C}_- \ \textit{(LHP)},$$

and any stable frequency response $H(j\omega)$, arbitrarily close approximation is achievable with

$$\left\| H(s) \ - \ \left(\sum_{i=1}^{N_1} \frac{\alpha_{1,i}}{(s-z_1)^i} \ + \ \cdots \ + \ \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{(s-z_n)^i} \right) \right\|_{\mathcal{H}^2} \leq \epsilon$$

by choosing any of the N_k's large enough



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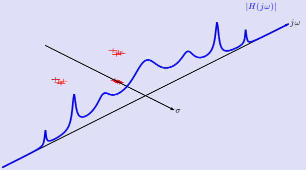


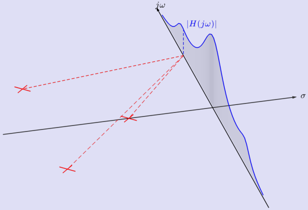
Remarks:

- No necessary relation between *pole locations* and *system resonances*
- ($\epsilon \to 0 \Rightarrow N_k \to \infty$), i.e. this is a *large-scale systems* phenomenon
- Large-scale systems: IO behavior not always predictable from modal behavior

Modal vs. Input-Output Response

However: Pole Locations $\not\leftrightarrow$ Frequency Response Peaks

MIMO case: $H(s) = (sI - A)^{-1}$

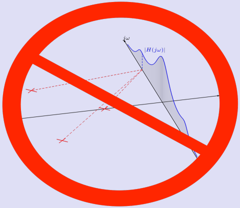


- *If A is normal* (has orthogonal eigenvectors), then

$$\sigma_{\max}\left((j\omega I - A)^{-1}\right) = \frac{1}{\text{distance}\,(j\omega,\ \text{nearest pole})}$$

- *If A is non-normal* : no clear relation between

  singular value plot $\not\leftrightarrow$ *eigs(A)*



Back to Fluids

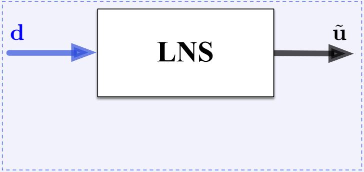
- Decompose the fields as

$$\mathbf{u} \;=\; \underset{\text{laminar}}{\underset{\uparrow}{\bar{\mathbf{u}}}} \;+\; \underset{\text{fluctuations}}{\underset{\uparrow}{\tilde{\mathbf{u}}}}$$

- Add a time-varying *exogenous disturbance* field $\mathbf{d}$ (e.g. random body forces)

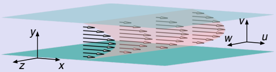
$$\begin{aligned} \partial_t \tilde{\mathbf{u}} &= -\nabla_{\bar{\mathbf{u}}} \tilde{\mathbf{u}} - \nabla_{\tilde{\mathbf{u}}} \bar{\mathbf{u}} - \text{grad}\, \tilde{p} + \tfrac{1}{R} \Delta \tilde{\mathbf{u}} - \nabla_{\tilde{u}} \tilde{u} + \mathbf{d} \\ 0 &= \text{div}\, \tilde{\mathbf{u}} \end{aligned}$$

- Neglect the feedback $\nabla_{\tilde{u}} \tilde{u}$



$$\partial_t \begin{bmatrix} \Delta \tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U''\partial_x - U\Delta\partial_x + \frac{1}{R}\Delta^2 & 0 \\ -U'\partial_z & -U\partial_x + \frac{1}{R}\Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_z & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \left(\partial_x^2 + \partial_z^2\right)^{-1} \begin{bmatrix} \partial_{xy} & -\partial_z \\ \partial_x^2 + \partial_z^2 & 0 \\ \partial_{zy} & \partial_x \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}$$



$$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$

Input-Output Analysis of the Linearized NS Equations

$$\partial_t \begin{bmatrix} \Delta\tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U''\partial_x - U\Delta\partial_x + \frac{1}{R}\Delta^2 & 0 \\ -U'\partial_z & -U\partial_x + \frac{1}{R}\Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_z & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

