Stochastic Superparameterization

Multiscale Methods in Ocean Modeling

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Principle Collaborators for Stochastic Superparameterization

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Some "big picture" comments

- There are many kinds of turbulence model, and many reasons for making them. Our goal is to improve the representation of large scales in numerical models.
- The most accurate models of turbulence are not always the most useful for numerical modeling, and vice versa, e.g. EDQNM, (Q)DIA vs Smagorinsky or GM.
- Stochastic SP is a multiscale model for turbulence, but we sacrifice small-scale accuracy to gain computational efficiency (without sacrificing large-scale accuracy).

OUTLINE

1. Background: multiple scales and superparameterization
2. General formulation of stochastic superparameterization
3. Stochastic superparameterization in QG models

Lots of multiscale models are based on multiple-scales asymptotics, e.g.

Several by Keith & collaborators (myself included)

Also Rupert Klein and Andy Majda

Multiple-scales asymptotics is a bit delicate: it doesn't always lead to a closed multiscale model

Once you have a multiscale model (from asymptotics or otherwise), how do you turn it into an efficient numerical simulation?

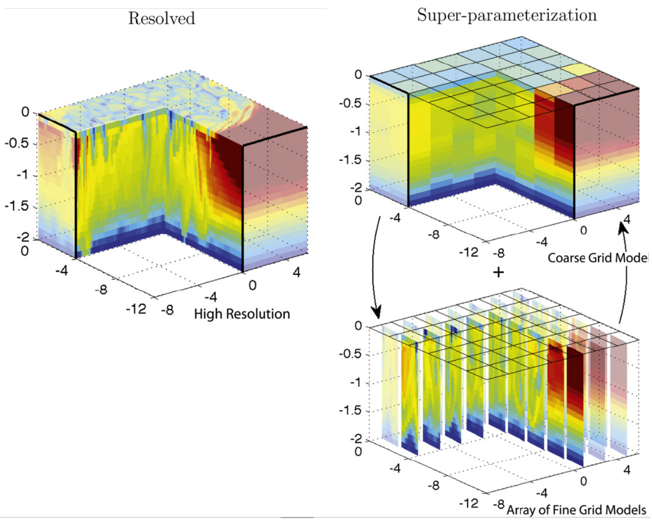
If you have a multiscale mathematical model with coupled large-scale and small-scale equations (e.g. from asymptotics), and you run a simulation of these equations with periodic small-scale domains I will call it "Superparameterization" (SP).

The name SP comes from the work of Grabowski & Smolarkiewicz in atmospheric moist convection.

The next slide illustrates SP in oceanic convection.

SUPERPARAMETERIZATION

J.-M. Campin et al. / Ocean Modelling 36 (2011) 90–101



SUPERPARAMETERIZATION

You need some tricks to make the computation more efficient, e.g.

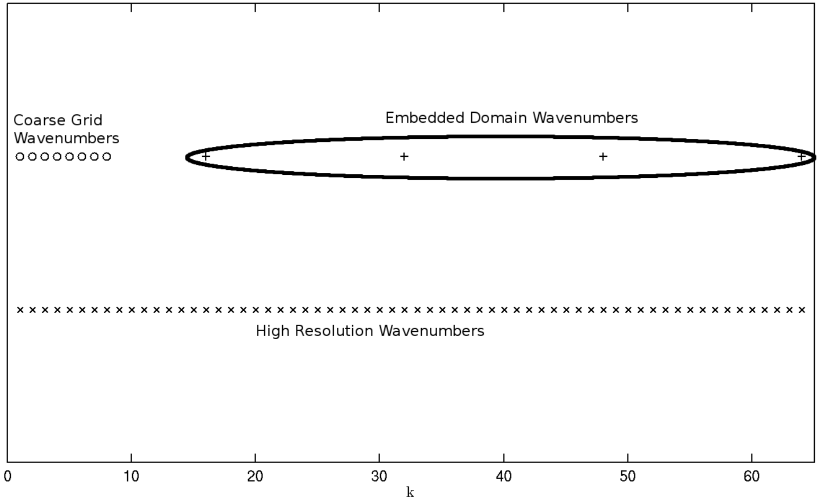
- Make the small-scale computations 2D
- Make the small-scale domains smaller than the coarse grid scale
- Run small-scale simulations shorter than the coarse-grid time step
- Use fewer small-scale domains than coarse grid points

