Rapidly Rotating Compressible Convection and the Breakdown of the Anelastic Approximation

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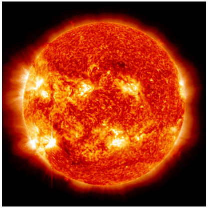
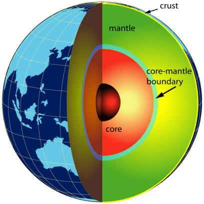
Outline

Compressibility in Planets and Stars

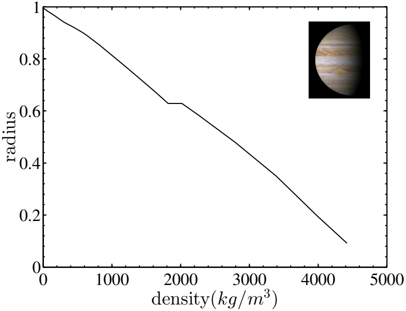
- Most geophysical and astrophysical fluid systems are stratified to some degree.
- Large depths leads to large variations in state variables across convection zone.





Jupiter's Internal Structure

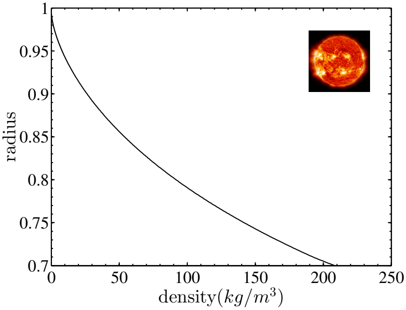
- Very sparse direct measurements on interior of Jupiter (i.e. one, Galileo).
- *Ab initio* simulations have provided insight into interior structure.
- $N_\rho = \ln\left[\frac{\rho(bottom)}{\rho(top)}\right] \approx 7$



French et al., 2012

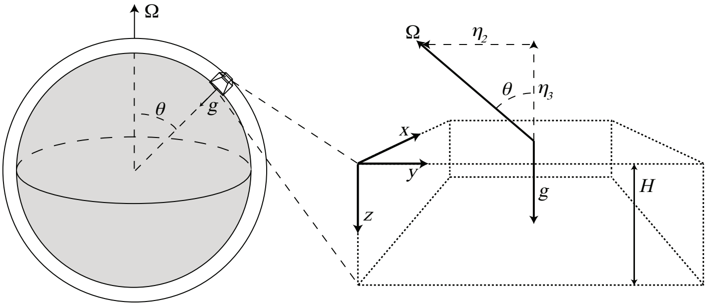
The Sun's Internal Structure

- Stellar structure models are coupled with helioseismic observations to provide insight into the Sun.
- For the convection zone, $N_\rho \approx 5$
- Compressibility cannot be rigorously removed from the governing equations.



Christensen-Dalsgaard et al., 1996

Geometry: The Tilted f-plane



- Rotating plane layer of gas
- Colatitude θ gives angle between rotation and gravity vectors
- Stress-free, isothermal boundary conditions

Modeling Compressible Convection: Perfect Gas

Navier-Stokes equations:

$$\rho\left(D_t\mathbf{u}+\sqrt{\frac{PrTa}{Ra}}\widehat{\boldsymbol{\eta}}\times\mathbf{u}\right)=-H_s\nabla p+H_s\rho\widehat{\mathbf{z}}+\sqrt{\frac{Pr}{Ra}}\mathbf{F}_\nu$$

Continuity:

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0$$

Energy:

$$\rho\vartheta D_t S=\frac{1}{\sqrt{PrRa}}\nabla^2\vartheta+\frac{1}{H_a}\sqrt{\frac{Pr}{Ra}}\Phi$$

State, Entropy:

$$p=H_a\left(\frac{\gamma-1}{\gamma}\right)\rho\vartheta,\quad S=\ln\left(\frac{p^{1/\gamma}}{\rho}\right)$$