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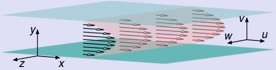

$$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$

- eigs $(\mathcal{A})$: determine stability

  (standard technique in *Linear Hydrodynamic Stability*)

- Transfer Function $\mathbf{d} \longrightarrow \tilde{\mathbf{u}}$: determines response to disturbances

  (an "open system")

Input-Output Analysis of the Linearized NS Equations

$$\partial_t \begin{bmatrix} \Delta\tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U''\partial_x - U\Delta\partial_x + \frac{1}{R}\Delta^2 & 0 \\ -U'\partial_z & -U\partial_x + \frac{1}{R}\Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_z & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

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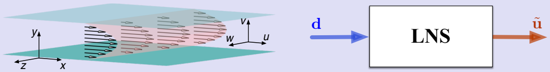
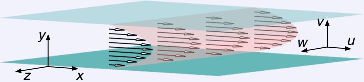


$$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$

Surprises:

- Even when $\mathcal{A}$ is stable — the gain $\mathbf{d} \longrightarrow \tilde{\mathbf{u}}$ can be very large
  ($(H^2 \text{ norm})^2$ scales with R^3)
- Input-output resonances — very different from least-damped modes of $\mathcal{A}$

Translation invariance in x & z implies



- *Impulse Response* *(Green's Function)*

$$\tilde{\mathbf{u}}(t,x,y,z) = \int G(t-\tau, x-\xi, \text{y,y'}, z-\zeta)\, \mathbf{d}(\tau,\xi,y',\zeta)\, d\tau d\xi dy' d\zeta$$

$$\tilde{\mathbf{u}}(t,x,.,z) = \int \mathcal{G}(t-\tau, x-\xi, z-\zeta)\, \mathbf{d}(\tau,\xi,.,\zeta)\, d\tau d\xi d\zeta$$

$\mathcal{G}(t,x,z)$: *Operator-valued* impulse response

- *Frequency Response*

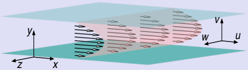
$$\tilde{\mathbf{u}}(\omega,k_x,k_z) = \mathcal{G}(\omega,k_x,k_z)\, \mathbf{d}(\omega,k_x,k_z)$$

$\mathcal{G}(\omega,k_x,k_z)$: *Operator-valued* frequency response (Packs lots of information!)

- *Spectrum of $\mathcal{A}$*:

$$\sigma(\mathcal{A}) = \overline{\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x,k_z)\right)}$$

Modal vs. Input-Output Analysis



- $$\begin{aligned} \partial_t \Psi &= \mathcal{A}\,\Psi + \mathcal{B}\,\mathbf{d} \\ \tilde{\mathbf{u}} &= \mathcal{C}\,\Psi \end{aligned}$$
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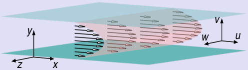

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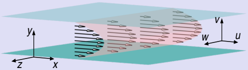
Modal Analysis: Look for unstable eigs of $\mathcal{A}$ $\left(\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x, k_z)\right)\right)$

Flow type	Classical linear theory R_c	Experimental R_c
Channel Flow	5772	$\approx$ 1,000-2,000
Plane Couette	∞	$\approx$ 350
Pipe Flow	∞	$\approx$ 2,200-100,000

Modal vs. Input-Output Analysis



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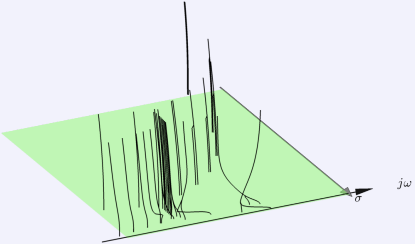
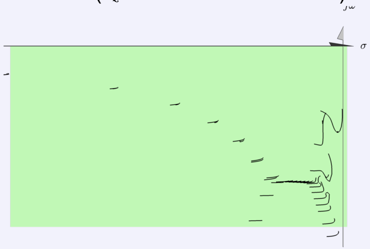
Modal Analysis: Look for unstable eigs of $\mathcal{A}$ $\left(\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x, k_z)\right)\right)$

