Wall Turbulence as an Open Dynamical System
The Input-Output View

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Control of Boundary Layer Turbulence

"passive" control

active control with sensor/actuator arrays

Other "open loop" schemes:
- Oscillating walls
- Body force traveling waves

Caveat: Plant's dynamics are not well understood

obstacles \{ not only device technology
also: dynamical modeling and control design \}
Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:

\[
\begin{align*}
\partial_t u &= -\nabla u \cdot u - \text{grad } p + \frac{1}{R} \Delta u \\
0 &= \text{div } u
\end{align*}
\]

- **Hydrodynamic Stability:** view NS as a dynamical system
- **laminar flow** \( \bar{u} \) := a stationary solution of the NS equations (an equilibrium)
The Navier-Stokes (NS) equations:

\[ \partial_t u = -\nabla u - \text{grad } p + \frac{1}{R} \Delta u \]
\[ 0 = \text{div } u \]

- **Hydrodynamic Stability:**
  - view NS as a dynamical system
  - *laminar flow* \( \bar{u} \) := a stationary solution of the NS equations (an equilibrium)

- laminar flow \( \bar{u} \) stable \( \iff \) i.c. \( u(0) \neq \bar{u} \), \( u(t) \xrightarrow{t \to \infty} \bar{u} \)
  - typically done with dynamics linearized about \( \bar{u} \)
  - various methods to track further "non-linear behavior"
Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:

\[ \partial_t u = - \nabla u \cdot u - \nabla p + \frac{1}{R} \Delta u \]

\[ 0 = \text{div} \ u \]

- **Hydrodynamic Stability:** view NS as a dynamical system
- A very successful (*phenomenologically predictive*) approach for many decades
- However: *it fails badly in the special (but important) case of streamlined flows*
Mathematical Modeling of Transition: Incorporating Uncertainty

- Decompose the fields as
  \[ u = \bar{u} + \tilde{u} \]
  \[ \uparrow \quad \text{laminar} \quad \uparrow \quad \text{fluctuations} \]

- Add a time-varying *exogenous disturbance* field \( d \) (e.g. random body forces)

\[
\begin{align*}
\partial_t \tilde{u} & = -\nabla \bar{u} \tilde{u} - \nabla \tilde{u} \bar{u} - \text{grad} \tilde{p} + \frac{1}{R} \Delta \tilde{u} - \nabla \tilde{u} \tilde{u} + d \\
0 & = \text{div} \tilde{u}
\end{align*}
\]

**IPAM, Nov 2014 4 / 24**
Decompose the fields as
\[ u = \bar{u} + \tilde{u} \]
\[ \uparrow \]
laminar fluctuations

Add a time-varying *exogenous disturbance* field \( d \) (e.g. random body forces)

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\partial_t \tilde{u} = -\nabla \bar{u} \tilde{u} - \nabla \tilde{u} \bar{u} - \text{grad} \tilde{p} + \frac{1}{R} \Delta \tilde{u} - \nabla \tilde{u} \tilde{u} + d
\]

0 = \text{div} \tilde{u}
Mathematical Modeling of Transition: Incorporating Uncertainty

- Decompose the fields as
  \[ \mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}} \]
  \[ \uparrow \text{laminar} \quad \uparrow \text{fluctuations} \]

- Add a time-varying exogenous disturbance field \( \mathbf{d} \) (e.g. random body forces)
  \[ \partial_t \tilde{\mathbf{u}} = - \nabla \bar{\mathbf{u}} \tilde{\mathbf{u}} - \nabla \tilde{\mathbf{u}} \bar{\mathbf{u}} - \text{grad} \tilde{\rho} + \frac{1}{R} \Delta \tilde{\mathbf{u}} - \nabla \tilde{\mathbf{u}} \tilde{\mathbf{u}} + \mathbf{d} \]
  \[ 0 = \text{div} \tilde{\mathbf{u}} \]

Input-Output view of the Linearized NS Equations

*Farrell, Ioannou, ’93 PoF*

*BB, Dahleh, ’01 PoF*

*Jovanovic, BB, ’05 JFM*
Internal Modes vs. External Resonances
A Detour
Internal Modes vs. External Resonances

\[ x(t) \]

\[ 2\pi/\omega \]
Internal Modes vs. External Resonances

- Free response
  
- Harmonic excitation
  
- Frequency response

\[ x(t) \]

\[ f \]

\[ \bar{\omega} \]
Internal Modes vs. External Resonances

- **free response**
- **harmonic excitation**
- **parametric excitation**

- **frequency response**
- **stability diagram**

- $x(t)$
- $2\pi/\omega$
- $f$
- $\omega$
- $\bar{\omega}$
Internal Modes vs. External Resonances

Typically:

internal modes frequencies  $\leftrightarrow$  externally excited response frequencies
Internal Modes vs. External Resonances

Does this correspondence hold for large-scale systems?
Internal Modes vs. External Resonances

Does this correspondence hold for large-scale systems?

typically:

internal modes $\leftrightarrow$ external resonances
Internal Modes vs. External Resonances

Does this correspondence hold for large-scale systems?