Governing Equations and Parameters

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- Background state:

$$H_a = \frac{c_p T_o}{gH}, \quad H_s = \frac{T_o}{\beta H}$$

Governing Equations and Parameters

- In the absence of convection we have:

$$\partial_z \overline{p} = \overline{\rho}, \quad 0 = \partial_z^2 \overline{\vartheta}, \quad \overline{p} = H_a \left(\frac{\gamma - 1}{\gamma} \right) \overline{\rho} \overline{\vartheta}$$

$$\Rightarrow \overline{\vartheta} = 1 - \left(H_a^{-1} + H_s^{-1} \right) z, \quad \overline{\rho} = \overline{\vartheta}^n, \quad \overline{p} = H_a \left(\frac{\gamma - 1}{\gamma} \right) \overline{\vartheta}^{n+1}$$

- The polytropic index is defined by

$$n = \left(\frac{\gamma}{\gamma - 1} \right) \frac{H_s}{H_a + H_s} - 1$$

- Adiabatic background state: $H_s \to \infty$, $n_a = (\gamma - 1)^{-1}$.

Challenges for Solving the Compressible Equations

- The equations are numerically stiff.
- Most systems are characterized by $(Ra, Ta) \gg 1$.
- For most gases $Pr < 1$.
  - Jupiter: $10^{-2} \lesssim Pr \lesssim 10^{-1}$
  - Sun: $10^{-6} \lesssim Pr \lesssim 10^{-3}$

Challenges for Solving the Compressible Equations

- Pros: you're not missing any physics.
- Cons: you're not missing any physics.
- Compressible equations have it all: fast and slow inertial waves, acoustic waves, etc.
- Do we need to solve for all the physics? Not necessarily.

The Anelastic Equations

- First derived by Batchelor (1953), then Ogura & Phillips (1962).
- Weak convective fluctuations are much weaker in magnitude than the *adiabatic* background state:

$$\rho(\mathbf{x},t) = \rho_0(z) + \epsilon_1 \rho_1(\mathbf{x},t) + \cdots , \quad \epsilon_1 = H_s^{-1} \to 0$$

- Why adiabatic background? Hydrostatic balance to leading order.

The Anelastic Equations

- At first order we have the background state:

$$\partial_z p_0 = \rho_0, \quad \partial_{zz}\vartheta_0 = 0, \quad p_0 = H_a\left(\frac{\gamma - 1}{\gamma}\right)\rho_0\vartheta_0.$$

$$\Rightarrow \vartheta_0 = 1 - H_a^{-1} z$$

- Prognostic equations at $\mathcal{O}(\epsilon_1)$:

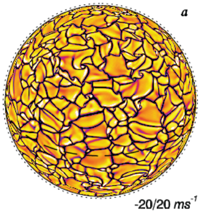
$$\rho_0\left(D_t\mathbf{u}_0 + \sqrt{\frac{PrTa}{Ra}}\widehat{\boldsymbol{\eta}} \times \mathbf{u}_0\right) = -\nabla p_1 + \rho_1\,\widehat{\mathbf{z}} + \sqrt{\frac{Pr}{Ra}}\mathbf{F}_\nu$$

$$\nabla\cdot(\rho_0\mathbf{u}_0) = 0$$

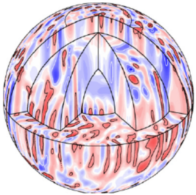
- The anelastic equations are *soundproof*: $\epsilon_1\partial_t\rho_1$ is subdominant.

Simulating the Anelastic Equations

- Many investigations have been in spherical geometries.
- Began with Gilman and Glatzmaier (1981).
- Used for (1) atmospheric and stellar convection; (2) dynamos.



Miesch et al., 2008



Jones et al., 2011

Do the Anelastic Equations Work?

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- Question: how do the anelastic and compressible equations compare as $Ta \to \infty$?

Linear Compressible and Anelastic Convection

- Our starting point: linear stability.
- The method:
  - Variables decomposed into normal modes, e.g.

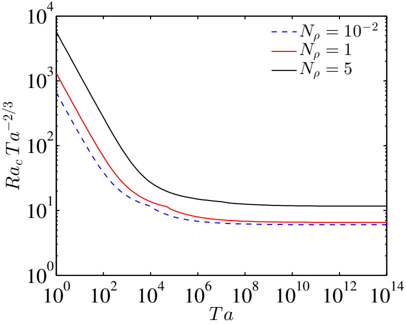
$$\rho' = \widehat{\rho}(z) \exp\left[i\left(\mathbf{k}_\perp \cdot \mathbf{x} - \omega t\right)\right]$$

  - $\mathbf{k}_\perp$ with minimum value of Ra gives (Ra_c, k_c, ω_c).