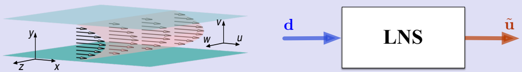
- Channel Flow @ $R = 2000$, $k_x = 1$, (k_z = vertical dimension):





top view

Modal vs. Input-Output Analysis



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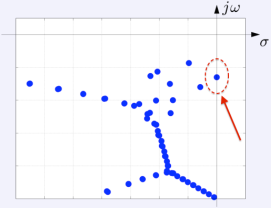
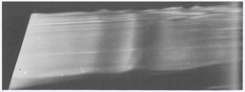
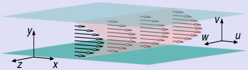
Modal Analysis: Look for unstable eigs of $\mathcal{A}$ $\left(\bigcup_{k_x,k_z} \sigma\left(\hat{\mathcal{A}}(k_x, k_z)\right)\right)$

- Channel Flow @ $R = 6000$, $k_x = 1$, $k_z = 0$:



- Flow structure of corresponding eigenfunction: Tollmein-Schlichting (TS) waves





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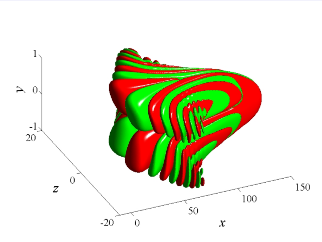
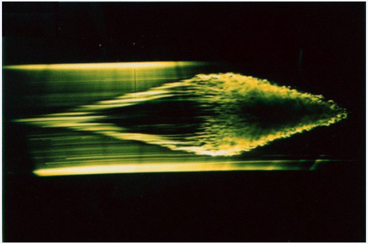
Impulse Response Analysis: Channel Flow @ $R = 2000$



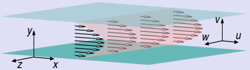


cf. “turbulent spots”

Jovanovic, BB, ’01 ACC,

more movies and pics at `http://engineering.ucsb.edu/~bamieh/pics/impulse_page.html`

Modal vs. Input-Output Analysis



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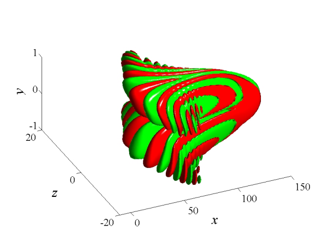
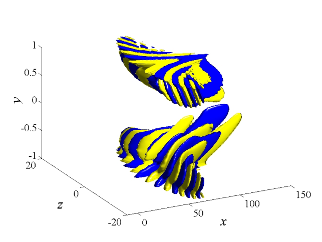
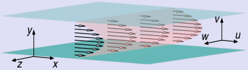


streamwise velocity



streamwise vorticity

Modal vs. Input-Output Analysis



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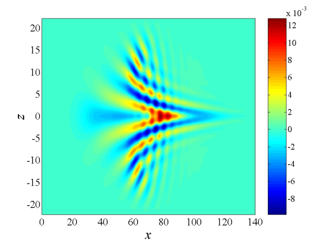
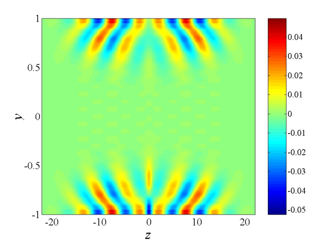
Impulse Response Analysis: Channel Flow @ $R = 2000$