However: this may not hold in general even in linear systems!
Modal vs. Input-Output Response

Typically: underdamped poles $\leftrightarrow$ frequency response peaks

cf. The “rubber sheet analogy”:

ODE (state space model)
\[
\begin{align*}
\dot{\psi}(t) &= A \psi(t) + B d(t) \\
\tilde{u}(t) &= C \psi(t)
\end{align*}
\]

Transfer Function
\[
H(s) = C (sI - A)^{-1} B
\]

$\text{eigs}(A) = \text{poles}(H(s))$
However:

**Theorem:** Given any desired pole locations

\[ z_1, \ldots, z_n \in \mathbb{C}^-(LHP), \]

and any stable frequency response \( H(j\omega) \), arbitrarily close approximation is achievable with

\[
\left\| H(s) - \left( \sum_{i=1}^{N_1} \frac{\alpha_{1,i}}{(s - z_1)^i} + \cdots + \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{(s - z_n)^i} \right) \right\|_{\mathcal{H}^2} \leq \epsilon
\]

by choosing any of the \( N_k \)'s large enough.
However:

**Theorem:** Given any desired pole locations

\[ z_1, \ldots, z_n \in \mathbb{C}^-(LHP), \]

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\]

by choosing any of the \( N_k \)'s large enough

**Remarks:**

- No necessary relation between pole locations and system resonances
- \( (\epsilon \to 0 \Rightarrow N_k \to \infty) \), i.e. this is a large-scale systems phenomenon
- Large-scale systems: IO behavior not always predictable from modal behavior
Modal vs. Input-Output Response

However:

Pole Locations ↔ Frequency Response Peaks

MIMO case: \( H(s) = (sI - A)^{-1} \)

- If \( A \) is normal (has orthogonal eigenvectors), then
  \[
  \sigma_{\text{max}} \left( (j\omega I - A)^{-1} \right) = \frac{1}{\text{distance} (j\omega, \text{nearest pole})}
  \]

- If \( A \) is non-normal: no clear relation between singular value plot ↔ eigs(A)
Back to Fluids
Decompose the fields as
\[ u = \bar{u} + \tilde{u} \]
upward
laminar fluctuations

Add a time-varying *exogenous disturbance* field \( d \) (e.g. random body forces)

\[
\partial_t \tilde{u} = -\nabla \bar{u} \tilde{u} - \nabla \tilde{u} \bar{u} - \text{grad} \ \tilde{p} + \frac{1}{R} \Delta \tilde{u} - \nabla \tilde{u} \tilde{u} + d
\]

0 = div \( \tilde{u} \)

Neglect the feedback \( \nabla \tilde{u} \tilde{u} \)
Input-Output Analysis of the Linearized NS Equations

\[
\partial_t \begin{bmatrix} \Delta \tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U'' \partial_x - U \Delta \partial_x + \frac{1}{R} \Delta^2 & 0 \\ -U' \partial_z & -U \partial_x + \frac{1}{R} \Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{xy} \\ \partial_x & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}
\]

\[
\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \left( \partial_x^2 + \partial_z^2 \right)^{-1} \begin{bmatrix} \partial_{xy} & -\partial_z \\ \partial_x^2 + \partial_z^2 & 0 \\ \partial_{zy} & \partial_x \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}
\]

\[
\partial_t \Psi = A \Psi + B d
\]

\[
\tilde{u} = C \Psi
\]
Input-Output Analysis of the Linearized NS Equations

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\end{align*}
\]

- \textbf{eigs} \( (A) \): determine stability
  
  (standard technique in \textit{Linear Hydrodynamic Stability})

- \textbf{Transfer Function} \( d \rightarrow \tilde{u} \): determines response to disturbances
  