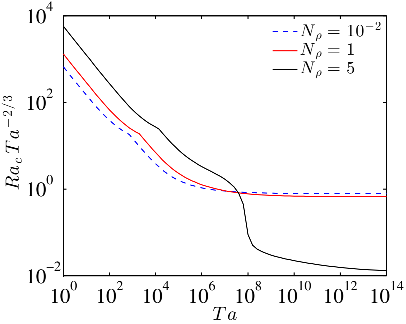
Linear Compressible and Anelastic Convection

- Parameter space is given by (Ta, Pr, N_ρ, n).
- We fix $\gamma = 5/3$, such that $n_a = 1.5$.
- For compressible convection, we require $n < n_a$.
- We consider values up to $n = 1.49$.

Compressible Convection Results



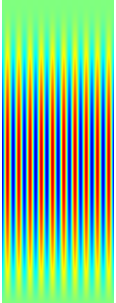
$Pr = 0.5, n = 1.49$



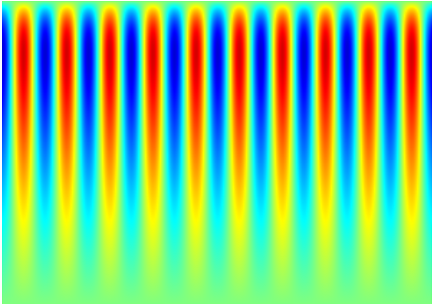
$Pr = 0.1, n = 1.49$

- Asymptotic behavior is observed for all stratification levels as $Ta \to \infty$: $Ra_c \sim Ta^{2/3}$.
- For $Pr = 0.5$, Ra_c increases with stratification; the opposite is true for $Pr = 0.1$.

Compressible Convection Results



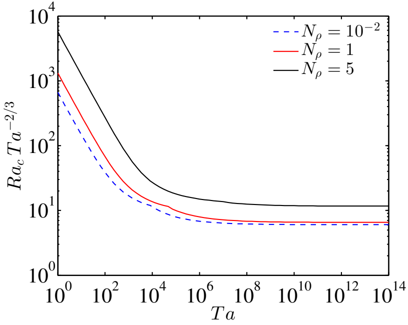
$Pr = 0.5$: vertical velocity



$Pr = 0.1$: vertical velocity

- For $Pr \sim \mathcal{O}(1)$, lower velocities are observed near the top of the layer; the opposite is true as the Prandtl number is reduced.

What about the Anelastic Equations?



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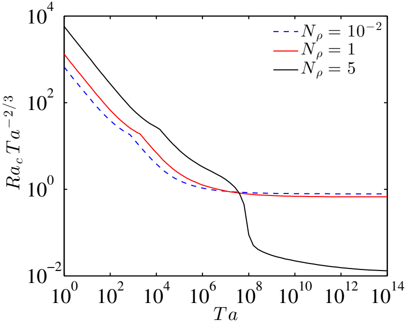

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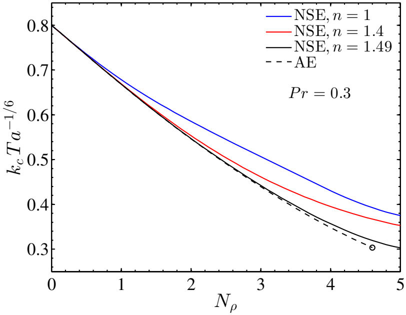
- Anelastic equations can reproduce $Pr \gtrsim 0.5$ results, but fail for lower Prandtl numbers.

Anelastic Shortcomings: $Ta = 10^{12}$



- NSE = Navier-Stokes Equations, AE = Anelastic Equations
- Open circle denotes the final resting place of the AE.