E.g. Xing, Majda, & Grabowski (2009), Malecha, Chini, & Julien (2014)

Next slide illustrates a difficulty of SP

SUPERPARAMETERIZATION

SCALE SEPARATION



Stochastic Superparameterization

Andy and I developed stochastic SP to

- Provide a multiscale mathematical model when other approaches (asymptotics) fail
- Significantly reduce the cost of small-scale simulations

STOCHASTIC SUPERPARAMETERIZATION

POINT APPROXIMATION

Start from equations governing dynamics at all scales, e.g.

$$\partial_t \boldsymbol{u} = L\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u})$$

Use Reynolds average to get large-scale mean and small-scale eddy equations

$$\partial_t \overline{\boldsymbol{u}} = L\overline{\boldsymbol{u}} + B(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}) + \overline{B(\boldsymbol{u}', \boldsymbol{u}')}$$
$$\partial_t \boldsymbol{u}' = (L + B(\overline{\boldsymbol{u}}, \cdot) + B(\cdot, \overline{\boldsymbol{u}}))\boldsymbol{u}' + B(\boldsymbol{u}', \boldsymbol{u}')'$$

(Works with any polynomial nonlinearity)

STOCHASTIC SUPERPARAMETERIZATION

POINT APPROXIMATION

Apply '**point approximation**':

1. Eddy variables depend on new independent coordinates $\partial_t \boldsymbol{u}' \to \partial_\tau \boldsymbol{u}'$, $\partial_x \boldsymbol{u}' \to \partial_{\tilde{x}} \boldsymbol{u}'$, etc.
2. Interpret overbar as average over the new coordinates

Point-approximation eddy equations are

$$\partial_\tau \boldsymbol{u}' = \mathcal{L}\boldsymbol{u}' + B(\boldsymbol{u}', \boldsymbol{u}')'$$

where $\mathcal{L} = L + B(\overline{\boldsymbol{u}}, \cdot) + B(\cdot, \overline{\boldsymbol{u}})$ is a constant-coefficient linear differential operator.

Point approximation is similar to asymptotics, but without requiring scale separation.

Could be used as a basis for SP, with aforementioned computational tricks to increase efficiency.

STOCHASTIC SUPERPARAMETERIZATION

GAUSSIAN CLOSURE

Apply **Gaussian closure**: replace eddy-eddy nonlinearity by stochastic forcing and dissipation

$$\partial_\tau \boldsymbol{u}' = \mathcal{L}\boldsymbol{u}' + F - \Gamma\boldsymbol{u}'.$$

F is spatially-correlated Gaussian white noise and Γ is a positive-definite operator.

The properties of the stochastic forcing and dissipation are chosen such that, in the absence of $\mathcal{L}$, the eddies have generic/universal properties.

The Fourier coefficients are complex Ornstein-Uhlenbeck processes; constant-coefficient $\mathcal{L}$ implies Fourier modes decouple.

Gaussian Closure

- This is not an a priori model! You need to know enough about the small scale dynamics to make a reasonable stochastic approximation.
- This stochastic model makes most sense for turbulent (vs weakly nonlinear) small scale dynamics. You could use a different kind of stochastic model in other circumstances.
- The stochastic model is a crude model of small scales, but this is OK if the method gives good large-scale dynamics.
- Stochastic model not correct for long times; need to re-initialize small scales to correct.
- Easy to compute solutions of the small-scale dynamics because the Fourier dynamics are linear and decoupled.
- You can use any Fourier modes you want; this alleviates gaps in the spectrum discussed above for SP.

