Approaching the Asymptotic Regime of Rapidly Rotating Convection: Boundary Layer vs Interior Dynamics

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Mathematics of Turbulence Long Program
Geophysical and Astrophysical Turbulence
IPAM, UCLA
Reduced Modeling & Simulation (Colorado & Berkeley)
Edgar Knobloch, Jeff Weiss
Ian Grooms, Antonio Rubio,
Geoff Vasil, Michael Calkins, Philippe Marti

Lab Experiments
Jon Aurnou
Eric King, Jonathan Cheng

DNS (U. Muenster)
Stephan Stellmach
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Monday, November 10, 14
Rotating Rayleigh-Benard Convection

\[ Ra = \frac{g \alpha \Delta T H^3}{\nu \kappa}, \quad \sigma = \frac{\nu}{\kappa} \]

Ekman number

\[ E = \frac{\nu}{2 \Omega H^2} \]

Convective Rossby number

\[ Ro = \sqrt{\frac{Ra}{\sigma} E} = \frac{\sqrt{g \alpha \Delta T / H}}{2 \Omega} \]

Ra \gg 1: conduction to turbulent motions

- Mixed interiors with thermal boundary layers
- Rich in flow morphologies
Rotationally Constrained Parameter Space: Low $Ro$, High $Ra$

\[ E^{-2} = T a \propto \Omega^2 \text{ (rotation rate)} \]

Ocean & Astrophysical Systems: $Ra, T a \Rightarrow 10^{17} - 10^{20}$

$Ro_{\text{conv}} = 0.01 - 1$

Vorobieff & Ecke (2002)

Sakai (1997)

$Ro \sim 10^{-7}$

$Re \sim 10^8$

$E k \sim 10^{-15}$

$Ro \sim 10^{-2}$

$Re \sim 10^{16}$

$E k \sim 10^{-18}$
Characterization - Heat Transport
Low $Ro$ branch characterized by steep branch in $Nu$-$Ra$ space

$Ro_{crit} \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4/3}$

$$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left(\frac{Ra}{Ra_c}\right)^\beta$$

$\beta_{rot} > 1$

$\beta_{norot} = 2/7$

UCLA Spin-Lab: King et al. Nature 2009
U. Muenster: Stellmach DNS simulation effort
Characterization - Heat Transport
Low $Ro$ branch characterized by steep branch in $Nu-Ra$ space

\[ Ro_{\text{crit}} \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4/3} \]

\[ \Delta T_{Ro=1} / \Delta T_{Ro=Ro_c} \propto E^{-2/3} \]

$Nu = \frac{\text{Total Heat Transfer}}{\text{Conductive Transfer}} \sim \left( \frac{Ra}{Ra_c} \right)^\beta$

\[ \beta_{\text{rot}} > 1 \]

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Characterization - Heat Transport
Low $Ro$ branch characterized by steep branch in $Nu$-$Ra$ space

$$Rocrit \sim E^{1/3} < Ro < 1, \quad Ra \sim E^{-4/3}$$

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$$\frac{\Delta T_{Ro=1}}{\Delta T_{Ro=Ro_c}} \propto E^{-2/3}$$

$Nu$ vs $Ra$

Reduced Models

$$Nu \sim (Ra/Ra_c)^{\beta}$$

$$Nu = \frac{Total \ Heat \ Transfer}{Conductive \ Transfer} \sim \left(\frac{Ra}{Ra_c}\right)^{\beta}$$

$\beta_{rot} > 1$

$$\beta_{norot} = 2/7$$

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Governing Equations in the Rotationally Constrained Limit

\[ D_t \mathbf{u}_\perp + w \partial_Z \mathbf{u}_\perp + \frac{1}{E} \hat{Z} \times \mathbf{u}_\perp = -\nabla_\perp p + (\nabla_\perp^2 + \partial_Z Z) \mathbf{u}_\perp \]

\[ D_t w + w \partial_Z w = -\partial_Z p + \frac{Ra}{\sigma} T + (\nabla_\perp^2 + \partial_Z Z) w \]

\[ D_t T + w \partial_Z T = \frac{1}{\sigma} (\nabla_\perp^2 + \partial_Z Z) T \]

\[ \nabla_\perp \cdot \mathbf{u}_\perp + \partial_Z w = 0 \]

Incompressible Navier-Stokes

- Challenge: spanning the entire rotationally constrained branch
- Resolving BL’s, fast waves at large Ra as \( Ro, E \Rightarrow 0 \)

