Rapidly Rotating Compressible Convection and the Breakdown of the Anelastic Approximation

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Outline

Applications

Compressible Convection Background

Linear Stability of Compressible Convection
Compressibility in Planets and Stars

- Most geophysical and astrophysical fluid systems are stratified to some degree.
- Large depths leads to large variations in state variables across convection zone.
Jupiter’s Internal Structure

- Very sparse direct measurements on interior of Jupiter (i.e. one, Galileo).
- *Ab initio* simulations have provided insight into interior structure.
- \( N_\rho = \ln \left( \frac{\rho_{(bottom)}}{\rho_{(top)}} \right) \approx 7 \)

French et al., 2012
Stellar structure models are coupled with helioseismic observations to provide insight into the Sun.

For the convection zone, $N_\rho \approx 5$

Compressibility cannot be rigorously removed from the governing equations.

Christensen-Dalsgaard et al., 1996
Geometry: The Tilted $f$-plane

- Rotating plane layer of gas
- Colatitude $\theta$ gives angle between rotation and gravity vectors
- Stress-free, isothermal boundary conditions
Modeling Compressible Convection: Perfect Gas

Navier-Stokes equations:

\[ \rho \left( D_t u + \sqrt{\frac{Pr Ta}{Ra}} \hat{\eta} \times u \right) = -H_s \nabla p + H_s \rho \hat{z} + \sqrt{\frac{Pr}{Ra}} F \nu \]

Continuity:

\[ \partial_t \rho + \nabla \cdot (\rho u) = 0 \]

Energy:

\[ \rho \vartheta D_t S = \frac{1}{\sqrt{Pr Ra}} \nabla^2 \vartheta + \frac{1}{H_a} \sqrt{\frac{Pr}{Ra}} \Phi \]

State, Entropy:

\[ p = H_a \left( \frac{\gamma - 1}{\gamma} \right) \rho \vartheta, \quad S = \ln \left( \frac{p^{1/\gamma}}{\rho} \right) \]
Rotating compressible convection of a perfect gas is described by six (6) independent dimensionless parameters.

- \[ Ra = \frac{\rho_0 c_p g \beta H}{\mu k} \]

- \[ Ta = \left( \frac{2 \rho_0 \Omega H^2}{\mu} \right)^{2} \]

- \[ Pr = \frac{\mu c_p}{k} \]

- \[ \gamma = \frac{c_p}{c_v} \]

- Background state:
  - \[ H_a = c_p T_0 g H \]
  - \[ H_s = T_0 \beta H \]
Governing Equations and Parameters

- Rotating compressible convection of a perfect gas is described by six (6) independent dimensionless parameters.

- Forcing and fluid properties:

\[ Ra = \frac{\rho_o c_p g \beta H^4}{T_o \mu k}, \quad Ta = \left(\frac{2\rho_o \Omega H^2}{\mu}\right)^2, \quad Pr = \frac{\mu c_p}{k}, \quad \gamma = \frac{c_p}{c_v} \]
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Background state:

\[ H_a = \frac{c_p T o}{g H}, \quad H_s = \frac{T o}{\beta H} \]
In the absence of convection we have:

\[
\partial_z \bar{p} = \bar{\rho}, \quad 0 = \partial_z^2 \bar{\vartheta}, \quad \bar{p} = H_a \left( \frac{\gamma - 1}{\gamma} \right) \bar{\rho} \bar{\vartheta}
\]

\[
\Rightarrow \bar{\vartheta} = 1 - \left( H_a^{-1} + H_s^{-1} \right) z, \quad \bar{\rho} = \bar{\vartheta}^n, \quad \bar{p} = H_a \left( \frac{\gamma - 1}{\gamma} \right) \bar{\vartheta}^{n+1}
\]

The polytropic index is defined by

\[
n = \left( \frac{\gamma}{\gamma - 1} \right) \frac{H_s}{H_a + H_s} - 1
\]

Adiabatic background state: \( H_s \to \infty, \ n_a = (\gamma - 1)^{-1} \).
Challenges for Solving the Compressible Equations

- The equations are numerically stiff.
- Most systems are characterized by \((Ra, Ta) \gg 1\).
- For most gases \(Pr < 1\).
  - Jupiter: \(10^{-2} \lesssim Pr \lesssim 10^{-1}\)
  - Sun: \(10^{-6} \lesssim Pr \lesssim 10^{-3}\)
Pros: you’re not missing any physics.