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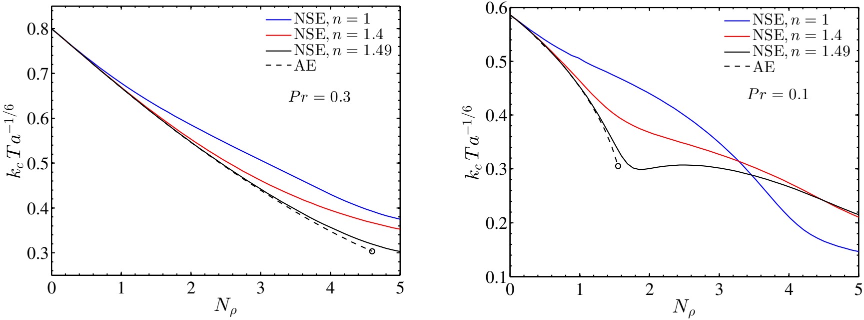

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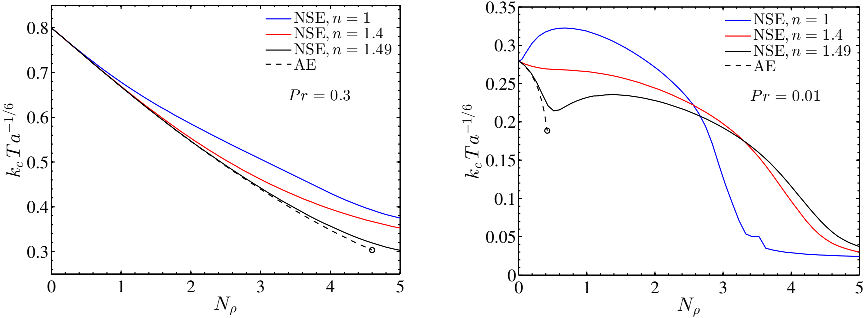

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Why do the Anelastic Equations Fail?

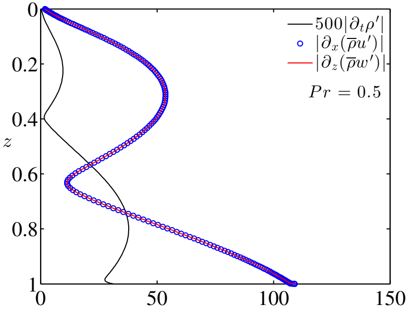
- For $n \approx n_a$, the background states for both equation sets are nearly the same: $\overline{\rho} \approx \rho_0$.
- The only difference between the two sets is the form of the mass conservation equation:

$$\partial_t \rho' + \partial_x \left(\overline{\rho} u'\right) + \partial_z \left(\overline{\rho} w'\right) = 0 \quad \text{and} \quad \partial_x \left(\rho_0 u_0\right) + \partial_z \left(\rho_0 w_0\right) = 0$$

Why do the Anelastic Equations Fail?

Axial profiles of each term in compressible mass conservation:

$$\partial_t \rho' + \partial_x (\overline{\rho} u') + \partial_z (\overline{\rho} w') = 0$$



$$Ta = 10^{12},\ N_\rho = 5$$

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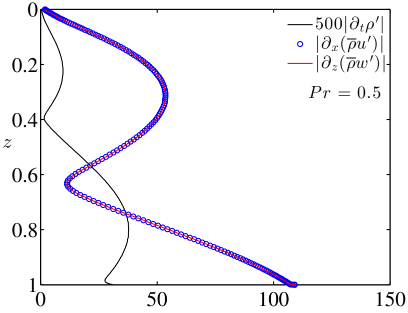
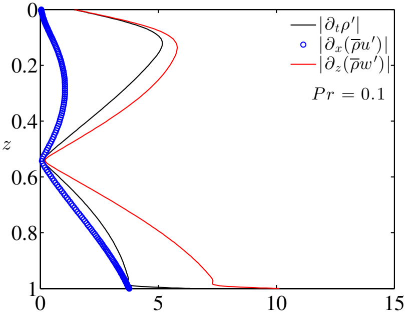


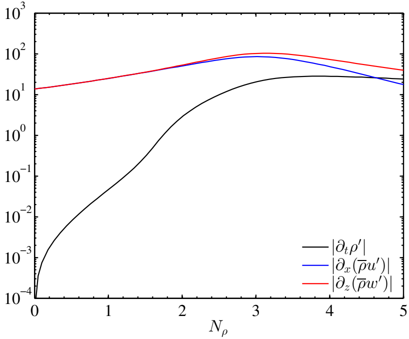


$$Ta = 10^{12},\ N_\rho = 5$$

Why do the Anelastic Equations Fail?

