Large Scale Dynamos at high Rm Do they work and can we get a statistical theory?

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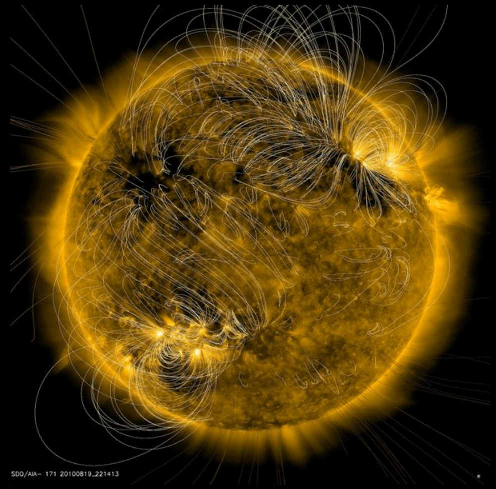
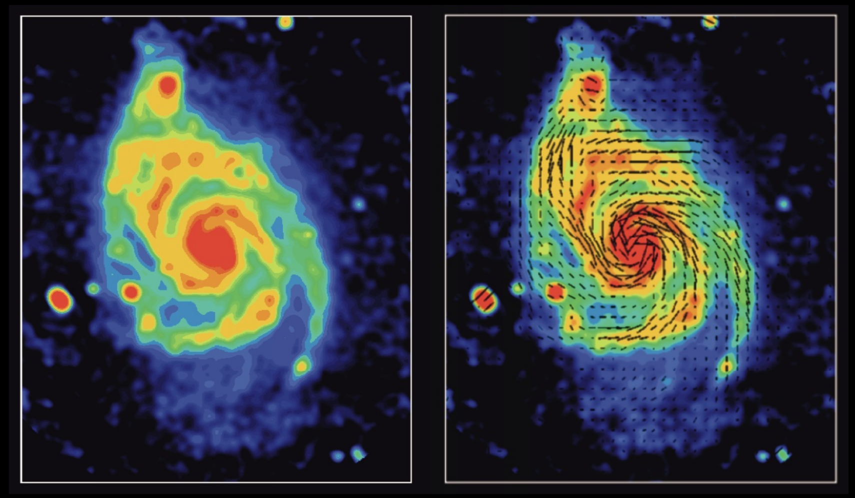
Yes (well perhaps) and Maybe

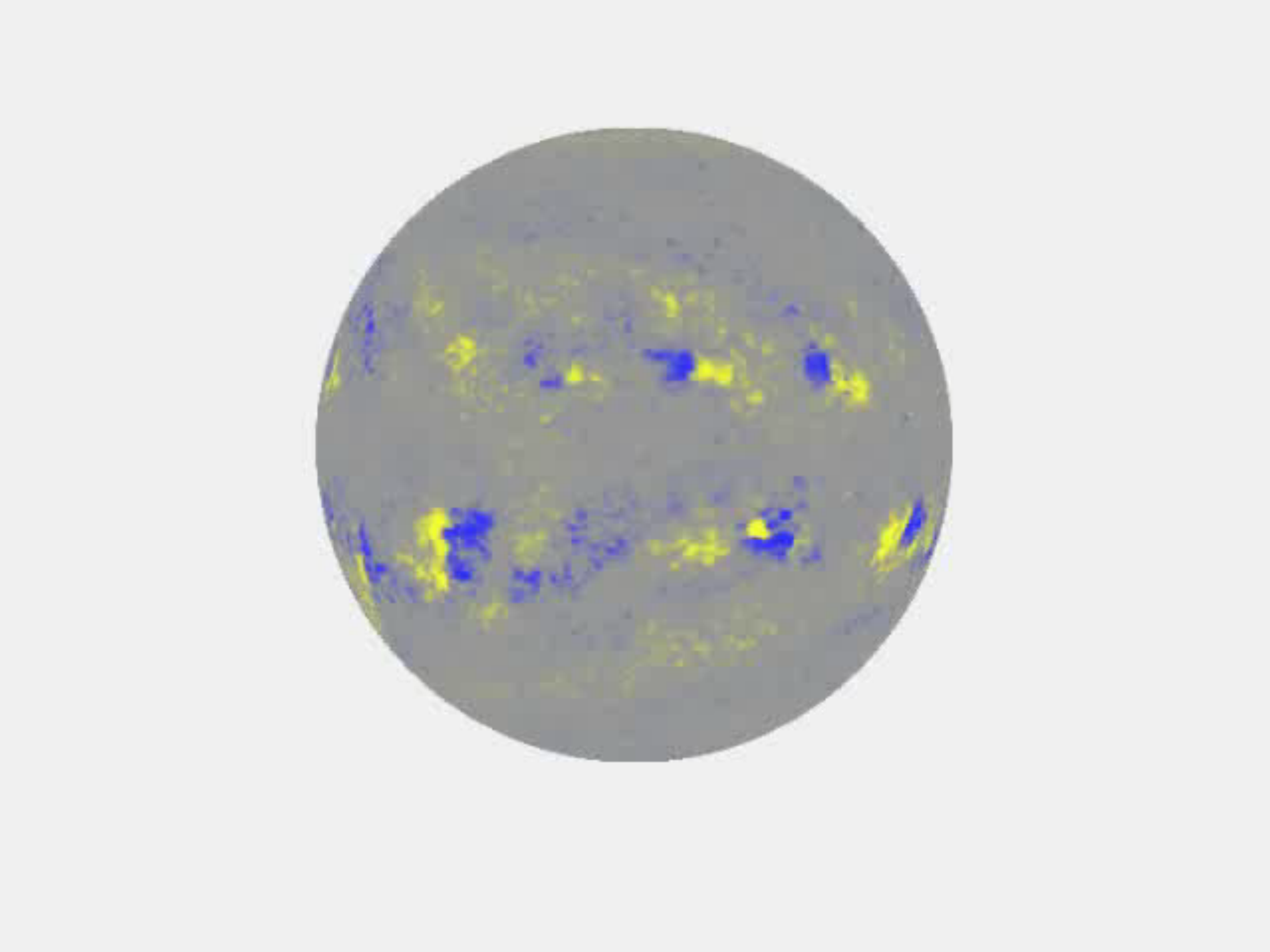


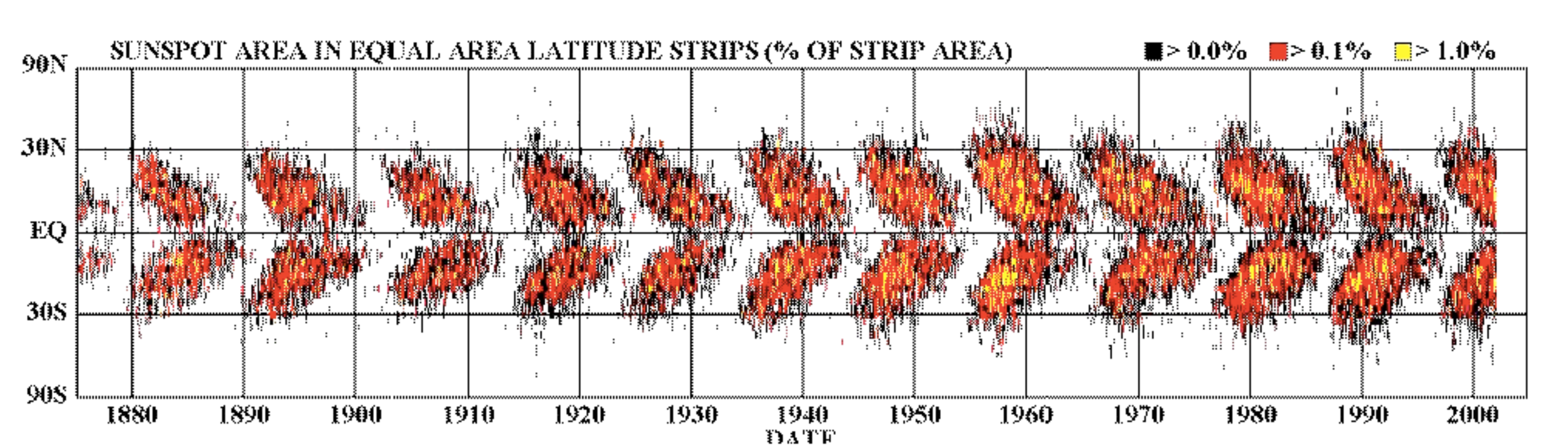


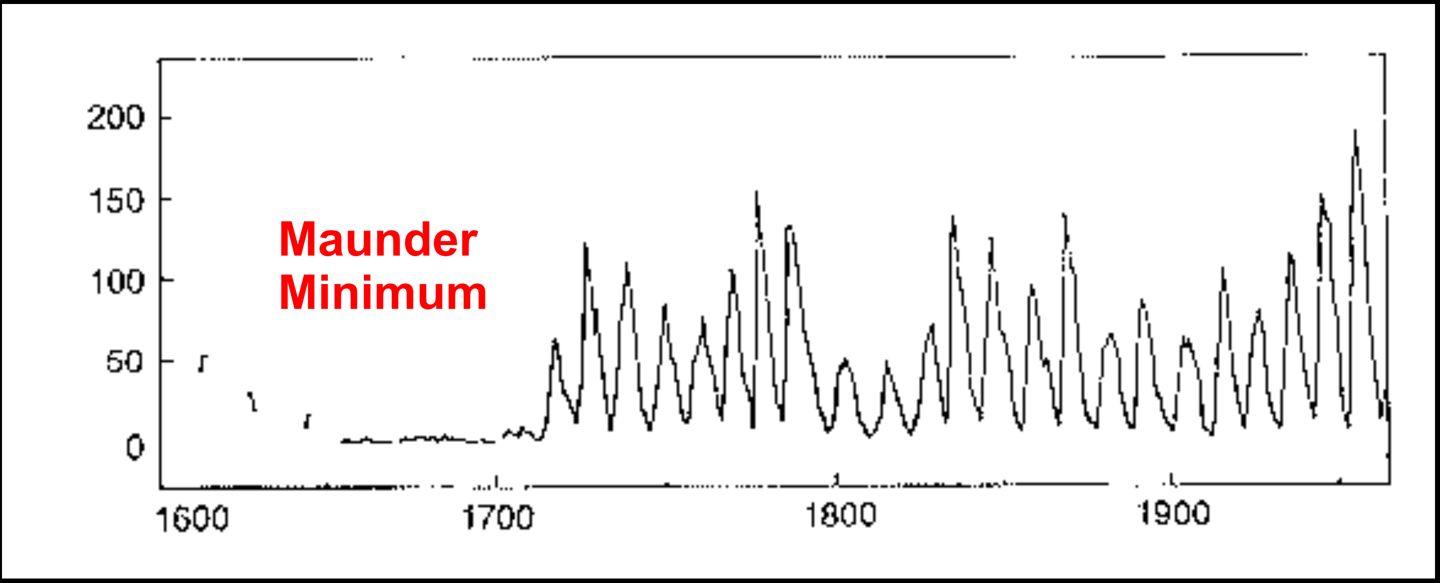
- ***Question:*** How can we understand the dynamo properties of turbulent flows with a large range of scales
  - Random fluctuations
  - Coherent structures
- ***Question:*** How can an astrophysical object such as a star or galaxy generate a systematic (large-scale) magnetic field *at high Rm*? How can it overcome its tendency to be dominated by fluctuations at the small scales?

Using our insight from answering this can we derive a (statistical?) theory that describes these interactions.