u in a horizontal plane

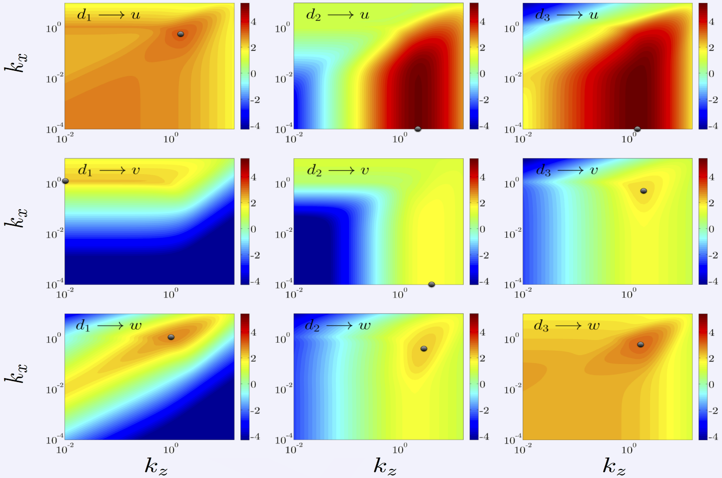


u in a vertical plane

Spatio-temporal Frequency Response

$\mathcal{G}(\omega, k_x, k_z)$ is a *LARGE* object! (very "data rich"!)

one visualization method: $\sup_\omega \sigma_{\max} \left(\mathcal{G}(\omega, k_x, k_z) \right)$



Jovanovic, BB, '05 JFM

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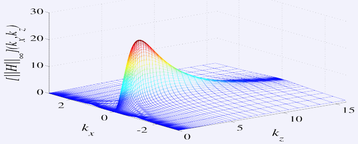
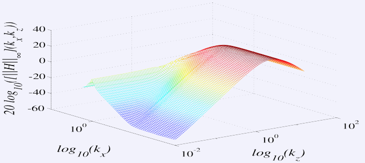
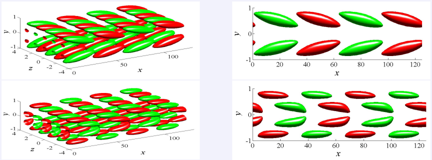
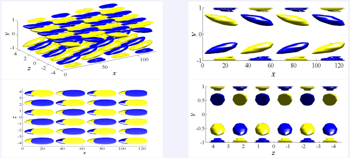




What do the corresponding flow structures look like?



streamwise velocity isosurfaces

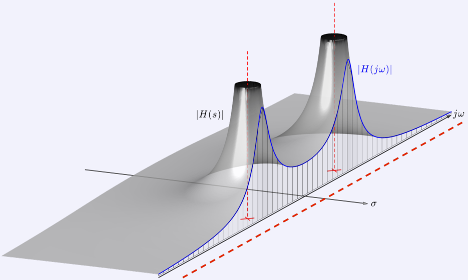


streamwise vorticity isosurfaces

Subcritical vs. Supercritical Frequency Response

Using "exponentially discounted" signal norms, e.g $\int_0^\infty \left\langle e^{-\alpha t}\tilde{\mathbf{u}}(t), e^{-\alpha t}\tilde{\mathbf{u}}(t) \right\rangle_E dt$

a proxy for finite-time-horizon energy integrals



Amounts to:

Frequency response is the Transfer Function on a shifted imaginary axis $(\alpha + j\omega)$

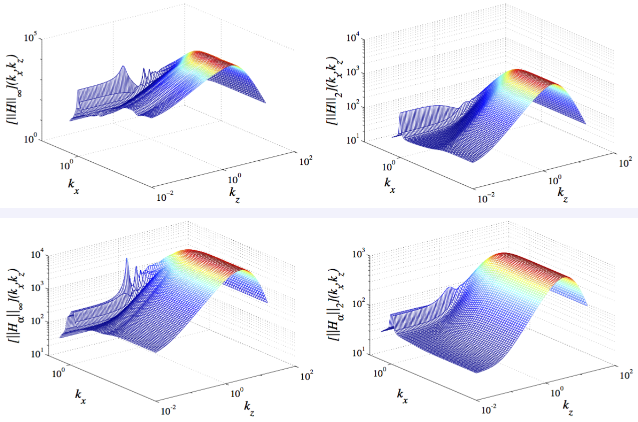
Subcritical vs. Supercritical Frequency Response