  (an “open system”)

\[
\partial_t \Psi = A \Psi + B \mathbf{d} \\
\tilde{u} = C \Psi
\]
\[
\begin{align*}
\partial_t \begin{bmatrix} \Delta \tilde{v} \\ \tilde{\omega} \end{bmatrix} &= \begin{bmatrix} U'' \partial_x - U \Delta \partial_x + \frac{1}{R} \Delta^2 & 0 \\ -U' \partial_z & -U \partial_x + \frac{1}{R} \Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_{xz} & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \\
\tilde{u} &\tilde{v} \tilde{w} = \left( \partial_x^2 + \partial_z^2 \right)^{-1} \begin{bmatrix} \partial_{xy} & \partial_x & -\partial_z \\ \partial_{xz} & 0 & \partial_x \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}
\end{align*}
\]

**Surprises:**

- Even when \( \mathcal{A} \) is stable, the gain \( d \rightarrow \tilde{u} \) can be very large (\( (H^2 \text{ norm})^2 \) scales with \( R^3 \))
- Input-output resonances very different from least-damped modes of \( \mathcal{A} \)
Translation invariance in $x$ & $z$ implies

- **Impulse Response** *(Green's Function)*
  \[
  \tilde{u}(t, x, y, z) = \int G(t - \tau, x - \xi, y, y', z - \zeta) \, d(\tau, \xi, y', \zeta) 
  d\tau d\xi dy' d\zeta
  \]
  \[
  \tilde{u}(t, x, .., z) = \int G(t - \tau, x - \xi, z - \zeta) \, d(\tau, \xi, .., \zeta) 
  d\tau d\xi d\zeta
  \]
  \[G(t, x, z) : \text{Operator-valued impulse response}\]

- **Frequency Response**
  \[
  \tilde{u}(\omega, k_x, k_z) = G(\omega, k_x, k_z) \, d(\omega, k_x, k_z)
  \]
  \[G(\omega, k_x, k_z) : \text{Operator-valued frequency response}\]
  (Packs lots of information!)

- **Spectrum of $A$:**
  \[
  \sigma(A) = \bigcup_{k_x,k_z} \sigma\left(\hat{A}(k_x, k_z)\right)
  \]
Modal vs. Input-Output Analysis

\[ \frac{\partial_t \Psi}{\partial t} = A \Psi + B d \]

\[ \hat{\mathbf{u}} = C \Psi \]

- IR: \( G(t, x, z) \)
- FR: \( G(\omega, k_x, k_z) \)
Modal Analysis: Look for unstable eigs of $\mathcal{A}$

$$\left( \bigcup_{k_x, k_z} \sigma \left( \hat{\mathcal{A}}(k_x, k_z) \right) \right)$$

<table>
<thead>
<tr>
<th>Flow type</th>
<th>Classical linear theory $R_c$</th>
<th>Experimental $R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel Flow</td>
<td>5772</td>
<td>$\approx 1,000-2,000$</td>
</tr>
<tr>
<td>Plane Couette</td>
<td>$\infty$</td>
<td>$\approx 350$</td>
</tr>
<tr>
<td>Pipe Flow</td>
<td>$\infty$</td>
<td>$\approx 2,200-100,000$</td>
</tr>
</tbody>
</table>
Modal vs. Input-Output Analysis

\[ \partial_t \Psi = A \Psi + B d \]
\[ \tilde{u} = C \Psi \]

IR: \( G(t, x, z) \)
FR: \( G(\omega, k_x, k_z) \)

**Modal Analysis**: Look for unstable eigs of \( A \)

- Channel Flow @ \( R = 2000, k_x = 1 \)

Channel Flow @ \( R = 2000 \), \( k_x = 1 \)

\( \left( \bigcup_{k_x, k_z} \sigma \left( \hat{A}(k_x, k_z) \right) \right) \)

\( (k_z = \text{vertical dimension}) \)}
Modal vs. Input-Output Analysis

\[
\frac{\partial_t \Psi}{\hat{u}} = A \Psi + B d \\
\hat{u} = C \Psi
\]

IR: \( \mathcal{G}(t, x, z) \)
FR: \( \mathcal{G}(\omega, k_x, k_z) \)

**Modal Analysis:** Look for unstable eigs of \( \mathcal{A} \) \( \bigcup_{k_x, k_z} \sigma \left( \hat{A} \left(k_x, k_z \right) \right) \)

- Channel Flow @ \( R = 6000, k_x = 1, k_z = 0 \):
- Flow structure of corresponding eigenfunction: Tollmein-Schlichting (TS) waves
Modal vs. Input-Output Analysis

\[ \frac{\partial_t \Psi}{\partial t} = A \Psi + B \, d \]
\[ \tilde{u} = C \Psi \]