STOCHASTIC SUPERPARAMETERIZATION FOR QG

We develop stochastic SP for two-layer quasigeostrophic (QG) dynamics on a β-plane forced by imposed baroclinic shear

$$\partial_t q_1 + \nabla \cdot (\boldsymbol{u}_1 q_1) + \partial_x q_1 + (k_\beta^2 + k_d^2) v_1 = \nu \nabla^8 q_1$$
$$\partial_t q_2 + \nabla \cdot (\boldsymbol{u}_2 q_2) - \partial_x q_2 + (k_\beta^2 - k_d^2) v_2 = -r \nabla^2 \psi_2 + \nu \nabla^8 q_2$$

$$\boldsymbol{u}_i = \nabla^\perp \psi_i, \;\; q_i = \nabla^2 \psi_i + \frac{k_d^2}{2}(\psi_j - \psi_i), \;\; i,j = 1,2, \; i \neq j$$

STOCHASTIC SUPERPARAMETERIZATION

FORMULATION OF SP

Apply Reynolds average* to governing equations

$$\partial_t \overline{q}_j = -\nabla \cdot (\overline{\boldsymbol{u}_j q_j}) + (-1)^j \partial_x \overline{q}_j - \Pi_j \partial_x \overline{\psi}_j - \delta_{j2} r \nabla^2 \overline{\psi}_j - \nu \nabla^8 \overline{q}_j,$$

$$\partial_t q_j' = -\nabla \cdot (\boldsymbol{u}_j' q_j')' - (\overline{\boldsymbol{u}}_j - (-1)^j \hat{\boldsymbol{x}}) \cdot \nabla q_j' - \boldsymbol{u}_j' \cdot \nabla \overline{Q}_j - \delta_{j2} r \nabla^2 \psi_j' - \nu \nabla^8 q_j'$$

where $\Pi_j = k_\beta^2 - k_d^2 (-1)^j$ and $\overline{Q}_j = \Pi_j y + \overline{q}_j$.

Note

$$\nabla \cdot (\overline{\boldsymbol{u}_j q_j}) = \nabla \cdot (\overline{\boldsymbol{u}}_j \overline{q}_j) + \nabla \cdot (\overline{\boldsymbol{u}_j' q_j'})$$

$$\nabla \cdot (\overline{\boldsymbol{u}_j' q_j'}) = \frac{k_d^2 (-1)^j}{2} \nabla \cdot (\overline{\boldsymbol{u}_j' (\psi_1' - \psi_2')}) + \left(\partial_x^2 - \partial_y^2\right) \overline{u_j' v_j'} + \partial_{xy} \left(\overline{(v_j')^2} - \overline{(u_j')^2}\right).$$

* low-pass filter is more appropriate but gets to the same place eventually.

STOCHASTIC SUPERPARAMETERIZATION: POINT APPROXIMATION

Apply 'point approximation' to get eddy equations for SP:

$$\partial_t q_j' = -\nabla \cdot (\boldsymbol{u}_j' q_j')' - (\overline{\boldsymbol{u}}_j - (-1)^j \hat{\boldsymbol{x}}) \cdot \nabla q_j' - \boldsymbol{u}_j' \cdot \nabla \overline{Q}_j - \delta_{j2} r \nabla^2 \psi_j' - \nu \nabla^8 q_j'$$

$$\downarrow$$

$$\partial_\tau q_j' = -\tilde{\nabla} \cdot (\boldsymbol{u}_j' q_j')' - (\overline{\boldsymbol{u}}_j - (-1)^j \hat{\boldsymbol{x}}) \cdot \tilde{\nabla} q_j' - \boldsymbol{u}_j' \cdot \nabla \overline{Q}_j - \delta_{j2} r \tilde{\nabla}^2 \psi_j' - \nu \tilde{\nabla}^8 q_j'$$

One could run an SP based on these equations. But it would be way too expensive and probably wouldn't work in some situations (ask me).