NonHydrostatic Quasi-Geostrophic Eqns

- Challenge: robustness of asymptotics as \( Ro, E \) increase away from 0

\[ D_t \zeta - \partial_Z w = \nabla_\perp^2 \zeta \]

\[ D_t w = -\partial_Z \psi + \frac{Ra E^{4/3}}{\sigma} \theta + \nabla_\perp^2 w \]

\[ D_t \theta + w \partial_Z \overline{T} = \frac{1}{\sigma} \nabla_\perp^2 \theta \]

\[ \partial_T \overline{T} + \partial_Z (\overline{w \theta}) = \frac{1}{\sigma} \partial_Z Z \overline{T} \]

\[ p = \psi, \quad \mathbf{u}_\perp = (-\partial_x \psi, \partial_y \psi, 0), \quad \zeta = \nabla_\perp^2 \psi \]
Governing Equations in the Rotationally Constrained Limit

\[ D_t^\perp \mathbf{u}_\perp + w \partial_Z \mathbf{u}_\perp + \frac{1}{E} \hat{z} \times \mathbf{u}_\perp = -\nabla_\perp p + (\nabla_\perp^2 + \partial_Z Z) \mathbf{u}_\perp \]

\[ D_t^\perp w + w \partial_Z w = -\partial_Z p + \frac{Ra}{\sigma} T + (\nabla_\perp^2 + \partial_Z Z) w \]

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Incompressible Navier-Stokes

- Challenge: spanning the entire rotationally constrained branch
- Resolving BL’s, fast waves at large Ra as \( Ro, E \to 0 \)

Boundary conditions

- Fixed Temperature
- Impenetrable \( \implies \) Stress-Free

Nonhydrostatic Quasi-Geostrophic Eqns

- Challenge: robustness of asymptotics as \( Ro, E \) increase away from 0
Governing Equations in the Rotationally Constrained Limit

\[ D_t^\perp \mathbf{u}_\perp + w \partial_Z \mathbf{u}_\perp + \frac{1}{E} \hat{\mathbf{z}} \times \mathbf{u}_\perp = -\nabla_\perp p + \left( \nabla_\perp^2 + \partial_{ZZ} \right) \mathbf{u}_\perp \]

\[ D_t^\perp w + w \partial_Z w = -\partial_Z p + \frac{Ra}{\sigma} T + \left( \nabla_\perp^2 + \partial_{ZZ} \right) w \]

\[ D_t^\perp T + w \partial_z T = \frac{1}{\sigma} \left( \nabla_\perp^2 + \partial_{ZZ} \right) T \]

\[ \nabla_\perp \cdot \mathbf{u}_\perp + \partial_Z w = 0 \]

Overlap in (Ra, Ro, E) permissible by Lab Exp’s, DNS and Reduced Models

Incompressible Navier-Stokes
- Challenge: spanning the entire rotationally constrained branch
- Resolving BL’s, fast waves at large Ra as Ro,E \to 0

NonHydrostatic Quasi-Geostrophic Eqns
- Challenge: robustness of asymptotics as Ro,E increase away from 0

\[ p = \psi, \quad \mathbf{u}_\perp = (-\partial_x \psi, \partial_y \psi, 0), \quad \zeta = \nabla_\perp^2 \psi \]
Four Flow Regimes as $Ra$ ↑: laminar to turbulent

Cells $\rightarrow$ CTC’s via TBL instability & synchronization of TBL’s

CTC’s: Shielded vortical columns with zero circulation

Geostrophic turbulence regime occurs when TBL are unable to synchronize

- Cells
- CTC’s via TBL instability & synchronization of TBL’s
- Geostrophic turbulence regime occurs when TBL are unable to synchronize
Geostrophic turbulence is the greatest throttle on heat transport

\[ Nu \propto \sigma^\alpha (Ra/Ra_c)^\beta, \quad Ra_c = E^{-4/3} \]

- marginally stable tbl's (Malkus, '54):
  \[ \beta = 3 \]

- depth independence (Priestley, '59):
  \[ \beta = 3 \]

- ultimate (dissipation-free) turbulent law (Kraichnan '63, Howard '63):
  \[ \beta = 3/2 < 3, \quad \alpha = -1/2 \]

\[ \sigma = \frac{\nu}{\kappa} \]

\[ Nu = \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g \alpha \Delta T H^3}{\nu \kappa}, \quad E = \frac{\nu}{2 \Omega H^2} \]
Four flow regimes captured by DNS

\[ \text{\( E k = 10^{-7} \)} \]
DNS RBC vs NH-QGE
Impenetrable Stress-Free Boundaries

Good quantitative agreement

- NH-QGE (Open Symbols)
- DNS (Closed Symbols) \( E_k = 10^{-7} \)
Experiments, GAFD vs NH-QGE?