Courtesy
D. Hathaway

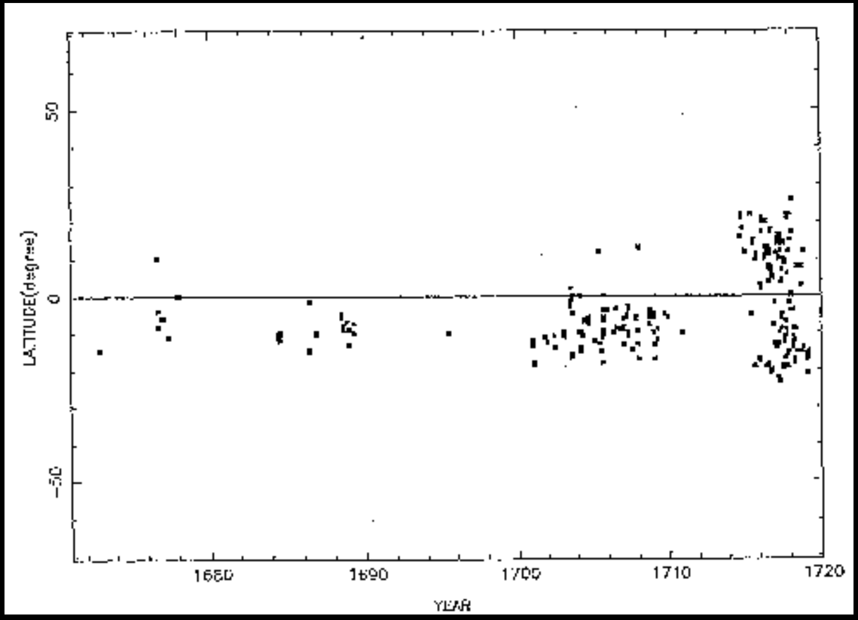






SUNSPOT NUMBER: last 400 years

Modulation of basic cycle amplitude (some modulation of frequency)
Gleissberg Cycle: ~80 year modulation
MAUNDER MINIMUM: Very Few Spots , Lasted a few cycles

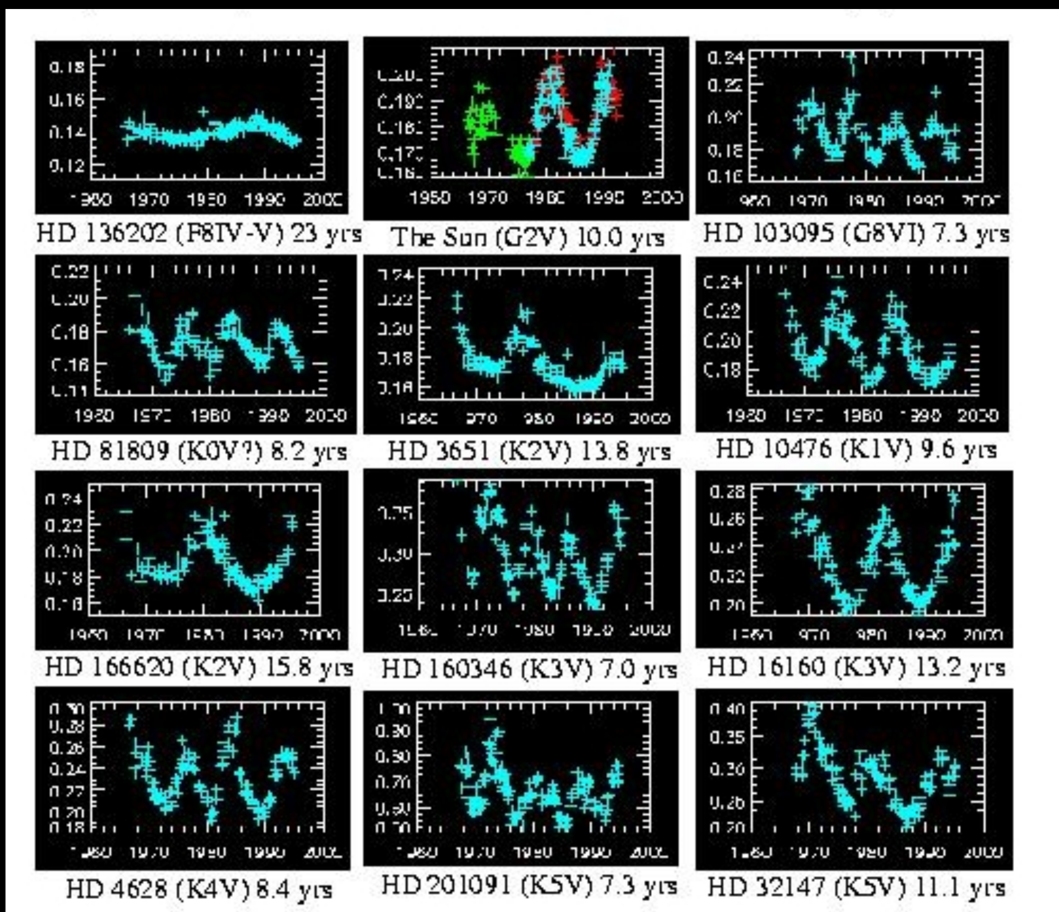


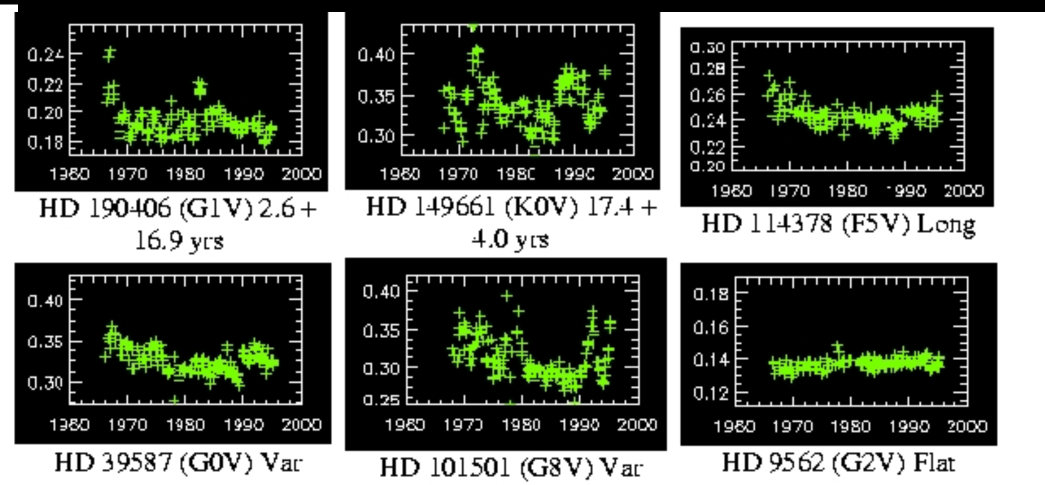
Observations: Stellar (Solar-Type Stars)

Stellar Magnetic Activity can be inferred by amount of Chromospheric Ca H and K emission

Mount Wilson Survey (see e.g. Baliunas)

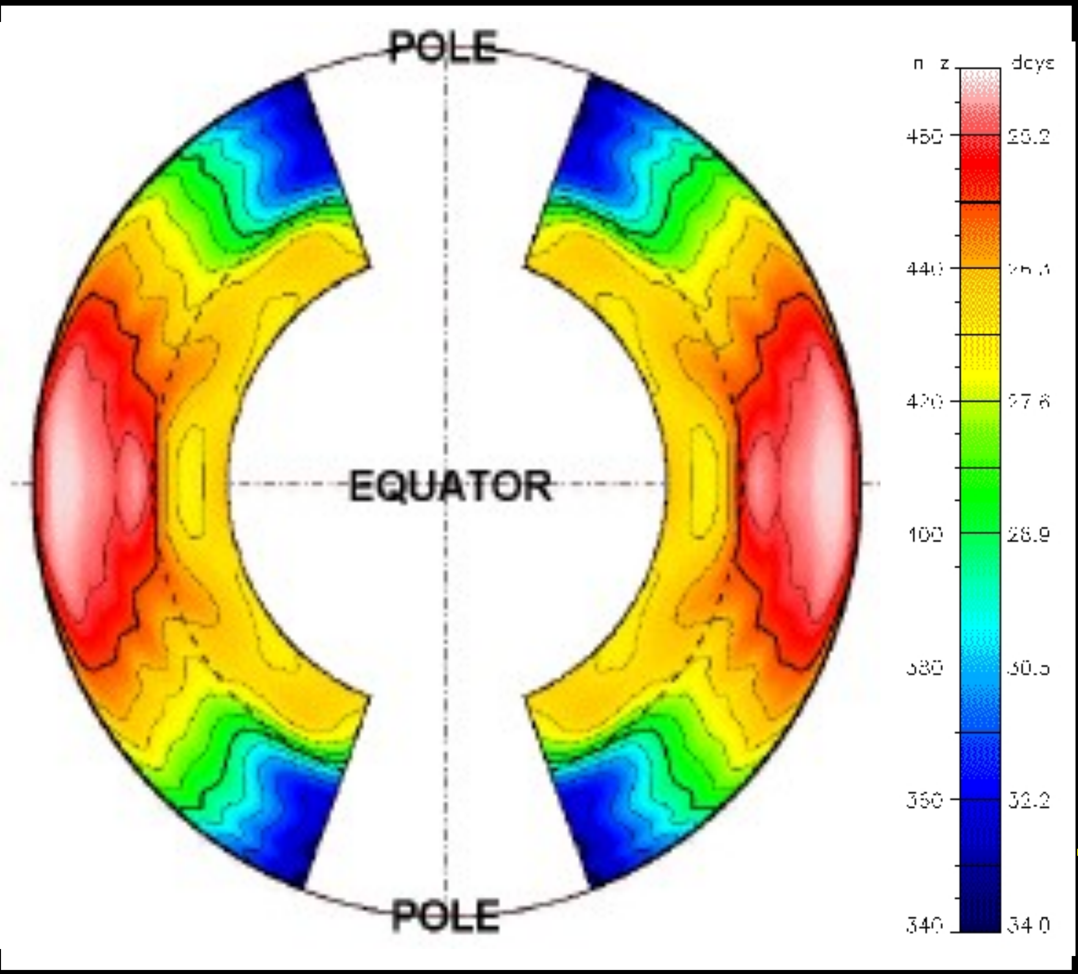
Solar-Type Stars show a variety of activity.





Small Scale Dynamos

The Large-Scale Solar Dynamo



Tachocline instabilities:
Knobloch & Spruit (1982) (GSF et al)
Gilman & Fox (1997)   (Joint diff rotn mag field)

Shear is very important

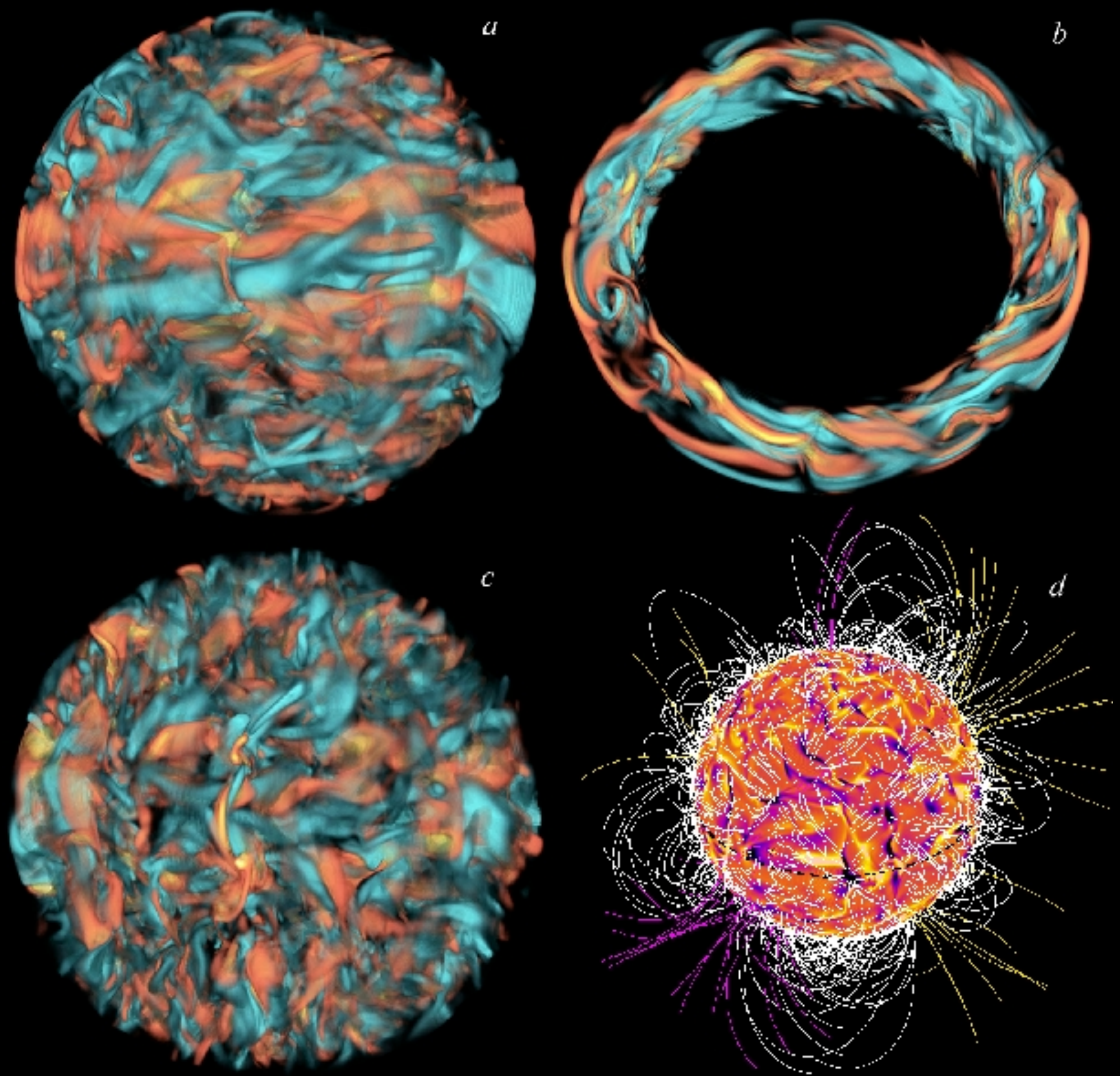
Helioseismology shows the internal structure of the Sun.