STOCHASTIC SUPERPARAMETERIZATION

FORMULATION OF SP

'Gaussian Closure' eddy equations:

$$\partial_\tau q_j' = -\tilde{\nabla}\cdot(\boldsymbol{u}_j' q_j')' - (\overline{\boldsymbol{u}}_j - (-1)^j\hat{\boldsymbol{x}})\cdot\tilde{\nabla}q_j' - \boldsymbol{u}_j'\cdot\nabla\overline{Q}_j - \delta_{j2}r\tilde{\nabla}^2\psi_j' - \nu\tilde{\nabla}^8 q_j'$$

$$\downarrow$$

$$\partial_\tau q_j' = F_j - \Gamma q_j' - (\overline{\boldsymbol{u}}_j - (-1)^j\hat{\boldsymbol{x}})\cdot\tilde{\nabla}q_j' - \boldsymbol{u}_j'\cdot\nabla\overline{Q}_j - \delta_{j2}r\tilde{\nabla}^2\psi_j' - \nu\tilde{\nabla}^8 q_j'$$

How do we specify the stochastic eddy model? We specify a "universal" solution (next slide) for the following equation

$$\partial_t q_i' = F_i' - \Gamma q_i'$$

This isn't enough, so we also specify Γ as follows:
Since Γ sets the decorrelation time of each Fourier mode, we set it using a standard dimensional argument from turbulence theory:

$$\Gamma e^{ikx} = \gamma_k e^{ikx}, \quad \gamma_k = \gamma_0 \sqrt{k^3 E(k)}$$

where γ_0 is a tunable parameter.

The "universal" eddy behavior:

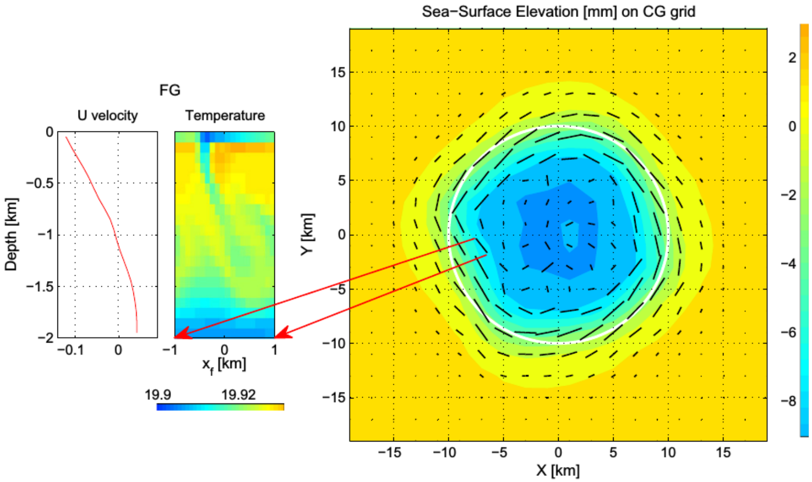
- isotropic energy spectrum $\Rightarrow$ zero mean Reynolds stress terms
- energy spectrum $\propto k^{-5/3}$ for $k < k_d$
- energy spectrum $\propto k^{-3}$ for $k > k_d$
- eddies do not generate heat flux in the absence of temperature gradient
- specify a constant ratio of barotropic and baroclinic energy at each k: either equipartition or 6 times more barotropic energy

The total energy in the eddies is a tunable parameter.

THE ALGORITHM

1. At the beginning of a coarse model time step, evaluate the large-scale variables that appear in the eddy equations, e.g. shear, vorticity gradient.
2. Pick a random direction for the eddies. (See next slide)
3. While holding the large-scale terms fixed, evolve the eddies for a fixed time of length ϵ^{-1}. This is cheap because eddy dynamics are linear.
4. Compute the eddy PV flux from step 3. This is also cheap because of simple Fourier analysis.
5. Update the large-scale variables.
6. Re-set the eddy variables to a 'climatological' state, i.e. forget the final state of the eddies from the end of step 3.
7. Repeat

SUPERPARAMETERIZATION



RANDOM DIRECTIONS

It turns out to be easier to compute the expected value of the eddy feedback terms than to compute a single realization (ODE vs SDE).

The expected value of the eddy feedback is not stochastic, and in the test case below the fluctuations about the mean are crucial.

To “randomize” the feedback, we take a page from conventional SP and make the eddy domains 2D (depth plus horizontal).