- **Laboratory Experiments & GAFD Flows**
  - Physical boundaries $\Rightarrow$ impenetrable no-slip boundaries

- **DNS**
  - Impenetrable no-slip in vertical, periodic in horizontal.

- **Asymptotic Reduced Model**
  - Impenetrable Stress-free vertical, periodic in horizontal
  - Linear Theory (Chandrasekhar, 1961; Niiler & Bisshopp, JFM 1965)
    - *differences between bc’s are asymptotically small*, confined to Ekman BL’s
  - Strongly NonLinear Theory (Julien & Knobloch JFM 1998)
    - EBL’s passive, Ekman pumping asymptotically weak, again no difference
Four flow regimes again captured by DNS

- Geostrophic vortex (LSV) suppressed
  
  $Ek = 10^{-7}$
DNS RBC vs NH-QGE
Impenetrable No-Slip Boundaries

Significant departures
- NH-QGE (Open Symbols)
- DNS (Closed Symbols)
- Exp (asterisks)

\[ \text{Ek} = 10^{-7} \]

Some evidence for convergence
- GT regime
- Interior throttles heat flux

Ekman layers are the only real difference between BC’s!
DNS RBC vs NH-QGE
LS: Diminishing influence of Ekman layers!

- Significant departures
  - NH-QGE (Open Symbols)
  - DNS (Closed Symbols)
  - Exp (asterisks)
  - $Ek = 10^{-7}$

- Some evidence for convergence
  - GT regime
  - Interior throttles heat flux

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On the influence of Coriolis force on onset of thermal convection

By P. P. NILLER* AND F. E. BISHOP
Division of Applied Mathematics, Brown University, Providence, Rhode Island

(Received 1 September 1964 and in revised form 29 January 1965)
NH-QG Theory for No-Slip Boundaries states Ekman layers are passive.

Are Ekman layers passive?

\[ \widehat{z} \times \mathbf{u}_{\perp}^{(e)} = \partial_\delta \delta \mathbf{u}_{\perp}^{(e)} \]

\[ u_0^{(e)}(x, y, \delta, t) = U_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \cos \frac{\delta}{\sqrt{2}} \right) - V_0^{(g)}(x, y, 0, t)e^{-\frac{\delta}{\sqrt{2}}} \sin \frac{\delta}{\sqrt{2}} \]

\[ v_0^{(e)}(x, y, \delta, t) = V_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \cos \frac{\delta}{\sqrt{2}} \right) + U_0^{(g)}(x, y, 0, t)e^{-\frac{\delta}{\sqrt{2}}} \sin \frac{\delta}{\sqrt{2}} \]

\[ w_{1/2}^{(e)}(x, y, \delta, t) = \frac{1}{\sqrt{2}} c_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \left[ \cos \frac{\delta}{\sqrt{2}} + \sin \frac{\delta}{\sqrt{2}} \right] \right) \]

\[ \theta_1^{(e)}(x, y, \delta, t) = \Theta_1^{(g)}(x, y, 0, t) \equiv 0 \]

\[ p_1^{(e)}(x, y, \delta, t) = \Psi_0^{(g)}(x, y, 0, t) \]
NH-QG Theory for No-Slip Boundaries
Predicts Ekman layers are passive

Are Ekman layers passive?

As in atmos. and ocean applications filter
Ekman layers by pumping/suction BC

\[ w_{1/2}^{(e)}(x, y, Z_{\pm}, t) = \pm \frac{1}{\sqrt{2}} \zeta_0^{(g)}(x, y, Z_{\pm}, t) \]

\[ \hat{z} \times u^{(e)}_\perp = \partial \delta \delta u^{(e)}_\perp \]

\[ u_0^{(e)}(x, y, \delta, t) = U_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \cos \frac{\delta}{\sqrt{2}} \right) - V_0^{(g)}(x, y, 0, t)e^{-\frac{\delta}{\sqrt{2}}} \sin \frac{\delta}{\sqrt{2}} \]