Using "exponentially discounted" signal norms, e.g $\int_0^\infty \left\langle e^{-\alpha t}\tilde{\mathbf{u}}(t), e^{-\alpha t}\tilde{\mathbf{u}}(t) \right\rangle_E dt$

a proxy for finite-time-horizon energy integrals

$R{=}5700$
$\alpha = 0$

$R{=}10000$
$\alpha = .004$



$\|.\|_\infty := \sup_\omega(.)$

a worst case measure

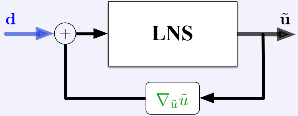
$\|.\|_2^2 := \int (.)^2 d\omega$

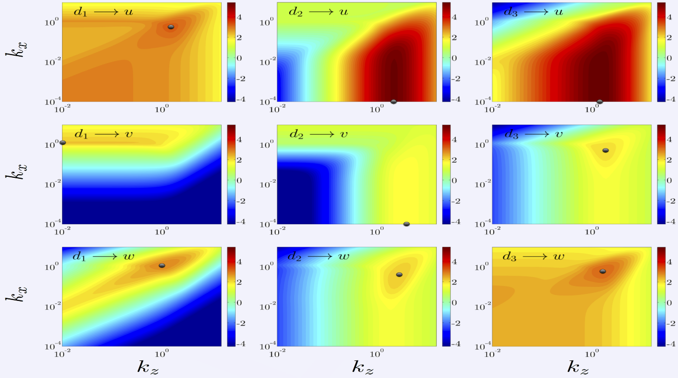
an average measure, variances

Spatio-temporal Frequency Response

How to view $\mathcal{G}(\omega, k_x, k_z)$?

how to better visualize it so that
the role of the $\nabla_{\tilde{u}}\tilde{u}$ feedback is clarified?





Spatio-temporal Frequency Response

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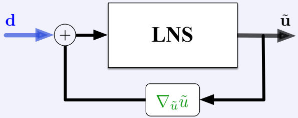
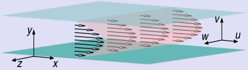


The *Linearized* Navier-Stokes equations are still not fully explored!

Implications for Turbulence

For large-scale systems: IO behavior not predictable from modal behavior



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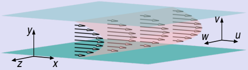

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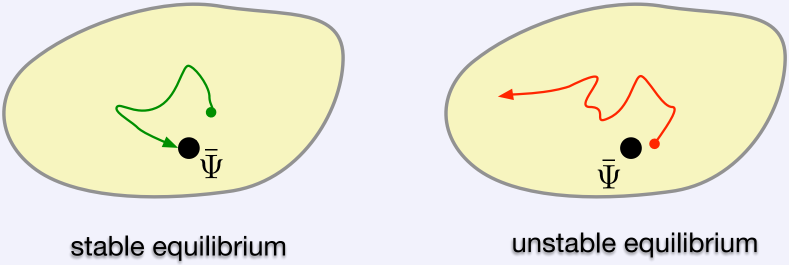
- "modal behavior": Stability due to *initial condition uncertainty*
- "IO behavior": behavior in the presence of *ambient uncertainty*
  - forcing terms from wall roughness and/or vibrations
  - Free-stream disturbances in boundary layers
  - Thermal (Langevin) forces
  - uncertain dynamics

Reexamining Stability Theory

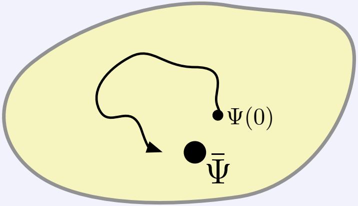
If starting "near" equilibrium, does system come back to it??