Surface Differential Rotation is maintained throughout the Convection zone

Solid body rotation in the radiative interior

Thin matching zone of shear known as the tachocline at the base of the solar convection zone (just in the stable

Large-Scale Computation



cf Toomre, Miesch, Brun, Browning, Brown (ASH code)

Basics for the Sun

Dynamics in the solar interior is governed by the following equations of MHD

INDUCTION

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \qquad (\nabla.\mathbf{B} = 0),$$

MOMENTUM

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u}\right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} + \mathbf{F}_{viscous} + \mathbf{F}_{other},$$

CONTINUITY

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0,$$

ENERGY

$$\frac{D(p\rho^{-\gamma})}{Dt} = \text{loss terms},$$

GAS LAW

$$p = R\rho T.$$

Basics for the Sun

	BASE OF CZ	PHOTOSPHERE
$Ra \equiv g\Delta\nabla d^4 / \nu\chi H_P$	10^{20}	10^{16}
$\mathrm{Re} \equiv UL/\nu$	10^{13}	10^{12}
$\boxed{Rm \equiv UL/\eta}$	$\boxed{10^{10}}$	$\boxed{10^{6}}$
$\mathrm{Pr} = \nu/\chi$	10^{-7}	10^{-7}
$\beta = 2\mu_0 p / B^2$	10^{5}	1
$Pm = \nu/\eta$	10^{-3}	10^{-6}
$M = U/c_s$	10^{-4}	1
$Ro = U/2\Omega L$	0.1-1	10^{-3}-0.4

Ossendrijver (2003)

The Puzzle…

Dynamo models at moderate rotation rates…

At low Rm, or for short correlation time turbulence can get large-scale systematic magnetic fields.

As Rm is increased (still not nearly close to astrophysical values) systematic magnetic field breaks down and small-scale dynamos emerge.

<u>Can we fix this?</u>

Basics for the Sun

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$$p = R\rho T.$$

Small-Scale Dynamos

Small-scale dynamos rely on *chaotic stretching* and reinforcement of the field (see e.g. Childress & Gilbert 1995)

- More coherent (in time) the velocity the better the stretching (usually)

Any sufficiently chaotic flow will tend to generate magnetic field on the resistive scale.

Interesting questions do remain…

- e.g. Low Pm problem.
  - What happens when magnetic field dissipates in inertial range of the turbulence?
- Coherent structures versus random flows

Large-Scale Dynamos

Starting point is the magnetic induction equation of MHD:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},}$$

where **B** is the magnetic field, **u** is the fluid velocity and η is the magnetic diffusivity (assumed constant for simplicity).

Assume scale separation between large- and small-scale field and flow:

$$\boxed{\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{U} = \mathbf{U}_0 + \mathbf{u},}$$

where **B** and **U** vary on some large length scale L, and **u** and **b** vary on a much smaller scale l.

$$\boxed{\langle \mathbf{B} \rangle = \mathbf{B}_0, \quad \langle \mathbf{U} \rangle = \mathbf{U}_0,}$$

where averages are taken over some intermediate scale l « a « L.

For simplicity, ignore large-scale flow, for the moment.

Induction equation for mean field:

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times \mathcal{E} + \eta \nabla^2 \mathbf{B}_0$$

where mean emf is $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$

This equation is exact, but is only useful if we can relate $\mathcal{E}$ to $\mathbf{B}_0$

Consider the induction equation for the fluctuating field:

$$\boxed{\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b},}$$

Where $\mathbf{G} = \mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle$. “pain in the neck term”

Traditional approach is to assume that the fluctuating field is driven solely by the large-scale magnetic field.

i.e. in the absence of $\mathbf{B_0}$ the fluctuating field decays.

i.e. No small-scale dynamo (not really appropriate for high Rm turbulent fluids)

Under this assumption, the relation between $\mathbf{b}$ and $\mathbf{B}_0$ (and hence between $\mathcal{E}$ and $\mathbf{B}_0$) is linear and homogeneous.

Postulate an expansion of the form:

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \ldots$$

where α_{ij} and β_{ijk} are pseudo-tensors, determined by the statistics of the turbulence.

Simplest case is that of isotropic turbulence, for which $\alpha_{ij} = \alpha\delta_{ij}$ and $\beta_{ijk} = \beta\varepsilon_{ijk}$.
Then mean induction equation becomes:

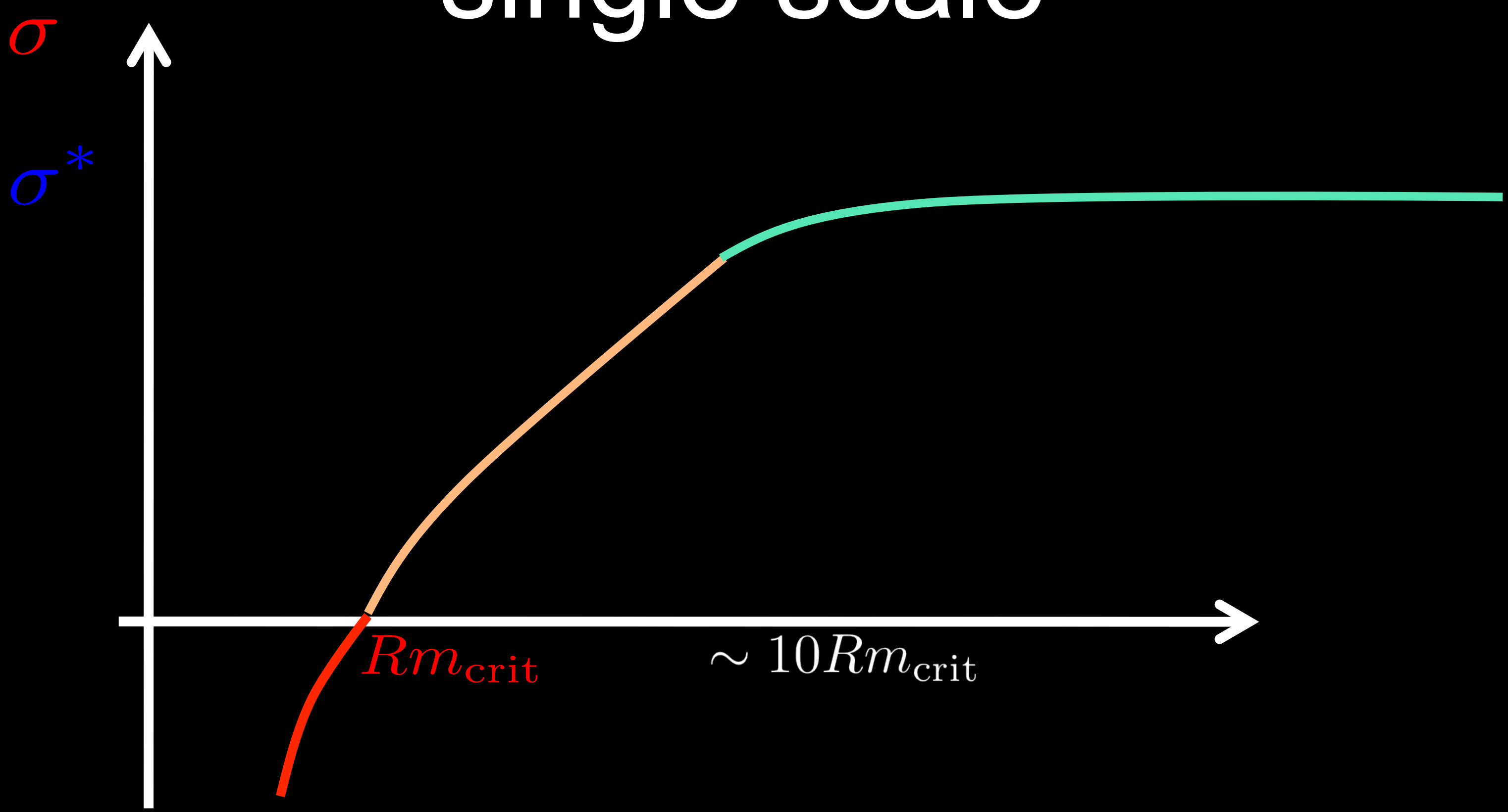
$$\boxed{\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\alpha \mathbf{B}_0) + (\eta + \beta)\nabla^2 \mathbf{B}_0.}$$

α: regenerative term, responsible for large-scale dynamo action.

Since $\mathcal{E}$ is a polar vector whereas $\mathbf{B}_0$ is an axial vector then α **can be non-zero only for turbulence lacking reflexional symmetry** (i.e. possessing handedness).

β: turbulent diffusivity.

Growth-rates for dynamos at a single scale



A *fast* dynamo is has an asymptotic growth rate as Rm gets large

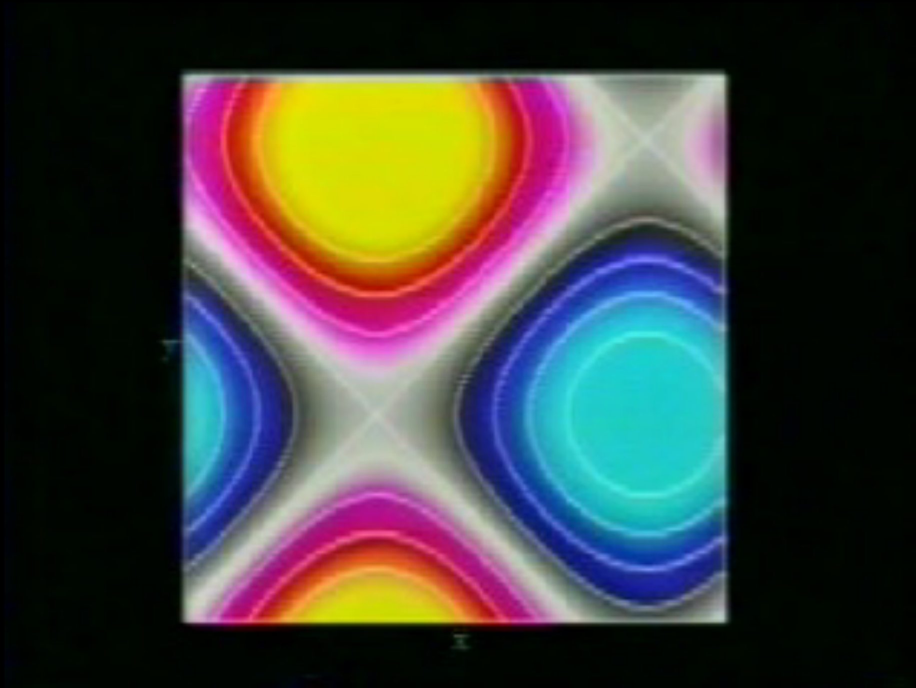
A *quick* dynamo is one which reaches this maximum growth rate close to $Rm_{\rm crit}$ (Tobias & Cattaneo, JFM, 2008)

We define high Rm to be well into "the green zone"

- Certainly the case for astrophysical flows
- Not usually true for numerical simulations (if we are talking about small-scale flows)