\[ v_0^{(e)}(x, y, \delta, t) = V_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \cos \frac{\delta}{\sqrt{2}} \right) + U_0^{(g)}(x, y, 0, t)e^{-\frac{\delta}{\sqrt{2}}} \sin \frac{\delta}{\sqrt{2}} \]

\[ w_{1/2}^{(e)}(x, y, \delta, t) = \frac{1}{\sqrt{2}} \zeta_0^{(g)}(x, y, 0, t) \left( 1 - e^{-\frac{\delta}{\sqrt{2}}} \right) \left[ \cos \frac{\delta}{\sqrt{2}} + \sin \frac{\delta}{\sqrt{2}} \right] \]

\[ \theta_1^{(e)}(x, y, \delta, t) = \Theta_1^{(g)}(x, y, 0, t) \equiv 0 \]

\[ p_1^{(e)}(x, y, \delta, t) = \Psi_0^{(g)}(x, y, 0, t) \]

J. & Knobloch, JFM '98; NH-QGE EBL

Stellmach .et al.

Ra = 50, \sigma = 7
Ra = 10^{11}, E = 10^{-7}
DNS RBC vs DNS with Parameterized Pumping

\[ w^{(e)}_{1/2}(x, y, Z\pm, t) = \pm \frac{1}{\sqrt{2}} \zeta_0^{(g)}(x, y, Z\pm, t) \]

\( \text{Ek} = 10^{-7} \)

• Filled symbols (DNS with No-Slip BC’s)
• Open symbols (DNS w/ Pumping BC’s)
• Convergence outside Ekman layers
• Ekman layers can be filtered
Conclusion:
• *Ekman pumping responsible for enhanced HT*

• Either $E$ to large
  as $E \to 0$, DNS $\Rightarrow$ NHQGE

OR

• Pumping remains important as $E \to 0$
  as $E \to 0$, non-convergence

- Filled symbols (DNS with No-Slip BC’s)
- Open symbols (DNS w/ Pumping BC’s)
Heat Transport: Exponent vs Ekman number

\[ Nu \propto \left( \frac{Ra}{Ra_c} \right)^{\beta} \]
nonconvergence: exponent appears to increase with decreasing \( E \! \)

Conclusion:
- \( O(E^{1/6} H) \) EBL’s having leading order affect on heat transport
- Not captured in asymptotic reduced model

Results from UCLA SpinLab
Courtesy Jon Aurnou, Jon Cheng
Asymptotic Development
Requires TBL Analysis of NH-QGE

NH-QGE

\[ D_t \zeta_0 - \partial_Z w_0 = \nabla_\perp^2 \zeta_0 \]

\[ D_t^\perp w_0 = - \partial_Z \psi_0 + \frac{Ra E^{4/3}}{\sigma} \theta_1 + \nabla_\perp^2 w_0 \]

\[ D_t^\perp \theta_1 + w_0 \partial_Z \bar{T}_0 = \frac{1}{\sigma} \nabla_\perp^2 \theta_1 \]

\[ \partial_T \bar{T}_0 + \partial_Z (w_0 \bar{\theta}_1) = \frac{1}{\sigma} \partial_Z \bar{T}_0 \]

Magnitudes:

\[ \Rightarrow \theta_1 \sim w_0 \partial_Z \bar{T}_0 \] thermal fluct.

\[ \Rightarrow w_0 \theta_1 \sim \partial_Z \bar{T}_0 \] heat flux
Asymptotic Development
Requires TBL Analysis of NH-QGE

\[ D_t^\perp \zeta_0 - \partial_Z w_0 = \nabla_{\perp}^2 \zeta_0 \]

\[ D_t^\perp w_0 = - \partial_Z \psi_0 + \frac{Ra E^{4/3}}{\sigma} \theta_1 + \nabla_{\perp}^2 w_0 \]

\[ D_t^\perp \theta_1 + w_0 \partial_Z \overline{T}_0 = \frac{1}{\sigma} \nabla_{\perp}^2 \theta_1 \]

\[ \partial_T \overline{T}_0 + \partial_Z (w_0 \theta_1) = \frac{1}{\sigma} \partial_Z w_0 \overline{T}_0 \]

Magnitude:
\[ \theta_1 \sim w_0 \partial_Z \overline{T}_0 \quad \text{thermal fluct.} \]
\[ w_0 \theta_1 \sim \partial_Z \overline{T}_0 \quad \text{heat flux} \]

