Turbulent flows
in libration-driven ellipsoids

B. Favier

Collaborators: A.J. Barker, M. Le Bars, A. Grannan, J. Aurnou

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Mathematics of Turbulence: Geophysical and Astrophysical Turbulence
Institute for Pure and Applied Mathematics
Outline

1 Introduction

2 Model

3 Results
   • Transition to turbulence
   • Developed turbulence regime

4 Future works and conclusions
Libration as a source of motion in planetary interiors

- Due to gravitational interactions, celestial bodies are affected by various mechanical forcings:
  - Precession
  - Tides
  - Libration

- These forcings can extract a fraction of the huge rotational energy and generate intense motions in the fluid layers
  - Waves
  - Zonal flows
  - Turbulence
Libration as a source of motion in planetary interiors

Due to gravitational interactions, celestial bodies are affected by various mechanical forcings:

- Precession
- Tides
- Libration

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- Waves
- Zonal flows
- Turbulence
Previous studies about libration

- **Zonal flows**: nonlinear interactions in Ekman layers lead to zonal flows both in axisymmetric (Busse 2010, Calkins et al. 2010) and non-axisymmetric containers (Zhang et al. 2011, Grannan et al. 2014).

- **Libration-driven elliptical instability**: triadic resonance of two inertial modes with the base flow (Kerswell & Malkus 1998) shown both experimentally (Noir et al. 2012, Grannan et al. 2014) and numerically (Cébron et al. 2012, Wu & Roberts 2013, Zhang et al. 2013) and generalized to multi-polar deformations (Cébron et al. 2014).
Grannan, Le Bars, Cébron & Aurnou, *Global-scale turbulent flows in libration-driven ellipsoids*, submitted to PoF.
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Librating ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

\[ \Omega(t) = \Omega_0 + \Delta \phi \omega_l \sin (\omega_l t) \]

- No-slip boundary conditions
- Incompressible fluid with constant \( \nu \)
Governing equations and parameters

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + 2 [1 + \epsilon \sin(ft)] \hat{z} \times u = -\nabla p + E \nabla^2 u - \epsilon f \cos(ft) \hat{z} \times r
\]

\[\nabla \cdot \mathbf{u} = 0\]

- Eccentricity: \( \beta = \frac{a^2 - b^2}{a^2 + b^2} = 0.34 \)
- Aspect ratio: \( \frac{c}{b} = 1 \)
- Librating frequency: \( f = \omega_l / \Omega_0 = 4 \)
- Libration amplitude: \( \epsilon = \Delta \phi f = 0.8 \)
- Ekman number: \( E = \frac{\nu}{\Omega_0 a} \approx 10^{-3} - 10^{-5} \)
Governing equations and parameters

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Growth rate of the elliptical instability:

\[ \approx \epsilon \beta - K \sqrt{E} \]
Numerical approach

Spectral element code **Nek5000**

http://nek5000.mcs.anl.gov

- \( E \) isoparametric hexahedral elements
- \( N^3 \) tensor-product Gauss-Lobatto Legendre collocation points
- Algebraic convergence with \( E \)
- Exponential convergence with \( N \)
- 3\textsuperscript{rd} order explicit Adams-Bashforth scheme for convective terms
- 3\textsuperscript{rd} order implicit Backward Differentiation scheme for diffusive and pressure terms
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Example close to threshold: $E = 5 \times 10^{-4}$

Vertical kinetic energy

$$
\bar{A}(t) = \frac{1}{T} \int_{t}^{t+T} A(\tau) d\tau
$$
Example close to threshold: \( E = 5 \times 10^{-4} \)

**Vertical kinetic energy**

![Graph showing vertical kinetic energy over time with a threshold example.](image)

\( E_z(t) \) and \( \dot{E}_z(t) \) are plotted against time, with a threshold at \( 5 \times 10^{-4} \).
Example close to threshold: $E = 5 \times 10^{-4}$

Vertical kinetic energy

$E_z(t)$

$\bar{E}_z(t)$

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Librating ellipsoid

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Example close to threshold: \( E = 5 \times 10^{-4} \)

Vertical kinetic energy

\[ E_z(t), \bar{E}_z(t) \]

Time

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Spectral analysis

![Graph showing spectral analysis with time on the x-axis and power spectrum on the y-axis. The graph displays two curves labeled $E_z(t)$ and $\bar{E}_z(t)$.]