IR: \( G(t, x, y, -1, z) \)
FR: \( G(\omega, k_x, k_z) \)

Impulse Response Analysis: Channel Flow @ \( R = 2000 \)

cf. “turbulent spots”

Jovanovic, BB, '01 ACC,
more movies and pics at [http://engineering.ucsb.edu/~bamieh/pics/impulse_page.html](http://engineering.ucsb.edu/~bamieh/pics/impulse_page.html)
Modal vs. Input-Output Analysis

\[ \partial_t \Psi = A \Psi + B \mathbf{d} \]
\[ \mathbf{\tilde{u}} = C \Psi \]

IR: \( G(t, x, y, -1, z) \)
FR: \( G(\omega, k_x, k_z) \)

**Impulse Response Analysis:** Channel Flow @ \( R = 2000 \)

- streamwise velocity
- streamwise vorticity
Modal vs. Input-Output Analysis

\[
\begin{align*}
\partial_t \Psi &= \Psi + B \mathbf{d} \\
\tilde{\mathbf{u}} &= C \Psi
\end{align*}
\]

IR:
FR: \( G(\omega, k_x, k_z) \)

Impulse Response Analysis: Channel Flow @ \( R = 2000 \)

\( u \) in a horizontal plane
\( u \) in a vertical plane
Spatio-temporal Frequency Response

$G(\omega, k_x, k_z)$ is a LARGE object! (very “data rich”!)

one visualization method: $\sup_\omega \sigma_{\text{max}} \left( G(\omega, k_x, k_z) \right)$

Jovanovic, BB, ’05 JFM
Spatio-temporal Frequency Response

$\mathcal{G}(\omega, k_x, k_z)$ is a LARGE object!  
(very “data rich”!)

one visualization method: $\sup_{\omega} \sigma_{\text{max}} \left( \mathcal{G}(\omega, k_x, k_z) \right)$

What do the corresponding flow structures look like?

streamwise velocity isosurfaces  
streamwise vorticity isosurfaces
Subcritical vs. Supercritical Frequency Response

Using “exponentially discounted” signal norms, e.g.
\[
\int_0^\infty \langle e^{-\alpha t} \tilde{u}(t), e^{-\alpha t} \tilde{u}(t) \rangle_E dt
\]
a proxy for finite-time-horizon energy integrals

Amounts to:
Frequency response is the Transfer Function on a shifted imaginary axis \((\alpha + j\omega)\)
Subcritical vs. Supercritical Frequency Response

Using “exponentially discounted” signal norms, e.g.
\[
\int_0^\infty \left\langle e^{-\alpha t} \tilde{u}(t), e^{-\alpha t} \tilde{u}(t) \right\rangle_E dt
\]
a proxy for finite-time-horizon energy integrals

- \( R = 5700 \), \( \alpha = 0 \)
- \( R = 10000 \), \( \alpha = 0.004 \)

\[ \| . \|_{\infty} := \sup_\omega ( . ) \]
a worst case measure

\[ \| . \|_2^2 := \int ( . )^2 d\omega \]
an average measure, variances
How to view $G(\omega, k_x, k_z)$?

How to better visualize it so that the role of the $\nabla \tilde{u} \tilde{u}$ feedback is clarified?
Spatio-temporal Frequency Response

How to view $G(\omega, k_x, k_z)$?

The *Linearized* Navier-Stokes equations are still not fully explored!
Implications for Turbulence

For large-scale systems: IO behavior not predictable from modal behavior

\[
\begin{align*}
\partial_t \Psi &= A \Psi + B d \\
\tilde{u} &= C \Psi
\end{align*}
\]

- IR: \( G(t, x, z) \)
- FR: \( G(\omega, k_x, k_z) \)
Implications for Turbulence

For large-scale systems: IO behavior not predictable from modal behavior

- “modal behavior”: Stability due to *initial condition uncertainty*
- “IO behavior”: behavior in the presence of *ambient uncertainty*
  - forcing terms from wall roughness and/or vibrations
  - Free-stream disturbances in boundary layers
  - Thermal (Langevin) forces
  - uncertain dynamics

\[
\begin{align*}
\frac{\partial_t \Psi}{\Psi} &= A \Psi + B \mathbf{d} \\
\tilde{\mathbf{u}} &= C \Psi \\
\text{IR: } G(t, x, z) \\
\text{FR: } G(\omega, k_x, k_z)
\end{align*}
\]
Reexamining Stability Theory

If starting “near” equilibrium, does system come back to it??

stable equilibrium

unstable equilibrium

- An unstable equilibrium is not really an “equilibrium”
Reexamining Stability Theory

Lyapunov Stability deals with **uncertainty** in initial conditions.