Boundary layer scaling for columnar regime:

\[ Nu \sim \partial_Z \overline{T}_0 \sim \theta_1 \sim \overline{Ra}^\beta, \partial_Z \sim \overline{Ra}^{(1+\beta)/2} \]
\[ w_0 \sim \overline{Ra}^0, \zeta_0 \sim \psi_0 \sim \overline{Ra}^{(1+\beta)/2} \]

Empirical result
\[ \beta \approx 2 \]
J. et al GAFD 2012
Asymptotic Development
Ekman pumping drives bulk corrections

Pumping bc’s $\hat{w}^{(p)} = \pm E^{1/6} \zeta_0$ suggest the inclusion of higher order correction to NH-QGE

\[
\hat{w} = w_0 + \epsilon^{1/2} w_{1/2}^{(p)}, \quad \hat{\zeta} = \zeta_0 + \epsilon^{1/2} \zeta_{1/2}^{(p)}, \quad \hat{\psi} = \psi_0 + \epsilon^{1/2} \psi_{1/2}^{(p)},
\]
\[
\partial_Z \hat{T} = \partial_Z T_0 + \epsilon^{1/2} \partial_Z T_{1/2}, \quad \hat{\theta} = \epsilon \theta_1 + \epsilon^{3/2} \theta_{3/2}^{(p)}
\]
Asymptotic Development
Ekman pumping drives bulk corrections

Pumping bc’s \( \hat{w}^{(p)} = \pm E^{1/6} \zeta_0 \) suggest the inclusion of higher order correction to NH-QGE

\[
\text{NH-QGEcorr} \quad D_t^{1/2} \zeta_{1/2} + J[\psi_{1/2}, \zeta_0] - \partial_z w_{1/2} = \nabla^2_\perp \zeta_{1/2}
\]

\[
D_t^{1/2} w_{1/2} + J[\psi_{1/2}, w_0] = - \partial_z \psi_{1/2} + \frac{RaE^{4/3}}{\sigma} \theta_{3/2} + \nabla^2_\perp w_{1/2}
\]

\[
D_t^{1/2} \theta_{3/2} + J[\psi_{1/2}, \theta_1] + w_{1/2} \partial_z T_0 + w_0 \partial_z T_{1/2} = \frac{1}{\sigma} \nabla^2_\perp \theta_{3/2}
\]

\[
\partial_r T_{1/2} + \partial_z \left( w_0 \theta_{3/2} + w_{1/2} \theta_1 \right) = \frac{1}{\sigma} \partial_z z T_{1/2}
\]

Pose, with \( \epsilon = E^{1/3} \)

\[
\hat{w} = w_0 + \epsilon^{1/2} w_{1/2}^{(p)}, \quad \hat{\zeta} = \zeta_0 + \epsilon^{1/2} \zeta_{1/2}^{(p)}, \quad \hat{\psi} = \psi_0 + \epsilon^{1/2} \psi_{1/2}^{(p)},
\]

\[
\partial_z \hat{T} = \partial_z T_0 + \epsilon^{1/2} \partial_z T_{1/2}, \quad \hat{\theta} = \epsilon \theta_1 + \epsilon^{3/2} \theta_{3/2}^{(p)}
\]
Pumping bc’s \( w^{(p)} = \pm \zeta_0 \) suggest the inclusion of higher order correction to the NH-QGE

\[
\text{NH-QGEcorr} \quad D_t^{⊥} \zeta_{1/2} + J[\psi_{1/2}, \zeta_0] - \partial Z w_{1/2} = \nabla_{⊥}^2 \zeta_{1/2}
\]

\[
D_t^{⊥} w_{1/2} + J[\psi_{1/2}, w_0] = - \partial Z \psi_{1/2} + \frac{RaE^{4/3}}{\sigma} \theta_{3/2} + \nabla_{⊥}^2 w_{1/2}
\]

\[
D_t^{⊥} \theta_{3/2} + J[\psi_{1/2}, \theta_1] + w_{1/2} \partial Z \overline{T}_0 + w_0 \partial Z \overline{T}_{1/2} = \frac{1}{\sigma} \nabla_{⊥}^2 \theta_{3/2}
\]

\[
\partial_r \overline{T}_{1/2} + \partial Z \left( \frac{w_0 \theta_{3/2} + w_{1/2} \theta_1}{2} \right) = \frac{1}{\sigma} \partial_Z \overline{T}_{1/2}
\]