Dynamos at a single scale

Galloway & Proctor, (1992) Nature



$$\mathbf{u}(x, y, t) = (\psi_y, -\psi_x, w)$$

$$\psi = \sqrt{3/2}\,(\sin(x + \epsilon\cos\omega t) + \cos(y + \epsilon\sin\omega t))$$

$$w = \psi$$

$$\mathbf{B} = \hat{\mathbf{B}}(x, y)e^{ik_z z + \sigma t}$$

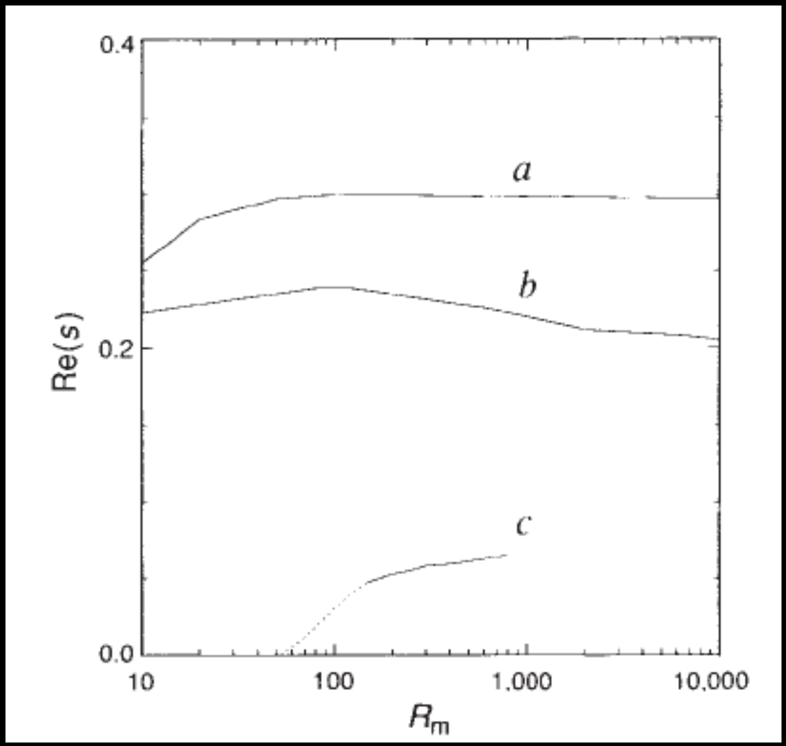
For a velocity field imposed at a finite scale

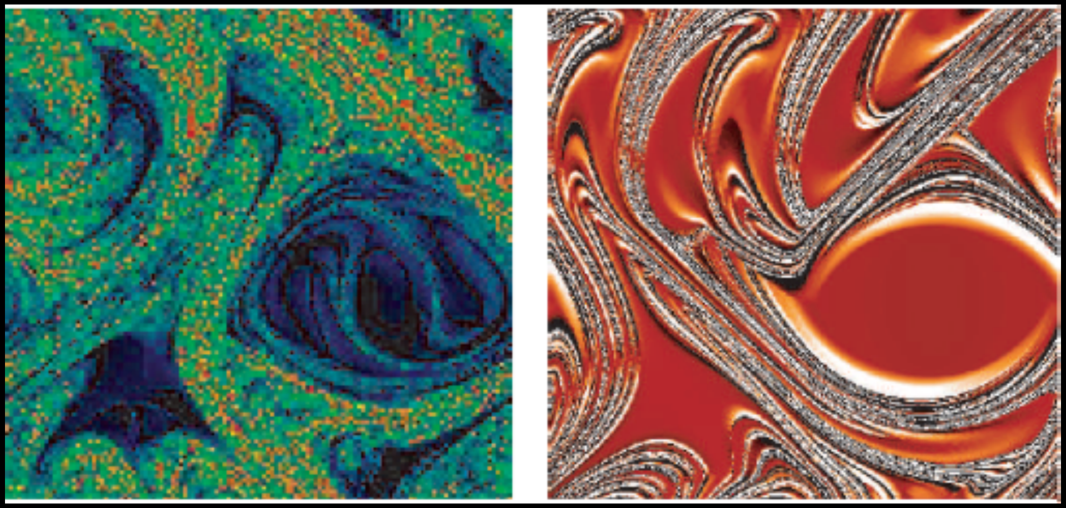
- Competition between stretching and diffusion.
- If stretching strong enough and coherent enough get exponential growth of field.
- Field is usually amplified at small scales
  - Resistive scale

$$l_B \sim Rm^{-1/2}$$

Note:This flow lacks reflexional symmetry (helical)
And should be a good large-scale dynamo

Dynamos at a single scale





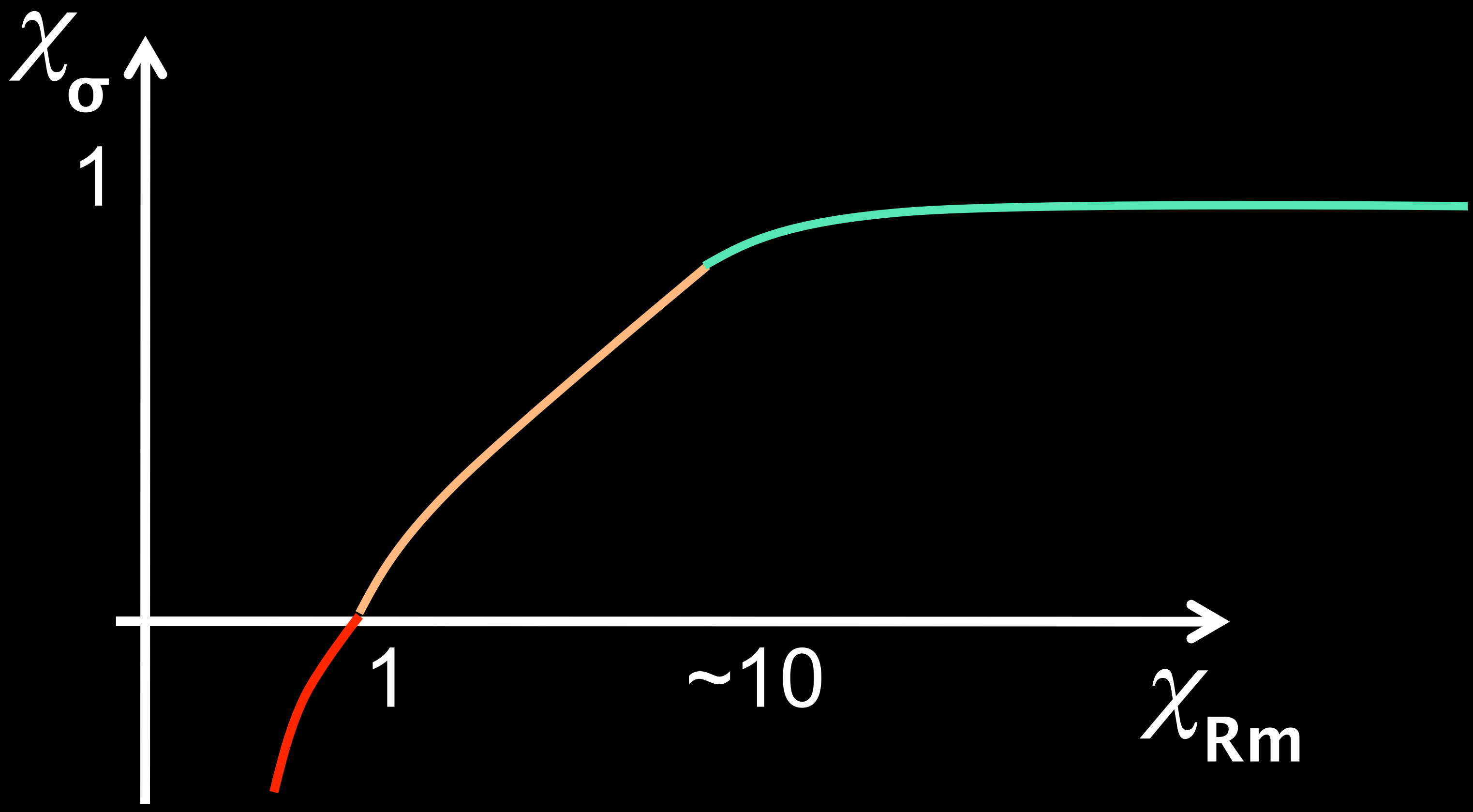
Field is amplified on local turnover time of the flow

- Independent of diffusion as Rm gets large (fast)

Relies on

- Chaotic stretching of fieldlines by velocity
  - Measured by the finite time Lyapunov exponent
- Not too much cancellation
  - measured by the cancellation exponent

Two important Ratios

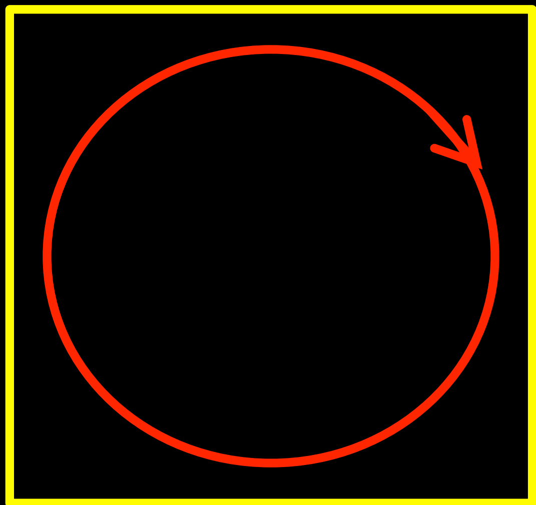


$\chi_\sigma = \sigma/\sigma_\star$

$\chi_{Rm} = Rm/Rm_{\rm crit}$

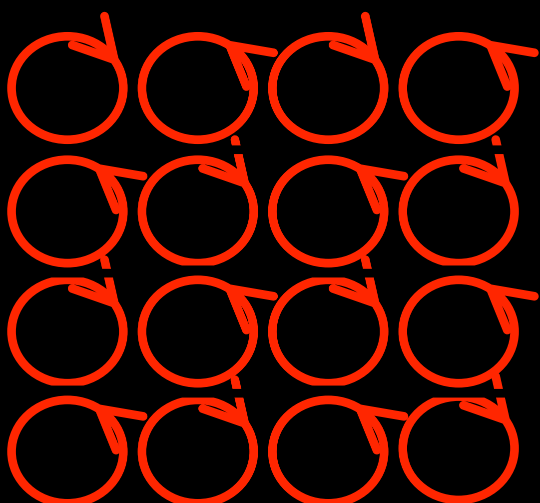
Dynamos at two scales

Cattaneo & Tobias, Physics of Fluids 2005



$$Rm = \frac{UL}{\eta}$$

$$\tau = \frac{L}{U}$$



$$Rm_k = \frac{U_k}{\eta k}$$

$$\tau_k = \frac{1}{kU_k}$$

Can get very interesting dynamics at high Rm

- Mode crossing between modes driven by large-scale flows and small-scale flows

Suppression of growth of small-scale field by large-scale flow at high Rm

- Shear enhanced dissipation
- Reduced stretching
  - Decrease of Lyapunov exponents
- Enhanced cancellation

The effect of a large-scale flow at high enough Rm is to decrease the efficiency of a small-scale dynamo.

Large-scale versus small-scale

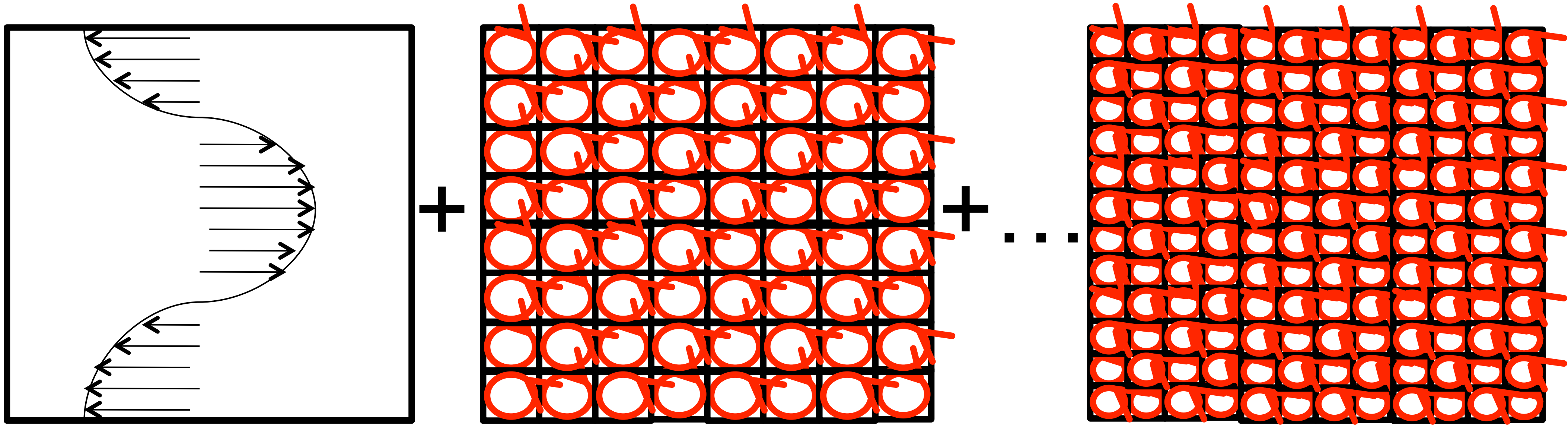
- All the dynamos described above have the ingredients required to be a large-scale dynamo
  - Lack of reflexional symmetry in the flow
  - Leads to the generation of a mean EMF

$$\mathcal{E} = \langle \mathbf{u}' \times \mathbf{b}' \rangle$$

- However at high R_{Rm} (in the green zone - and even in the amber zone) the large-scale magnetic field generated by this EMF is completely dominated by the small-scale fluctuations provided by the small-scale dynamo (cf Cattaneo & Hughes 2006)
- One idea is to use a shear flow to “boost” the EMF (and indeed the dynamo growth) via one of many effects (shear-current effect etc) (see e.g. Yousef et al 2008, Käpylä & Brandenburg 2009, Sridhar & Singh 2010, Hughes & Proctor 2013)
- Though see Courvoisier & Kim (2009)
- *An alternative is to use the shear to control the fluctuations*

High Rm effects of shear

Tobias & Cattaneo (2013, Nature) Cattaneo & Tobias (2014 ApJ)



$Rm_s \approx 0 \to 10^5$

$$Rm_k = \frac{U_k}{k\eta} \approx 2500$$

Need to get to very high Rm (so growth-rate is asymptotic for small-scale flow.)