- **Naive thought:**
  
  If $\Psi(0)$ is known to be **precisely** $\bar{\Psi}$, then $\Psi(t) = \bar{\Psi}, \ t \geq 0$

- We introduce the concept of Lyapunov stability because we can never be **infinitely certain** about the initial condition.
Lyapunov Stability

deals with uncertainty in initial conditions

Naive thought:
If $\Psi(0)$ is known to be precisely $\bar{\Psi}$, then $\Psi(t) = \bar{\Psi}, \ t \geq 0$

We introduce the concept of Lyapunov stability because we can never be infinitely certain about the initial condition.

Shortcomings of Lyapunov stability
- Perturbs only initial conditions
- Cares mostly about asymptotic behavior
Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]

- uncertain i.c.
- investigate

\[ \lim_{t \to \infty} \psi(t) = \psi(0) \]
Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]

- uncertain i.c.
- investigate

\[ \psi(0) \]

\[ \lim_{t \to \infty} \psi(t) \]

investigate transients

\[ \sup_{t \geq 0} \| \psi(t) \| \]

- introduce a norm (\(\| \cdot \|\)) on the state
- account for behavior at all times
Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]
- uncertain i.c.
- investigate

\[ \lim_{t \to \infty} \psi(t) \]

investigate transients

\[ \sup_{t \geq 0} \| \psi(t) \| \]

exogenous disturbances

\[ \dot{\psi}(t) = F(\psi(t), d(t)) \]

exogenous, spatio-temporally varying forcing fields, e.g.

- random body forces
- free-stream turbulence

Analysis of Uncertain Systems
Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]

- uncertain i.c. \( \psi(0) \)
- investigate \( \lim_{t \to \infty} \psi(t) \)

investigate transients
\[ \sup_{t \geq 0} \| \psi(t) \| \]

exogenous disturbances
\[ \dot{\psi}(t) = F(\psi(t), d(t)) \]

dynamical uncertainty
\[ \dot{\psi} = F(\psi) + \Delta(\psi) \]

“unmodeled dynamics”
- effects not modeled by NS equations
- unmodeled, dynamical wall-flow interactions
- etc.
Analysis of Uncertain Systems

Lyapunov Stability
\[ \dot{\psi} = F(\psi) \]
- uncertain i.c.
  \[ \psi(0) \]
- investigate
  \[ \lim_{t \to \infty} \psi(t) \]

investigate transients
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exogenous disturbances
\[ \dot{\psi}(t) = F(\psi(t), d(t)) \]

dynamical uncertainty
\[ \dot{\psi} = F(\psi) + \Delta(\psi) \]

combinations
\[ \dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t)) \]

increasing uncertainty
Analysis of Uncertain Systems

Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]
- uncertain i.c.
- investigate
\[ \lim_{t \to \infty} \psi(t) \]

investigate transients

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dynamical uncertainty

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increasing uncertainty

Eigenvalue Stability

\[ \dot{\psi} = A\psi \]
- uncertain i.c.
- investigate
\[ \lim_{t \to \infty} \psi(t) \]

linearized versions

combinations

\[ \dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t)) \]
Analysis of Uncertain Systems

Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]
- uncertain i.c.
  \[ \psi(0) \]
- investigate
  \[ \lim_{t \to \infty} \psi(t) \]

investigate transients

\[ \sup_{t \geq 0} \| \psi(t) \| \]

exogenous disturbances

\[ \dot{\psi}(t) = F \left( \psi(t), d(t) \right) \]

dynamical uncertainty

\[ \dot{\psi} = F(\psi) + \Delta(\psi) \]

combinations

\[ \dot{\psi}(t) = F \left( \psi(t), d(t) \right) + \Delta \left( \psi(t), d(t) \right) \]

Eigenvalue Stability

\[ \dot{\psi} = A\psi \]
- uncertain i.c.
  \[ \psi(0) \]
- investigate
  \[ \lim_{t \to \infty} \psi(t) \]

non-normal transient growth

\[ \sup_{t \geq 0} \| \psi(t) \| \]

input-output analysis

\[ \dot{\psi}(t) = A\psi(t) + Bd(t) \]

pseudo-spectrum

\[ \dot{\psi} = (A + \Delta)\psi \]

linearized versions
Analysis of Uncertain Systems

Lyapunov Stability
\[ \dot{\psi} = F(\psi) \]
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exogenous disturbances
\[ \dot{\psi}(t) = F \left( \psi(t), d(t) \right) \]

dynamical uncertainty
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increasing uncertainty