- An unstable equilibrium is not really an "equilibrium"

Reexamining Stability Theory

Lyapunov Stability deals with **uncertainty** in initial conditions



- Naive thought:

  If $\Psi(0)$ is known to be *precisely* $\bar{\Psi}$, then $\Psi(t) = \bar{\Psi}, \ t \geq 0$

- We introduce the concept of Lyapunov stability because we can never be *infinitely certain* about the initial condition

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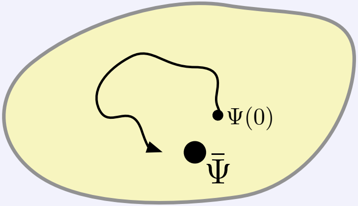

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Shortcomings of Lyapunov stability

- Perturbs only initial conditions
- Cares mostly about asymptotic behavior

Analysis of Uncertain Systems

Lyapunov Stability

$$\dot{\psi} = F(\psi)$$

- uncertain i.c.

$\psi(0)$

- investigate

$\lim_{t\to\infty} \psi(t)$

Analysis of Uncertain Systems

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$\lim_{t\to\infty} \psi(t)$

investigate transients

$\sup_{t\geq 0} \|\psi(t)\|$

- introduce a norm ($\|.\|$) on the state
- account for behavior at *all* times

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investigate transients

$\sup_{t \geq 0} \|\psi(t)\|$

exogenous disturbances

$$\dot{\psi}(t) = F\left(\psi\left(t\right), d\left(t\right)\right)$$

exogenous, spatio-temporally varying forcing fields, e.g.

- random body forces
- free-stream turbulence

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exogenous disturbances

$$\dot{\psi}(t) = F\left(\psi\left(t\right), d\left(t\right)\right)$$

dynamical uncertainty

$$\dot{\psi} \;=\; F(\psi) + \Delta(\psi)$$

"unmodeled dynamics"

- effects not modeled by NS equations
- unmodeled, dynamical wall-flow interactions
- etc.

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combinations

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increasing uncertainty →

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increasing uncertainty

linearized versions

Eigenvalue Stability

$$\dot{\psi} = A\psi$$

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non-normal transient growth

$$\sup_{t\geq 0} \|\psi(t)\|$$

input-output analysis

$$\dot{\psi}(t) = A\psi(t) + Bd(t)$$

pseudo-spectrum

$$\dot{\psi} = (A + \Delta)\psi$$

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increasing uncertainty

Robust Control Theory

linearized versions

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input-output analysis

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pseudo-spectrum

$$\dot{\psi} = (A + \Delta)\psi$$

combinations

$$\dot{\psi}(t) = (A + B\Delta C)\psi(t) + (F + G\Delta H)\, d(t)$$

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linearized versions

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great progress in the past 25 years

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**** great progress in the past 25 years***

**** yet the perception that: it's main utility is to help better understand the nonlinear stability problem***

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$$\sup_{t \geq 0} \|\psi(t)\|$$

exogenous disturbances

$$\dot{\psi}(t) = F\left(\psi(t), d(t)\right)$$

dynamical uncertainty

$$\dot{\psi} = F(\psi) + \Delta(\psi)$$

combinations

$$\dot{\psi}(t) = F\left(\psi(t), d(t)\right) + \Delta\left(\psi(t), d(t)\right)$$

increasing uncertainty →

linearized versions

Eigenvalue Stability

$$\dot{\psi} = A\psi$$

- uncertain i.c. $\psi(0)$
- investigate $\lim_{t \to \infty} \psi(t)$

non-normal transient growth

$$\sup_{t \geq 0} \|\psi(t)\|$$

input-output analysis

$$\dot{\psi}(t) = A\psi(t) + Bd(t)$$

pseudo-spectrum

$$\dot{\psi} = (A + \Delta)\psi$$

stability theory alone (linear or nonlinear)

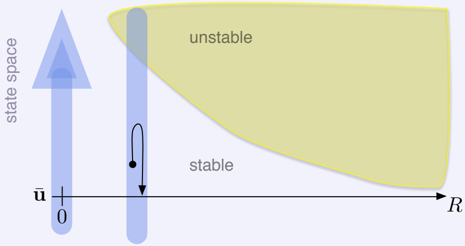
insufficient to capture the phenomenology of transition

The Nature of Turbulence

- Fluid flows are described by deterministic equations
- OLD QUESTION: why do fluid flows "look random" at high R?