Cons: you’re not missing any physics.

Compressible equations have it all: fast and slow inertial waves, acoustic waves, etc.

Do we need to solve for all the physics? Not necessarily.
The Anelastic Equations

First derived by Batchelor (1953), then Ogura & Phillips (1962).

Weak convective fluctuations are much weaker in magnitude than the adiabatic background state:

\[ \rho(x, t) = \rho_0(z) + \epsilon_1 \rho_1(x, t) + \cdots, \quad \epsilon_1 = H_s^{-1} \to 0 \]

Why adiabatic background? Hydrostatic balance to leading order.
The Anelastic Equations

- At first order we have the background state:

\[ \partial_z p_0 = \rho_0, \quad \partial_{zz} \vartheta_0 = 0, \quad p_0 = H_a \left( \frac{\gamma - 1}{\gamma} \right) \rho_0 \vartheta_0. \]

\[ \Rightarrow \vartheta_0 = 1 - H_a^{-1} z \]

- Prognostic equations at \( O(\epsilon_1) \):

\[ \rho_0 \left( D_t u_0 + \sqrt{\frac{Pr Ta}{Ra}} \hat{\eta} \times u_0 \right) = -\nabla p_1 + \rho_1 \hat{z} + \sqrt{\frac{Pr}{Ra}} F_v \]

\[ \nabla \cdot (\rho_0 u_0) = 0 \]

- The anelastic equations are soundproof: \( \epsilon_1 \partial_t \rho_1 \) is subdominant.
Many investigations have been in spherical geometries.


Used for (1) atmospheric and stellar convection; (2) dynamos.

Miesch et al., 2008

Jones et al., 2011
Surprisingly, no direct comparisons between compressible and anelastic rotating convection.
Do the Anelastic Equations Work?

- Surprisingly, no direct comparisons between compressible and anelastic rotating convection.

- Benchmarks have focused on comparing different codes, rather than investigating the accuracy of the anelastic approximation (e.g. Jones et al., 2011).
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Question: how do the anelastic and compressible equations compare as $Ta \to \infty$?
Our starting point: linear stability.

The method:
- Variables decomposed into normal modes, e.g.
  \[ \rho' = \hat{\rho}(z) \exp [i (\mathbf{k}_\perp \cdot \mathbf{x} - \omega t)] \]
- \( \mathbf{k}_\perp \) with minimum value of \( Ra \) gives \( (Ra_c, k_c, \omega_c) \).
Parameter space is given by \((Ta, Pr, N_\rho, n)\).

- We fix \(\gamma = \frac{5}{3}\), such that \(n_a = 1.5\).
- For compressible convection, we require \(n < n_a\).
- We consider values up to \(n = 1.49\).
Compressible Convection Results

Asymptotic behavior is observed for all stratification levels as $Ta \to \infty$: $Ra_c \sim Ta^{2/3}$.

For $Pr = 0.5$, $Ra_c$ increases with stratification; the opposite is true for $Pr = 0.1$. 

$Pr = 0.5, n = 1.49$

$Pr = 0.1, n = 1.49$
Compressible Convection Results

$Pr = 0.5$: vertical velocity

$Pr = 0.1$: vertical velocity

- For $Pr \sim \mathcal{O}(1)$, lower velocities are observed near the top of the layer; the opposite is true as the Prandtl number is reduced.
What about the Anelastic Equations?

Anelastic equations can reproduce \( Pr \gtrsim 0.5 \) results, but fail for lower Prandtl numbers.
Anelastic Shortcomings: $Ta = 10^{12}$

NSE = Navier-Stokes Equations, AE = Anelastic Equations

Open circle denotes the final resting place of the AE.
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Why do the Anelastic Equations Fail?