Very hard to do in 3D flow.

Resolutions up to 4096^2

Use multi-scale generalisation of the GP/CT2005 flow (2.5 D)

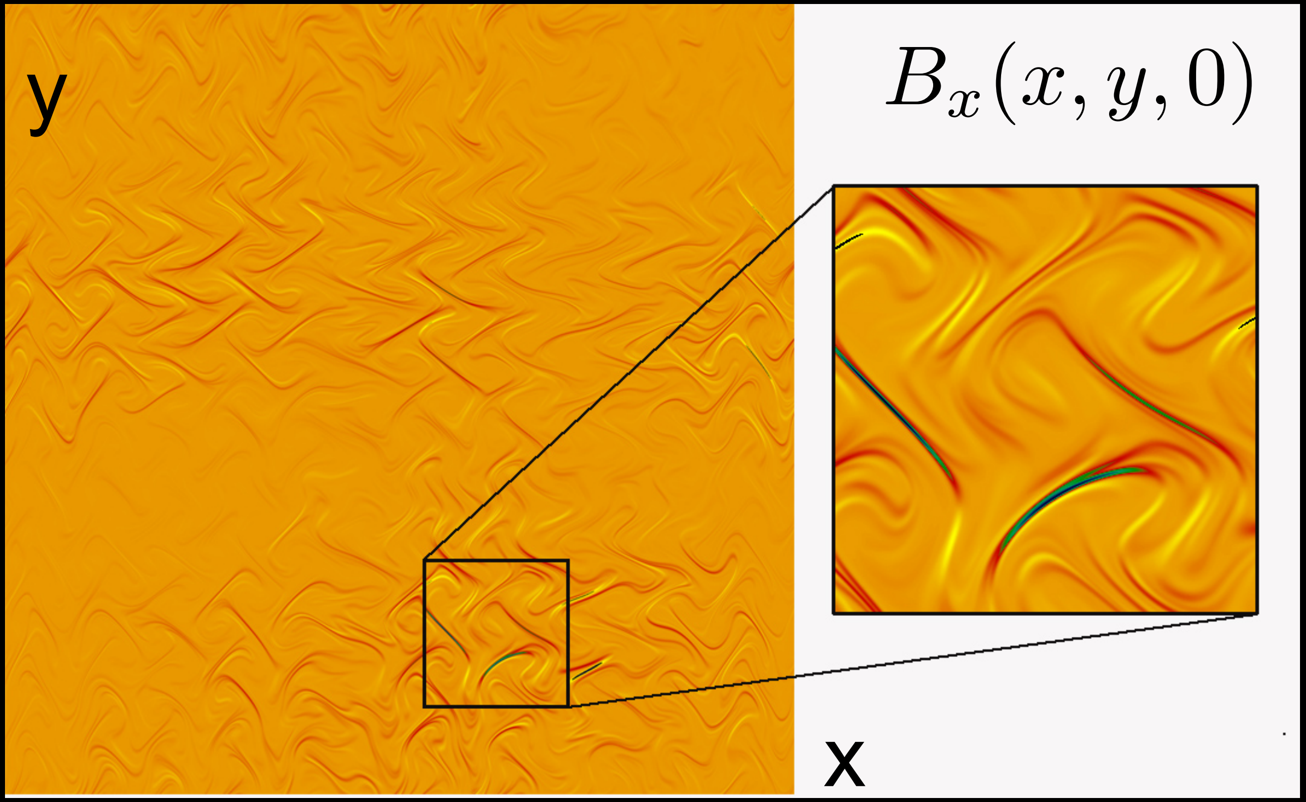
Velocity amplitude decreases with scale; shear rate and turnover frequency increase with scale

Scale dependent renewal time

comparable with local turnover time (in asymptotic regime)

much shorter than local turnover time (poor dynamo)

No shear: long correlation time



Tobias & Cattaneo (2013)

Computations using UK MHD Consortium Machine at University of Leeds

- ***Great small-scale dynamo (no real surprise)***

Filamentary field with

Length comparable to scale of velocity

Width controlled by diffusion $\propto Rm^{-1/2}$

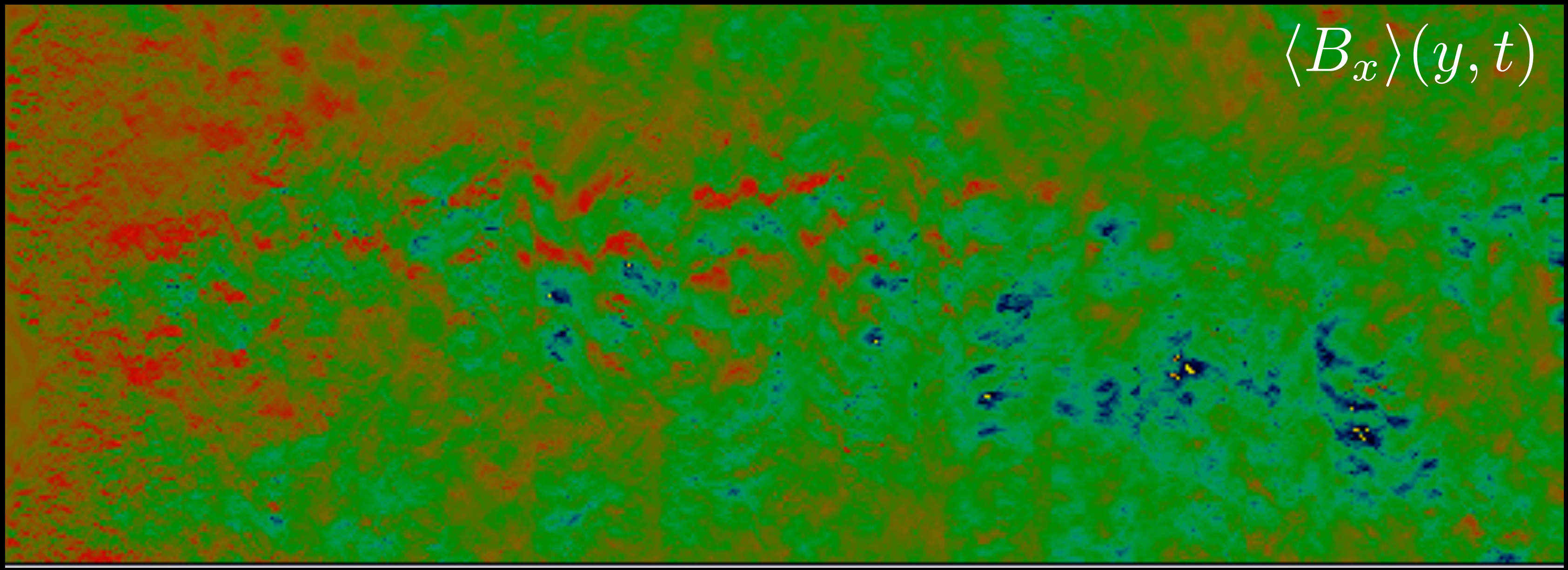
Overall pattern changes on the turnover time

Comparable with correlation time

No shear: Long correlation time

Tobias & Cattaneo (2013)