Zonal flow
Base flow

$0 < t < 500$

![Graph showing the power spectrum with frequency on the x-axis and power on the y-axis. The graph highlights a peak at $f = 4$.]
Spectral analysis

- **Time Graph:**
  - $E_z(t)$: Gray line
  - $\bar{E}_z(t)$: Red line
  - Time axis: $0 \leq t < 500$ and $900 < t < 1400$

- **Power Spectrum:**
  - Zonal flow
  - Inertial mode
  - Base flow
  - Frequency axis $f$

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Spectral analysis

The figure shows a plot of $E_z(t)$ and $\bar{E}_z(t)$ versus time. The plots indicate a significant change in the energy distribution over time, with peaks at certain time intervals.

The right side of the figure displays a power spectrum with frequency $f$ on the x-axis and power density on the y-axis. The spectrum is divided into three regions:

- $0 < t < 500$
- $900 < t < 1400$
- $3000 < t < 3500$

Key features include:

- Zonal flow
- Inertial mode
- Base flow
Spectral analysis

\[ \hat{u}(\omega_f, x) \approx \int_{t_i}^{t_f} u(x) e^{i\omega_f(t-t_i)} \, dt \]
High-order spectral analysis

In order to detect quadratic nonlinearities, we compute the bicoherence of the time signals:

\[ b^2(k, l) = \frac{\left| \sum_{i=1}^{N} u_i(k) u_i(l) u_i^*(k + l) \right|^2}{\sum_{i=1}^{N} |u_i(k) u_i(l)|^2 \sum_{i=1}^{N} |u_i(k + l)|^2} \]
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\]
Decreasing the Ekman number

$E = 5 \times 10^{-4}$
Decreasing the Ekman number

\[ E = 5 \times 10^{-4} \]

\[ E = 3.5 \times 10^{-4} \]
Decreasing the Ekman number

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\[ E = 3.5 \times 10^{-4} \]
\[ E = 2 \times 10^{-4} \]
Decreasing the Ekman number

\begin{align*}
E &= 5 \times 10^{-4} \\
E &= 3.5 \times 10^{-4} \\
E &= 2 \times 10^{-4} \\
E &= 10^{-4}
\end{align*}
Decreasing the Ekman number

\[ E = 5 \times 10^{-4} \]
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\[ E = 5 \times 10^{-5} \]

\[ E_z(t) \]

\[ E_z = 5 \times 10^{-4} \]
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Power spectra

Base flow

Inertial mode

Power spectrum

Frequency

Inertial mode

Base flow

$E = 5 \times 10^{-4}$

$E = 2 \times 10^{-4}$

$E = 10^{-4}$

$E = 5 \times 10^{-5}$

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Librating ellipsoid

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2nd order structure function: $S_2(l) = \left\langle \left[ \hat{\mathbf{r}} \cdot (\mathbf{u}(x) - \mathbf{u}(x + l\mathbf{r})) \right]^2 \right\rangle$

![Graph showing $S_2(l)$ vs $l$ with linear and $l^{2/3}$ fits, and $E = 10^{-4}$]
Decaying simulations

After reaching a quasi-steady state, the libration amplitude is gradually reduced.
Volume rendering of the enstrophy for $E = 10^{-4}$
Lagrangian particles for $E = 10^{-4}$
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Varying the librating frequency: \( f = 2.4 \)

Similar behaviours observed by Grannan et al. 2014
Spherical inner core

\[ E_z(t) \]

\[ \eta = 0 \]
\[ \eta = 0.2 \]
\[ \eta = 0.4 \]

Time

Librating ellipsoid
Dynamo action?

![Graph showing the evolution of magnetic field strength over time for different Pm values. The graph compares the magnetic field strength over time for Pm = 1 and Pm = 2.]
Conclusions

- At low Ekman numbers, the saturation of the instability leads to turbulence. What is the difference between the sustained and intermittent regime? (similarity with shearing-box simulations? see Barker & Lithwick 2013).

- Kinematic dynamo action is very likely (see Wu & Roberts 2013) although this remains to be properly checked in our case! Saturation of such a libration-driven dynamo?

- Stress-free simulations, extension to other mechanical forcings (in particular tides) and of course comparisons with the experiment.
Thank you for your attention!