Axial norm of mass conservation: $\partial_t \rho' + \partial_x (\overline{\rho} u') + \partial_z (\overline{\rho} w') = 0$

- Shows $|\partial_t \rho'|$ increase with N_ρ when $Pr < 1$.
- Anelastic equations fail when $|\partial_t \rho'| \sim O(0.1)$



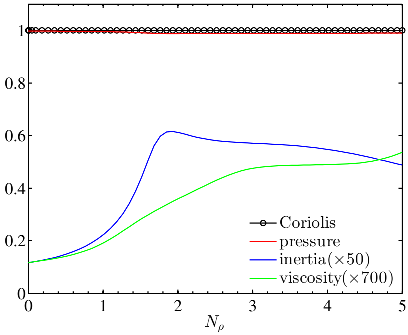
Axial norms for $Ta = 10^{12}$, $Pr = 0.1$

Geostrophic Balance in Compressible Convection

- The asymptotic scalings of the critical parameters suggests convection is geostrophically balanced:

$$Ro^{-1}\widehat{\boldsymbol{\eta}} \times \overline{\rho}\mathbf{u}_g' \approx -H_s \nabla_\perp p'$$

- Viscosity and inertia perturb the balance to allow convection.



$Pr = 0.1$, $Ta = 10^{12}$.

$$Ro = \sqrt{\frac{Ra}{PrTa}}$$

Geostrophic Balance in Compressible Convection

- Curling the geostrophic balance leads to horizontally non-divergent flow at leading order:

$$\nabla_\perp \cdot \left(\overline{\rho}\mathbf{u}'_g\right) \approx 0 \quad \Rightarrow \quad \nabla_\perp \cdot \mathbf{u}'_g \approx 0.$$

- To allow for three-dimensional mass conservation we require the presence of ageostrophic motions such that

$$\partial_t \rho' + \nabla_\perp \cdot \left(\overline{\rho}\mathbf{u}'_{ag}\right) + \partial_z \left(\overline{\rho} w'\right) = 0.$$

Vortex Stretching in Compressible Convection: Compressional-inertial Oscillations

- For Boussinesq quasi-geostrophic (QG) theory, vortex stretching is given by

$$\nabla_\perp \cdot \mathbf{u}'_{ag} = -\partial_z w'$$

- For compressible flows we have

$$\nabla_\perp \cdot \left(\bar{\rho}\mathbf{u}'_{ag}\right) = -\partial_z \left(\bar{\rho} w'\right) - \partial_t \rho'$$

- Fluid compression now acts as a source (or sink) of axial vorticity.
- In rapidly rotating, low Prandtl number compressible convection, the fundamental modes are *compressional-inertial oscillations*.

Summary

- Anelastic equations yield spurious results for rapidly rotating, low Prandtl number fluids.
- Soundproof equation sets are thus inappropriate for these flows.
- Now what do we do?
  $\Rightarrow$ Compressible QG convection equations: can be solved as efficiently as anelastic equations and not limited to adiabatic background.

Open Question

- Do the anelastic equations fail for *turbulent* rotating compressible convection with $\mathcal{O}(1)$ Prandtl numbers?
    - Compressional-inertial modes may be nonlinearly excited.

Relevant work

- Calkins, M.A., Julien, K., Marti, P. Onset of Rotating and Non-rotating Convection in Compressible and Anelastic Ideal Gases. *Geophys. Astrophys. Fluid Dyn.*, in review.
- Calkins, M.A., Julien, K., Marti, P. Rapidly rotating compressible convection and the shortcomings of the anelastic equations. Submitted to *Proc. R. Soc. A.* Preprint: http://arxiv.org/abs/1409.1959
- Calkins, M.A., Julien, K., Marti, P. A model for compressible quasi-geostrophic convection on the tilted f-plane. *J. Fluid Mech.*. In prep.