The direction of the eddy domains is a random field on the coarse grid. This is ad hoc; similar results could presumably be obtained by computing individual realizations of the stochastic eddies, but it would be more expensive.

STOCHASTIC SUPERPARAMETERIZATION

Summary

Stochastic SP involves three main tunable parameters:

1. the eddy amplitude/energy
2. the eddy decorrelation timescale γ_0^{-1}
3. the eddy averaging time ϵ^{-1}.

#1 Could be closed using an EKE model.

#2 Could probably be modeled using turbulence phenomenology.

#3 The optimal value of ϵ seems to depend on the details of the stochastic eddy model; probably should remain tunable.

Stochastic Backscatter

To understand the effect of the Reynolds stress terms consider their effect for $\epsilon = \infty$, i.e. just sampling the eddy initial condition without allowing the eddies to respond to the local mean flow.
In this case the Reynolds stress terms become

$$\overline{u'_j v'_j} \to \frac{E_0}{2}\sin(2\theta), \quad \overline{(v'_j)^2} - \overline{(u'_j)^2} \to E_0 \cos(2\theta)$$

where θ is the eddy 'angle.'

It can be shown that the Reynolds stress terms in the large-scale PV equations correspond to a random forcing with k^5 spectrum.

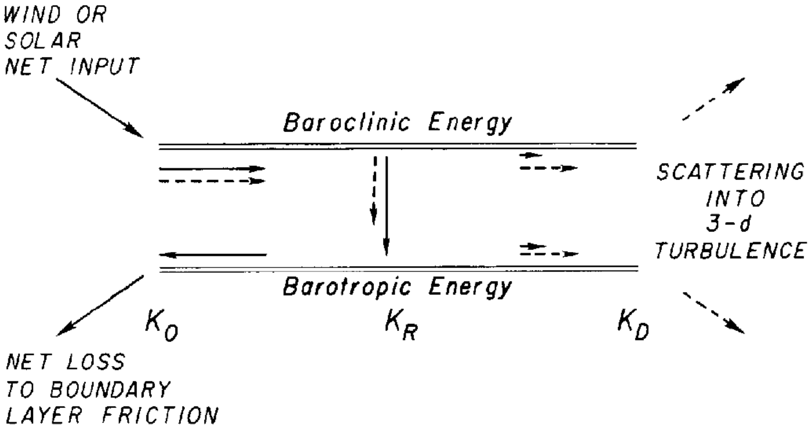
The expected value of the eddy feedback is zero – no backscatter.

Randomized GM

To understand the effect of the 'thickness flux' terms consider their response to a zonal shear on an f-plane.

In this case the thickness flux is directed orthogonal to θ and down the mean gradient.

The amplitude of the flux depends on the amplitude of the shear (and on γ_0 and ϵ), which makes it look like a randomized GM parameterization with $\kappa \propto F(\partial_z u)$ (cf Visbeck/Stone where $\kappa \propto |\partial_z \boldsymbol{u}|$).



Salmon Geo. Astro. Fluid Dyn. 1980

To test stochastic SP we first run eddy-resolving reference simulations with $k_d = 50$ and a 512×512 grid.

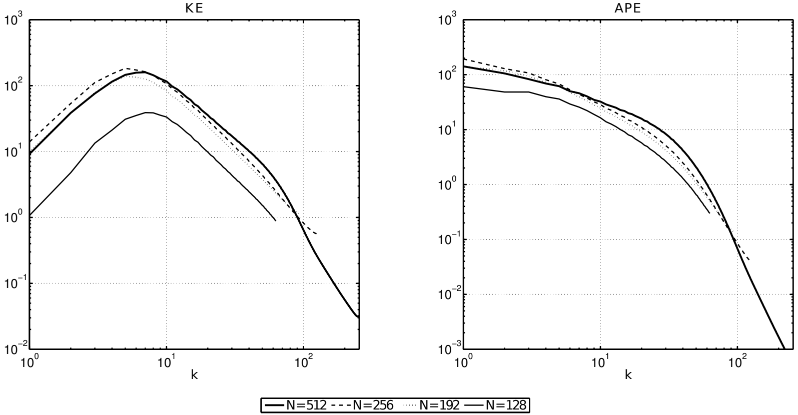
Then we run stochastic SP using a second-order FD discretization on a 96×96 grid – a factor of about 5 lower resolution.