y

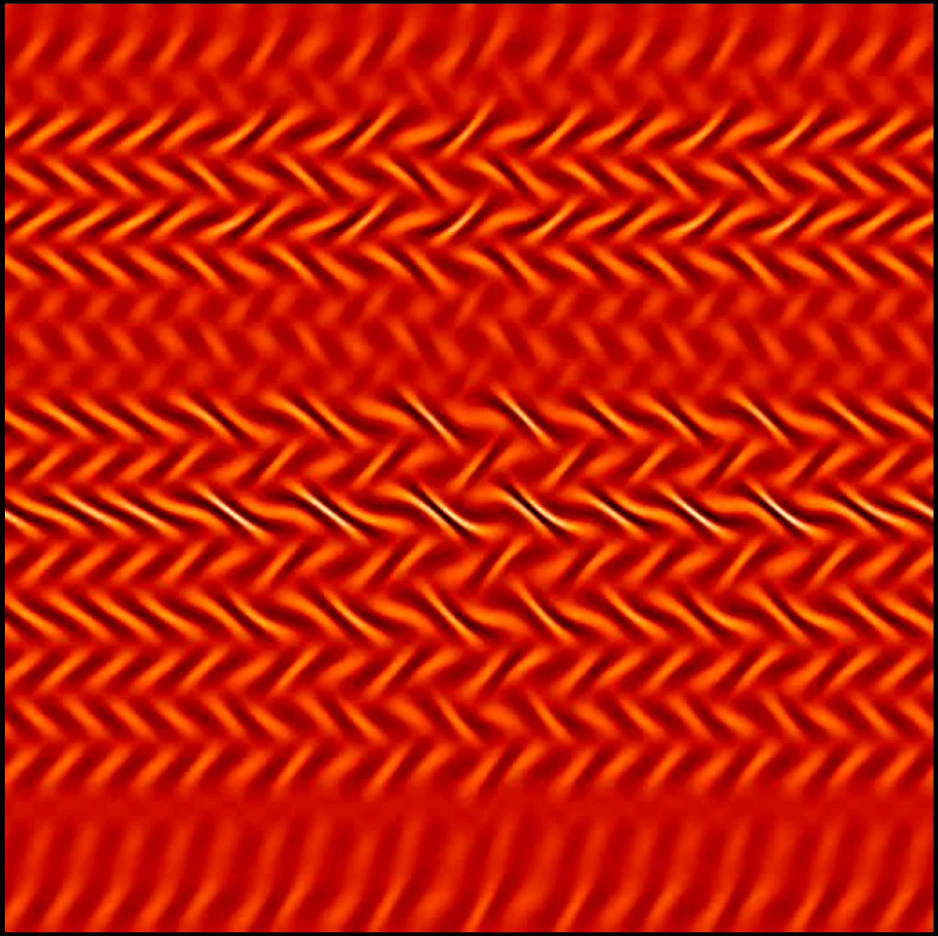


time

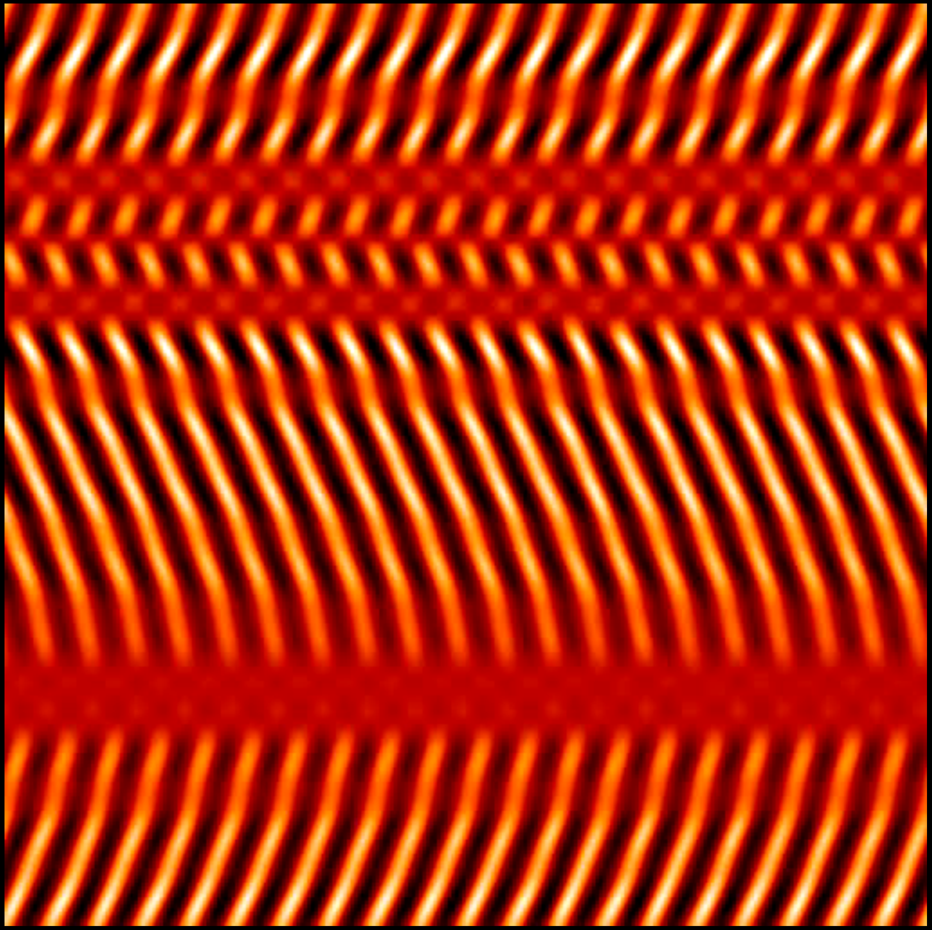
Exponential growth removed…

- No systematic large-scale behaviour
  - E.g. average B_x over x and plot as a function of y and t
- Can also construct a velocity field with ***no net helicity*** when averaged over time
  - This has comparable growth-rate as a small-scale dynamo
  - Similar stretching, similar cancellation, similar pictures…

Add some shear; long correlation time



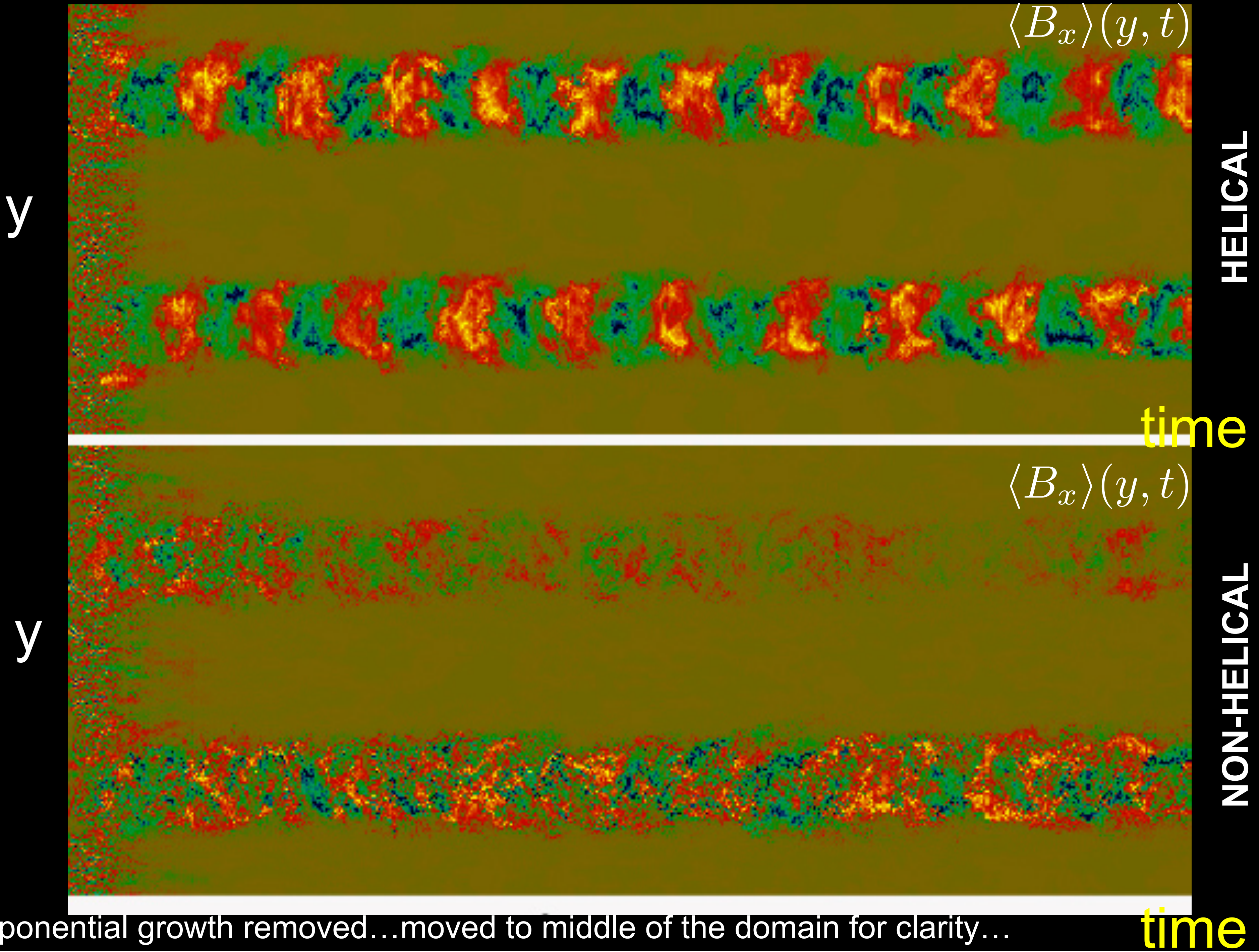
No net helicity



Helicity

With shear…

Tobias & Cattaneo (2013)



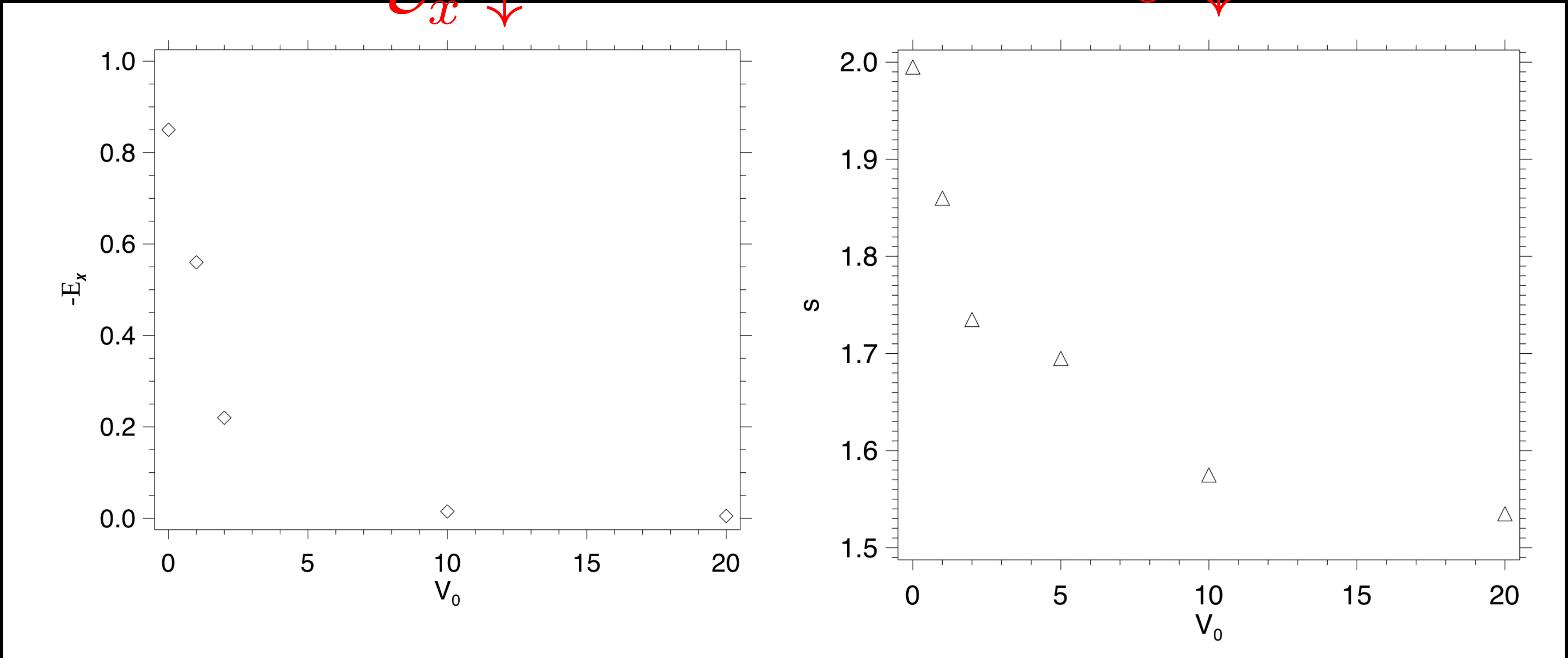
Exponential growth removed…moved to middle of the domain for clarity…

Mechanism? Long correlation time

Is it boosting the EMF?

$\mathcal{E}_x \downarrow$ $\sigma \downarrow$

Tobias & Cattaneo (2013)



Nope it is suppressing the small-scale dynamo (CT05)

This only happens for high enough Rm

Flows in the green zone!

Really for efficient enough small-scale dynamos

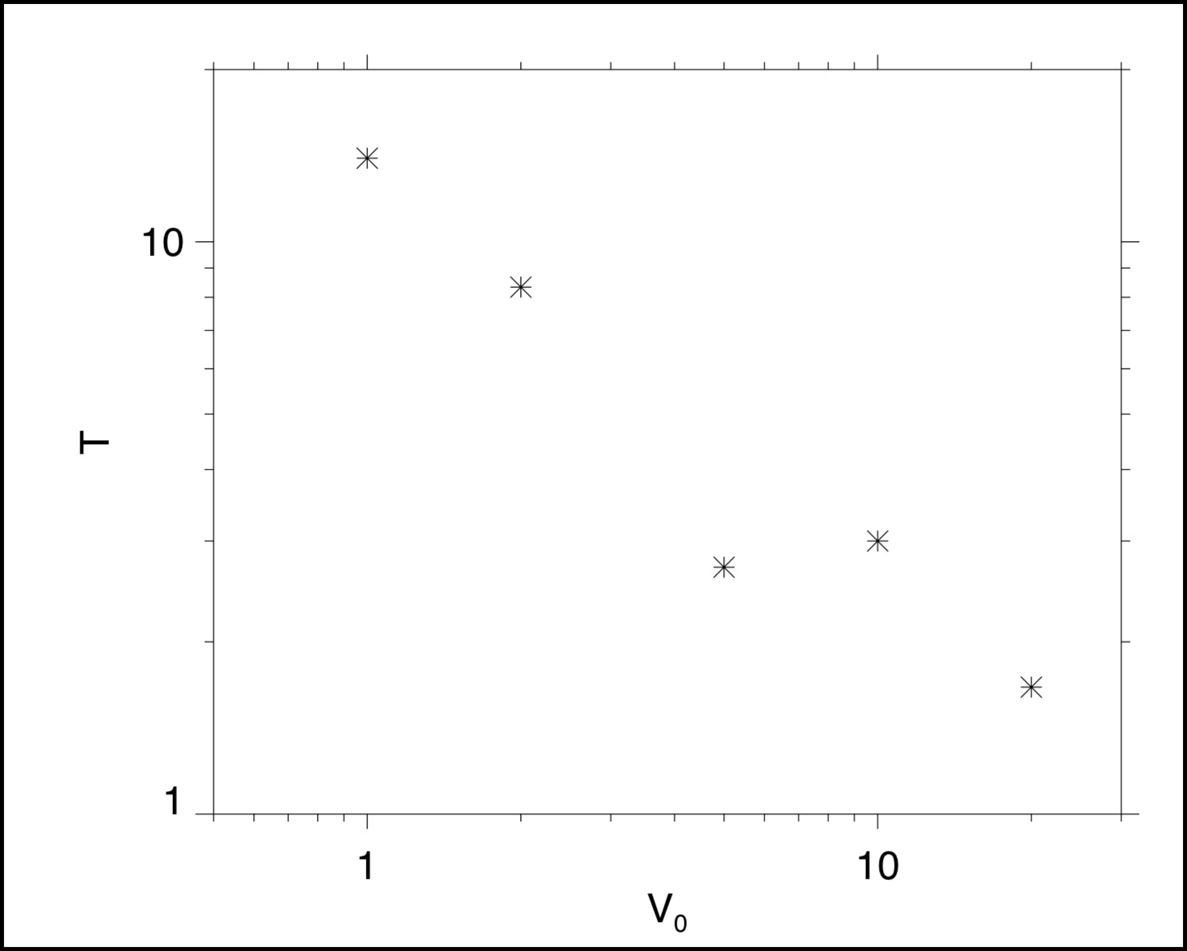
Are these really dynamo waves?