For $n \approx n_a$, the background states for both equation sets are nearly the same: $\bar{\rho} \approx \rho_0$.

The only difference between the two sets is the form of the mass conservation equation:

$$\partial_t \rho' + \partial_x (\bar{\rho} u') + \partial_z (\bar{\rho} w') = 0 \quad \text{and} \quad \partial_x (\rho_0 u_0) + \partial_z (\rho_0 w_0) = 0$$
Why do the Anelastic Equations Fail?

Axial profiles of each term in compressible mass conservation:

\[ \partial_t \rho' + \partial_x (\bar{\rho}u') + \partial_z (\bar{\rho}w') = 0 \]

\[ Ta = 10^{12}, \; N_{\rho} = 5 \]
Why do the Anelastic Equations Fail?

Axial profiles of each term in compressible mass conservation:

\[ \partial_t \rho' + \partial_x (\bar{\rho} u') + \partial_z (\bar{\rho} w') = 0 \]

\[ \begin{array}{c}
\text{\( P r = 0.5 \)} \\
500 |\partial_t \rho'| \\
|\partial_x (\bar{\rho} u')| \\
|\partial_z (\bar{\rho} w')| \\
\text{\( P r = 0.1 \)} \\
\end{array} \]

\[ Ta = 10^{12}, \ N\rho = 5 \]
Why do the Anelastic Equations Fail?

Axial norm of mass conservation: \( \partial_t \rho' + \partial_x (\bar{\rho}u') + \partial_z (\bar{\rho}w') = 0 \)

- Shows \( |\partial_t \rho'| \) increase with \( N_\rho \) when \( Pr < 1 \).
- Anelastic equations fail when \( |\partial_t \rho'| \sim O(0.1) \)

Axial norms for \( Ta = 10^{12}, Pr = 0.1 \).
The asymptotic scalings of the critical parameters suggests convection is geostrophically balanced:

\[ Ro^{-1} \hat{\eta} \times \bar{\rho} u'_g \approx -H_s \nabla \perp p' \]

Viscosity and inertia perturb the balance to allow convection.

\[ Pr = 0.1, \ Ta = 10^{12}. \]

\[ Ro = \sqrt{\frac{Ra}{Pr Ta}} \]
Curling the geostrophic balance leads to horizontally non-divergent flow at leading order:

\[ \nabla_{\perp} \cdot (\bar{\rho} \mathbf{u}_g') \approx 0 \quad \Rightarrow \quad \nabla_{\perp} \cdot \mathbf{u}_g' \approx 0. \]

To allow for three-dimensional mass conservation we require the presence of ageostrophic motions such that

\[ \partial_t \rho' + \nabla_{\perp} \cdot (\bar{\rho} \mathbf{u}_{ag}') + \partial_z (\bar{\rho} \mathbf{w}') = 0. \]
For Boussinesq quasi-geostrophic (QG) theory, vortex stretching is given by

\[ \nabla \cdot \nabla \cdot u'_{ag} = -\partial_z w' \]

For compressible flows we have

\[ \nabla \cdot (\rho u'_{ag}) = -\partial_z (\rho w') - \partial_t \rho' \]

Fluid compression now acts as a source (or sink) of axial vorticity.

In rapidly rotating, low Prandtl number compressible convection, the fundamental modes are \textit{compressional-inertial oscillations}. 
Synopsis

- Anelastic equations yield spurious results for rapidly rotating, low Prandtl number fluids.
- Soundproof equation sets are thus inappropriate for these flows.
- Now what do we do?
  ⇒ Compressible QG convection equations: can be solved as efficiently as anelastic equations and not limited to adiabatic background.
Open Question

- Do the anelastic equations fail for *turbulent* rotating compressible convection with $O(1)$ Prandtl numbers?
  - Compressional-inertial modes may be nonlinearly excited.
Relevant work