Eigenvalue Stability
\[ \dot{\psi} = A \psi \]
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non-normal transient growth
\[ \sup_{t \geq 0} \| \psi(t) \| \]

input-output analysis
\[ \dot{\psi}(t) = A\psi(t) + Bd(t) \]

pseudo-spectrum
\[ \dot{\psi} = (A + \Delta) \psi \]

combinations
\[ \dot{\psi}(t) = F \left( \psi(t), d(t) \right) + \Delta \left( \psi(t), d(t) \right) \]

Robust Control Theory
\[ \dot{\psi}(t) = (A + B \Delta C) \psi(t) + (F + G \Delta H) d(t) \]
Analysis of Uncertain Systems

**Lyapunov Stability**

\[
\dot{\psi} = F(\psi)
\]
- uncertain i.c. \( \psi(0) \)
- investigate \( \lim_{t \to \infty} \psi(t) \)

**investigate transients**

\[
\sup_{t \geq 0} \| \psi(t) \|
\]

**combinations**

\[
\dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t))
\]

**dynamical uncertainty**

\[
\dot{\psi} = F(\psi) + \Delta(\psi)
\]

**increasing uncertainty**

**Eigenvalue Stability**

\[
\dot{\psi} = A\psi
\]
- uncertain i.c. \( \psi(0) \)
- investigate \( \lim_{t \to \infty} \psi(t) \)

**non-normal transient growth**

\[
\sup_{t \geq 0} \| \psi(t) \|
\]

**input-output analysis**

\[
\dot{\psi}(t) = A\psi(t) + Bd(t)
\]

**pseudo-spectrum**

\[
\dot{\psi} = (A + \Delta)\psi
\]

*great progress in the past 25 years*
Analysis of Uncertain Systems

Lyapunov Stability

\[ \dot{\psi} = F(\psi) \]
- uncertain i.c.
- investigate
  \[ \lim_{t \to \infty} \psi(t) \]

investigate transients

\[ \sup_{t \geq 0} \| \psi(t) \| \]

exogenous disturbances

\[ \dot{\psi}(t) = F(\psi(t), d(t)) \]

dynamical uncertainty

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combinations

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linearized versions

*great progress in the past 25 years

*yet the perception that:
  it's main utility is to help better understand the nonlinear stability problem
Analysis of Uncertain Systems

Lyapunov Stability

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- uncertain i.c.
- investigate

\[ \lim_{t \to \infty} \psi(t) \]

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\[ \dot{\psi}(t) = A\psi(t) + Bd(t) \]

pseudo-spectrum

\[ \dot{\psi} = (A + \Delta)\psi \]

stability theory alone (linear or nonlinear)

insufficient to capture the phenomenology of transition

increasing uncertainty

combinations

\[ \dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t)) \]
The Nature of Turbulence

- Fluid flows are described by deterministic equations
- **OLD QUESTION:** why do fluid flows “look random” at high $R$?
The Nature of Turbulence

A common view of turbulence

\[\bar{u}_0\]

stable

unstable

state space

\[u\]

R

stable

unstable
The Nature of Turbulence

- **A common view of turbulence**

![Diagram of state space showing stable, unstable, and chaotic regions](image-url)
The Nature of Turbulence

- **A common view of turbulence**

![Diagram showing state space with stable, unstable, and chaotic dynamics]

- **Intuitive reasoning:**
  Complex, “statistical looking” behavior $\leftrightarrow$ chaotic dynamics
  “self-sustaining” cycle
The Nature of Turbulence

- **A common view of turbulence**

- **Intuitive reasoning:**
  Complex, “statistical looking” behavior $\leftrightarrow$ chaotic dynamics
  “self-sustaining” cycle

- Assumes NS eqs. with perfect BC, no disturbances or uncertainty (i.e. a closed system)
A driven (open) system

The NS equations act as an *amplifier of ambient uncertainty* at high $R$

*Qualitatively* similar to
The Nature of Turbulence (a mixed picture)

Dynamics of the NS Equations

more typical in bluff body flows

more typical in highly streamlined flows
The Nature of Turbulence (a mixed picture)

Dynamics of the NS Equations

more typical in
bluff body flows

more typical in
highly streamlined flows

There's probably a mixture of both mechanisms in most flows
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