Yes, they have all the right properties

Propagate in correct direction

Swap direction with direction of shear

Period Decreases with increasing shear.



Tobias & Cattaneo (2013)

Short correlation time

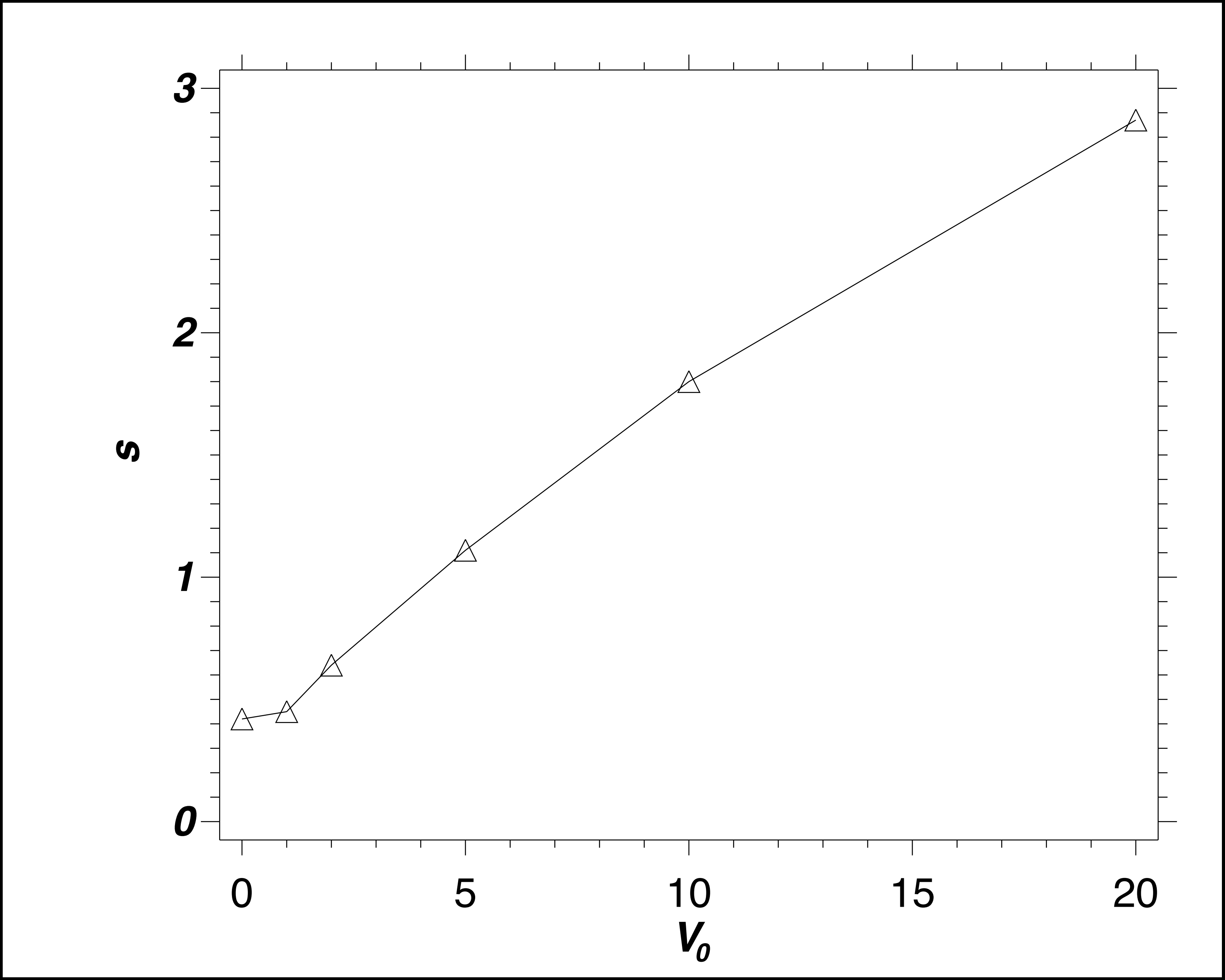
Of course reducing the correlation time reduces the efficiency of the small-scale dynamo (at fixed Rm)

so χ_{Rm} is decreased (even though Rm is still very high)

- This flow is a good mimic of current 3D numerical dynamo calculations
  - These small-scale flows are not usually optimised for small-scale dynamo action and are usually run at Rm close to onset - χ_{Rm} ***order unity***
  - Not usually in the asymptotic regime (green zone)
- Growth-time is again on local turnover time but gets longer as correlation time decreases.

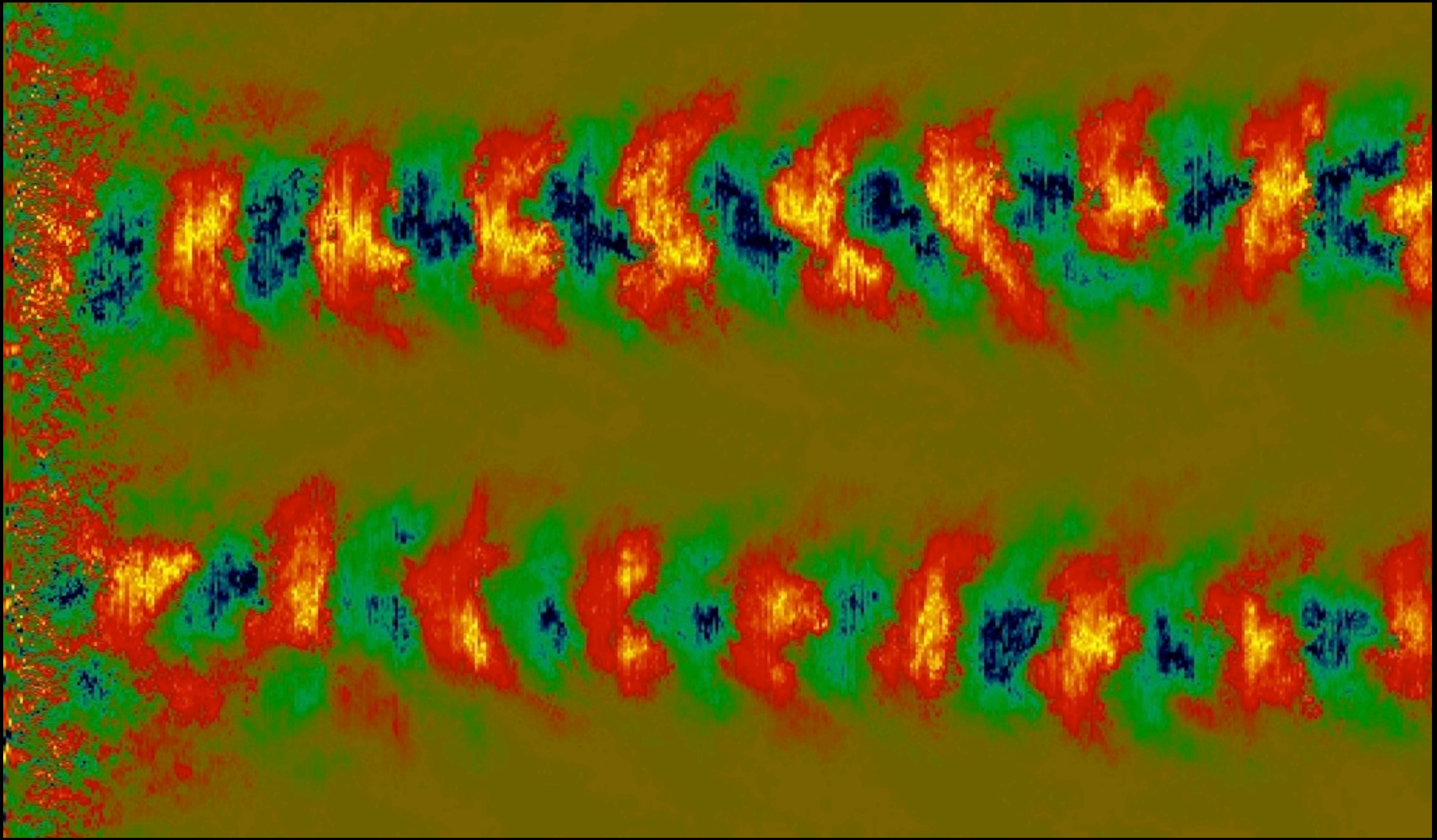
Cattaneo & Tobias (ApJ 2014)

Short correlation time



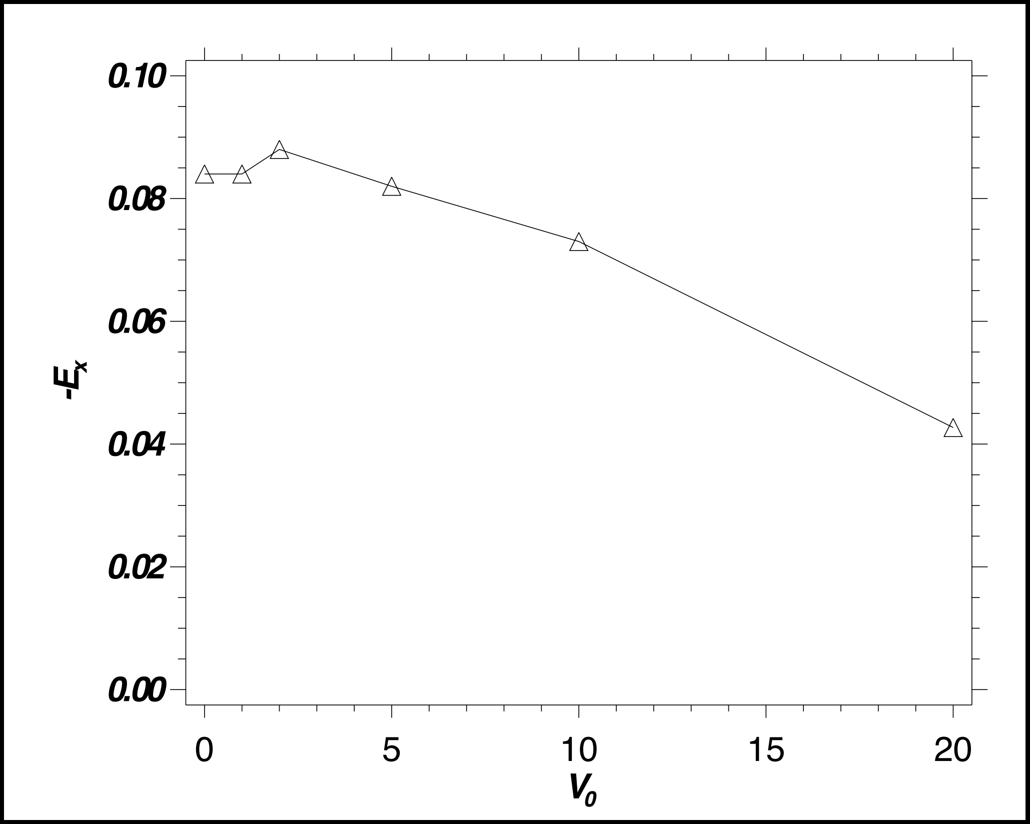
Cattaneo & Tobias (ApJ 2014)