Magnitudes:

- pumping induced thermal fluct.
  \[
  \theta_{3/2} \sim \epsilon^{1/2} w_{1/2} \partial Z \overline{T}_0
  \]
  \[
  \sim \epsilon^{1/2} \zeta_0 \partial Z \overline{T}_0
  \]
- heat flux
  \[
  w_0 \theta_{3/2} + w_{1/2} \theta_1 \sim \epsilon^{1/2} \partial Z \overline{T}_0
  \]
Asymptotic Development
Heat Transport by Ekman pumping is significant

Pumping bc’s $w^{(p)} = \pm \zeta_0$ suggest the inclusion of higher order correction to the NH-QGE

\[ D_t^\perp \zeta_{1/2} + J[\psi_{1/2}, \zeta_0] - \partial_Z w_{1/2} = \nabla^2_\perp \zeta_{1/2} \]

\[ D_t^\perp w_{1/2} + J[\psi_{1/2}, w_0] = - \partial_Z \psi_{1/2} + \frac{RaE^{4/3}}{\sigma} \theta_{3/2} + \nabla^2_\perp w_{1/2} \]

\[ D_t^\perp \theta_{3/2} + J[\psi_{1/2}, \theta_1] + w_{1/2} \partial_Z \overline{T}_0 + w_0 \partial_Z \overline{T}_{1/2} = \frac{1}{\sigma} \nabla^2_\perp \theta_{3/2} \]

\[ \partial_r \overline{T}_{1/2} + \partial_Z \left( w_0 \theta_{3/2} + w_{1/2} \theta_1 \right) = \frac{1}{\sigma} \partial_Z \overline{T}_{1/2} \]

Dynamics of higher system subdominant iff heat transport is ordered according to

\[ w_0 \theta_1 \gg \epsilon^{1/2} (w_0 \theta_{3/2} + w_{1/2} \theta_1) \gg \epsilon w_{1/2} \theta_{3/2} \]

\[ 1 \gg \epsilon^{1/2} \zeta_0 \gg \epsilon \zeta_0 \]

\[ \zeta_0(\pm) = \mathcal{O}(\epsilon^{-1/2}) = o(\epsilon^{-1}) \]

\[ \widetilde{Ra}_{thres} = E^{-1/3(1+\beta)} \]

Always violated!
Asymptotic Development
Estimates of transitional values

$$\zeta_0(\pm) = O(\epsilon^{-1/2}) = o(\epsilon^{-1}), \quad \epsilon = E^{1/3}$$

$$\tilde{Ra}_{thres} = E^{-1/3(1+\beta)} = E^{-1/9}, \quad \beta \approx 2$$

<table>
<thead>
<tr>
<th>DNS, Lab</th>
<th>$E = 10^{-7}, \quad \epsilon^{1/2} = 0.068, \quad \tilde{Ra}_t \approx 6.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E = 10^{-8}, \quad \epsilon^{1/2} = 0.046, \quad \tilde{Ra}_t \approx 7.7$</td>
</tr>
<tr>
<td></td>
<td>$E = 10^{-10}, \quad \epsilon^{1/2} = 0.022, \quad \tilde{Ra}_t \approx 12.9$</td>
</tr>
<tr>
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<td>$E = 10^{-12}, \quad \epsilon^{1/2} = 0.010, \quad \tilde{Ra}_t \approx 21.5$</td>
</tr>
<tr>
<td></td>
<td>$E = 10^{-14}, \quad \epsilon^{1/2} = 0.005, \quad \tilde{Ra}_t \approx 35.9$</td>
</tr>
<tr>
<td></td>
<td>$E = 10^{-15}, \quad \epsilon^{1/2} = 0.003, \quad \tilde{Ra}_t \approx 46.4$</td>
</tr>
</tbody>
</table>

| Earth’s core | $\tilde{Ra}_c = 8.7$ |
Asymptotic Development
Composite Reduced Model

\[ \hat{D}_t \hat{\zeta} - \partial_Z \hat{w} = \nabla^2_{\perp} \hat{\zeta} \]