The coarse-grid Nyquist wavenumber is 48, which is smaller than the deformation radius but larger than the peak of the KE spectrum (which is around $k = 5$).

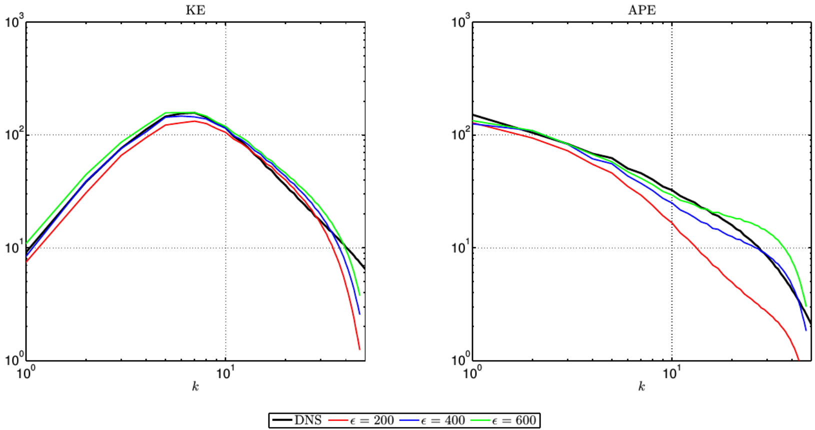
At this resolution & with these numerics we are in the “dual energy cascade” not the “forward enstrophy cascade.”

We run tests in three scenarios

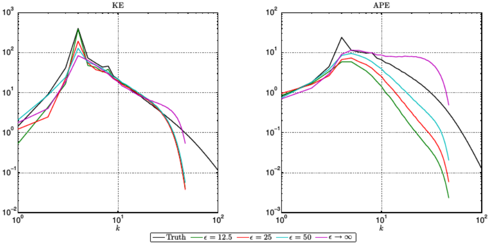
- "High Latitude" f-plane, $r = 16$
- "Mid Latitude" $\beta = Uk_d^2/4$, $r = 4$
- "Low Latitude" $\beta = Uk_d^2/2$, $r = 1$

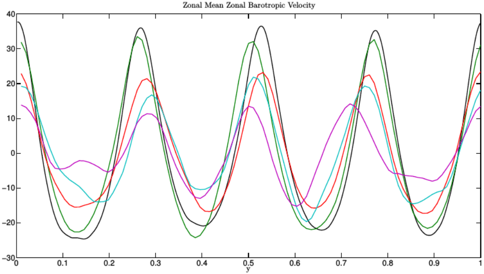


f-plane: unparameterized code at different resolutions, tuning ν for best results.

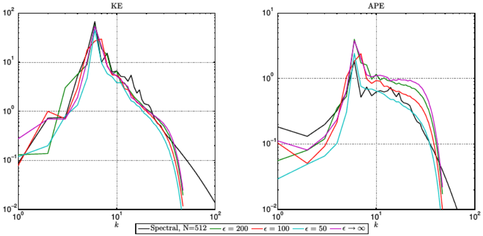


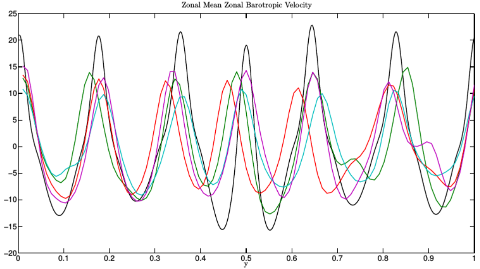
Time-averaged energy spectra from 512^2 reference simulation (black) and from 96^2 stochastic SP simulation. f-plane.





Midlatitude.





Low latitude.

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www.cims.nyu.edu/~grooms/