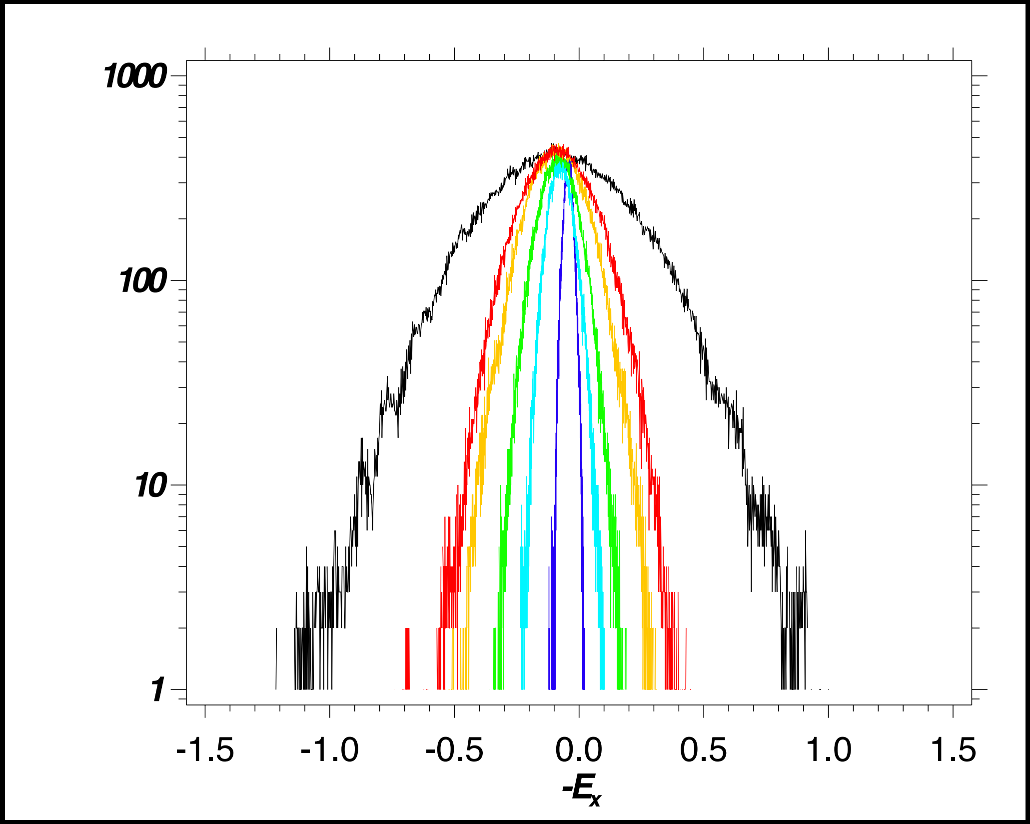
Short correlation time



Cattaneo & Tobias (ApJ 2014)

Short correlation time





Variance of EMF is a simple function of shear rate
(Tobias & Cattaneo 2014)

Cattaneo & Tobias (ApJ 2014)

Analysing Interactions

For geophysical and astrophysical (hydrodynamic) flows progress can be made by directly calculating the statistics of the flows via cumulant expansions

Direct Statistical Simulation

e.g. formation of jets (oceans, Jupiter)

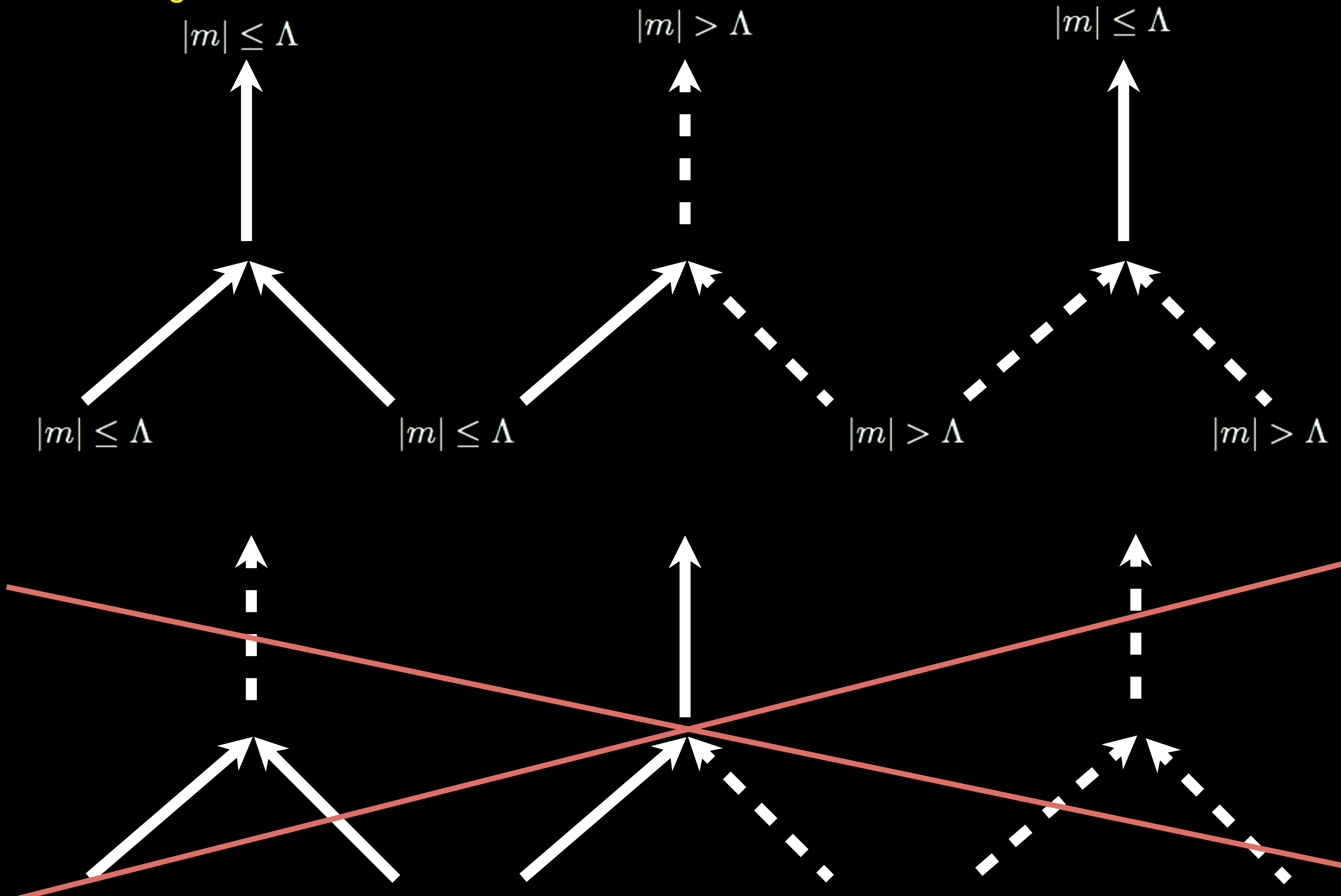
Barotropic and baroclinic instability

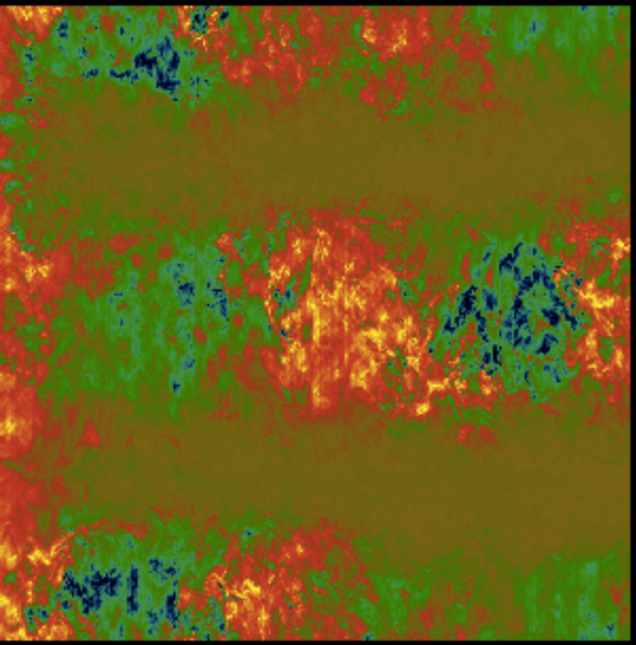
Srinivasan & Young (2012); Parker & Krommes (2014); T & Marston (2011); Bouchet, Nardini & Tangarife (2013); Farrell & Ioannou;

Importance of interactions can be analysed by generalising the definition of means and fluctuations and only keeping certain triad interactions (GCE2)

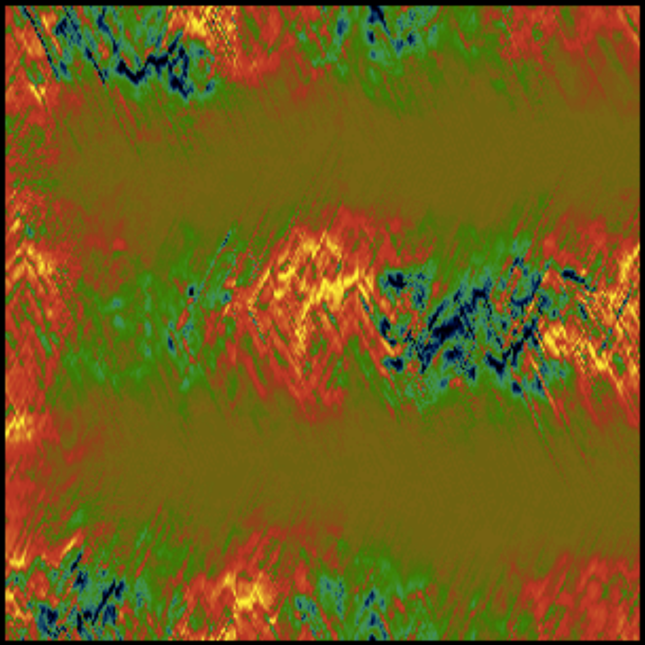
Separate Triads Into Long and Short Scales

m=along shear wavenumber

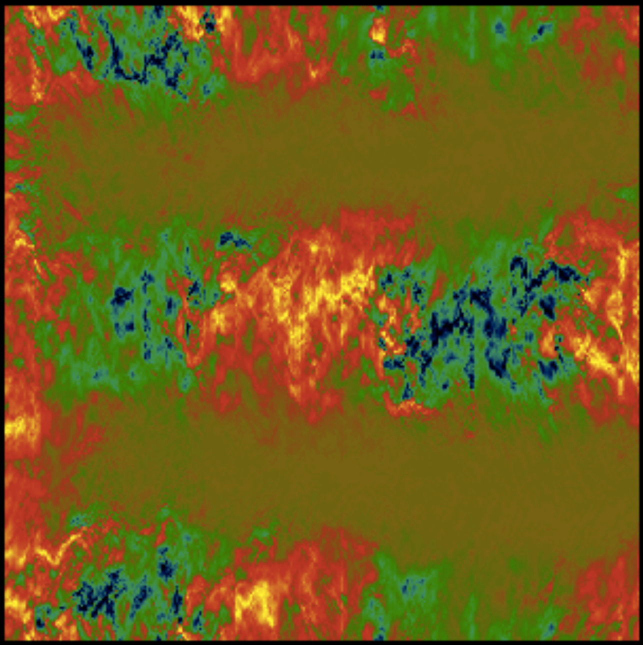




“Truth”



Λ=0



Λ=1

Severe truncations do quite well when shear is strong and distributions of EMF are narrow.
Slightly over-exaggerate mean to fluctuations
Linear equation: easier for truncations to do well.

Further Thoughts and Further Work

- Clearly there are scales in a turbulent cascade that are too small to
  - Feel the effects of a shear
  - Feel the effects of rotation
- These have very fast turnover times and low amplitudes and so will grow and saturate quickly?
- These will provide a background noise to the large-scale dynamo (cf the magnetic carpet and the solar cycle)
- Can only tell by performing a nonlinear calculation
  - Currently underway…

The Suppression Principle

(Cattaneo & Tobias ApJ 2014)

- It is all very well to try and boost the EMF to sustain large-scale dynamo action, but…
  - The small scale dynamo will win unless some agent acts to suppress it.
    - Shear
    - Nonlinear suppression (faster eddies have less energy)
    - Roughness of the turbulence (Subramanian & Brandenburg 2014, ArXiv)

Conclusions

At high Rm

Shear may suppress the small-scale dynamo

If small-scale dynamo is "as good as it gets" ratio χ_{Rm} large

If small-scale dynamo is "weak" then shear may help ratio χ_{Rm} small (2-3)

Shear may suppress the average EMF

But narrows distribution (reduced intermittency) potentially making statistical approaches more accurate.

So what do we mean by a high Rm small-scale dynamo?

One where adding a systematic large-scale flow ***decreases*** the efficiency of dynamo action by making the flow more integrable…

Large vs Small

- ***<u>So when do we get a large-scale dynamo and when a small-scale?</u>***

For a turbulent cascade…

(a) Can calculate which scales are active in creating small-scale dynamos (T&C JFM, 2007).

(b) Can calculate which scales contribute to mean and variance of EMF (T&C 2014)

If timescale of (a) << growth-time of large-scales in (b) then SSD wins

Note large-scale shear can modify both (b) and more importantly (a)!