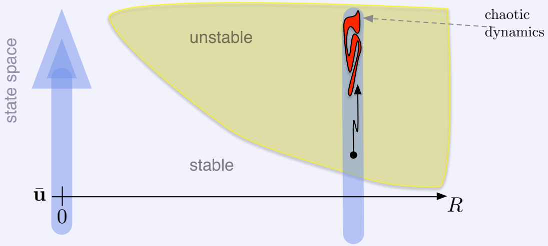
The Nature of Turbulence

- **A common view of turbulence**



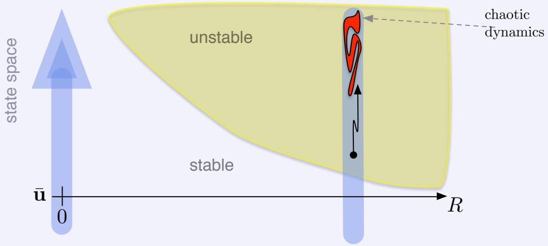
The Nature of Turbulence

- **A common view of turbulence**



The Nature of Turbulence

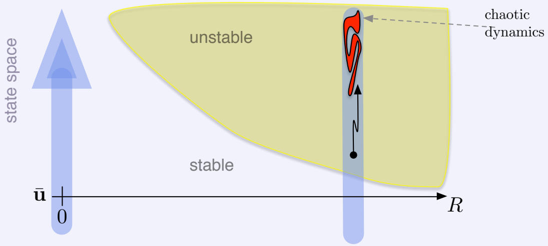
- **A common view of turbulence**



- **Intuitive reasoning:**
  Complex, "statistical looking" behavior $\longleftrightarrow$ chaotic dynamics
  "self-sustaining" cycle

The Nature of Turbulence

- **A common view of turbulence**

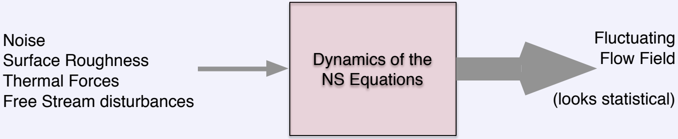


- **Intuitive reasoning:**
  Complex, "statistical looking" behavior $\longleftrightarrow$ chaotic dynamics
  "self-sustaining" cycle
- Assumes NS eqs. with perfect BC, no disturbances or uncertainty
  (i.e. a a closed system)

The Nature of Turbulence

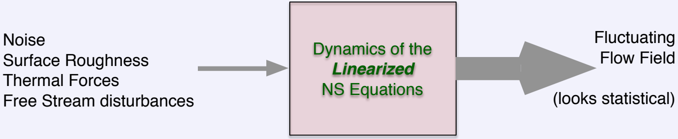
An Alternate Possibility

- A driven (open) system

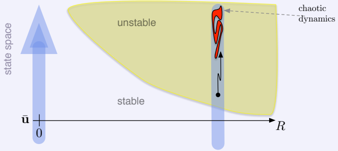


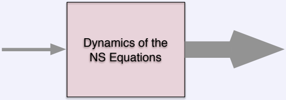
The NS equations act as an *amplifier of ambient uncertainty* at high R

- *Qualitatively* similar to



The Nature of Turbulence (a mixed picture)

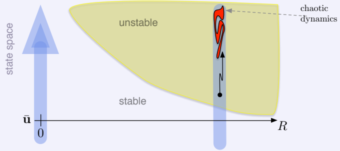


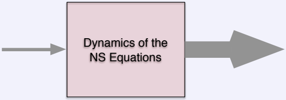


more typical in
bluff body flows

more typical in
highly streamlined flows

The Nature of Turbulence (a mixed picture)





more typical in
bluff body flows

more typical in
highly streamlined flows

There's probably a mixture of both mechanisms in most flows

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