CNH-QGE

\[ \hat{D}_t \hat{w} = - \partial_Z \hat{\psi} + \frac{Ra E^4/3}{\sigma} \hat{\theta} + \nabla^2_{\perp} \hat{w} \]

\[ \hat{D}_t \hat{\theta} + \hat{w} \partial_Z \hat{T} = \frac{1}{\sigma} \left( \nabla^2_{\perp} + \epsilon^2 \partial_Z Z \right) \hat{\theta} \]

\[ \partial_{\tau} \hat{T} + \partial_Z \left( \hat{w} \hat{\theta} \right) = \frac{1}{\sigma} \partial_Z Z \hat{T} \]

Heat flux correction due to Ekman pumping captured

\[ \hat{w} = \pm E^{1/6} \zeta_0 \]

\[ \overline{T}(0) = 1, \overline{T}(1) = 0 \]
Asymptotic Development
Composite Reduced Model

\[ \hat{D}^\perp_t \hat{\zeta} - \partial_Z \hat{w} = \nabla^2_\perp \hat{\zeta} \]

\[ \hat{D}^\perp_t \hat{w} = - \partial_Z \hat{\psi} + \frac{Ra E^{4/3}}{\sigma} \hat{\theta} + \nabla^2_\perp \hat{w} \]

\[ \hat{D}^\perp_t \hat{\theta} + \hat{w} \partial_Z \hat{T} = \frac{1}{\sigma} \left( \nabla^2_\perp + \epsilon^2 \partial_Z Z \right) \hat{\theta} \]

\[ \partial_\tau \hat{T} + \partial_Z \left( \hat{w} \hat{\theta} \right) = \frac{1}{\sigma} \partial_Z Z \hat{T} \]

Heat flux correction due to Ekman pumping captured

\[ \hat{w} = \pm E^{1/6} \zeta_0 \]

\[ \overline{T}(0) = 1, \overline{T}(1) = 0 \]

Preliminary investigation: single mode theory gives further reduction to a 1-D vertical model.
Results with Ekman pumping

Nu vs Ra

$E = 10^{-7}$, $\epsilon^{1/2} = 0.068$
$E = 10^{-8}$, $\epsilon^{1/2} = 0.046$
$E = 10^{-10}$, $\epsilon^{1/2} = 0.022$
$E = 10^{-12}$, $\epsilon^{1/2} = 0.010$
$E = 10^{-14}$, $\epsilon^{1/2} = 0.005$
$E = 10^{-\infty}$, $\epsilon^{1/2} = 0.0$
Results with Ekman pumping

$\text{Nu vs Ra}$

- Linear theory recovered, no slip easier to destabilize

$$\left( RaE^{4/3} \right)_{NS} \approx 8.7 SF - \mathcal{O} \left( E^{1/6} \right)$$
• NH-QGE recaptures as $E \to 0$ at fixed $Ra$.

• Linear theory recovered, no slip easier to destabilize

$$\left( Ra E^{4/3} \right)_{NS} \approx 8.7_{SF} - \mathcal{O} \left( E^{1/6} \right)$$
- Limit $E \to 0$ pumping always becomes important, GAFD implications
Results with Ekman pumping
Nu vs Ra

\[ E = 10^{-7}, \ \epsilon^{1/2} = 0.068 \]
\[ E = 10^{-8}, \ \epsilon^{1/2} = 0.046 \]
\[ E = 10^{-10}, \ \epsilon^{1/2} = 0.022 \]
\[ E = 10^{-12}, \ \epsilon^{1/2} = 0.010 \]
\[ E = 10^{-14}, \ \epsilon^{1/2} = 0.005 \]
\[ E = 10^{-\infty}, \ \epsilon^{1/2} = 0.0 \]
• Ekman pumping appears to have a non-diminishing effect
  - as $E \to 0$ contribution to heat transport remains $O(1)$
  - has implication for GAFD flows that exist on low Ro branch
• EBL Effect can be captured by parameterized pumping BC’s
  - DNS can be performed at lower E barring CFL constraints
  - NH-QGE can be extended to include pumping
• Effort requires synergy between Lab exp’s, DNS, and reduced modeling
Some Open Questions for NH-QGE

• Analysis (MTWK1):
  - Global Existence, Regularity, ... No, Nonlinear Vortex Stretching, Reynolds Closure prob.
  - Bounds on HT : (Stress-Free done. Grooms & Whitehead ’14)

• Mixing and Transport (MTWK2):
  - Mixing efficiency? RRBC has semi-analytic flows (Cells, CTCs)

• Engineering Applications (MTWK4)
  - Need greater understanding of EBL’s and TBL’s (Data analysis, Visualizations)