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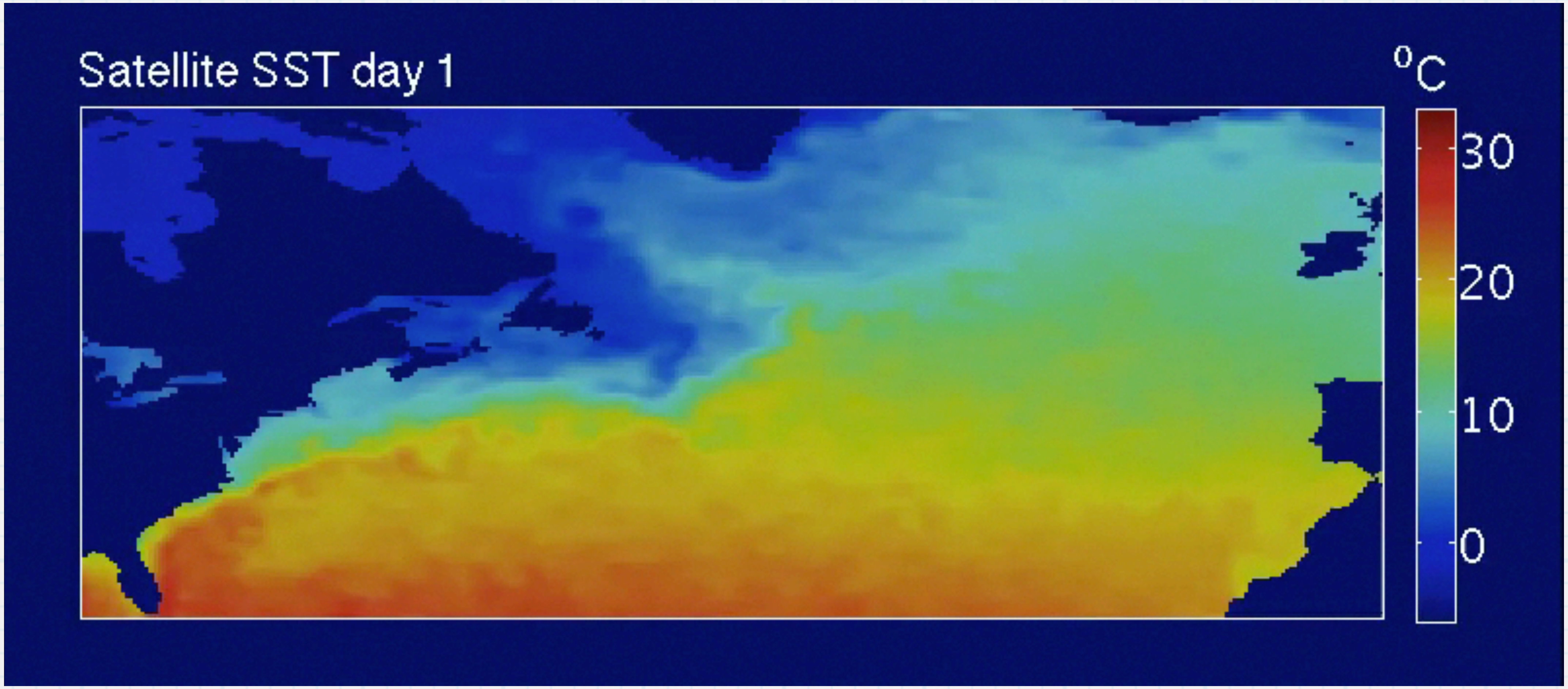
Diagnosing sea-surface temperature dynamics from stochastically-forced fluctuations



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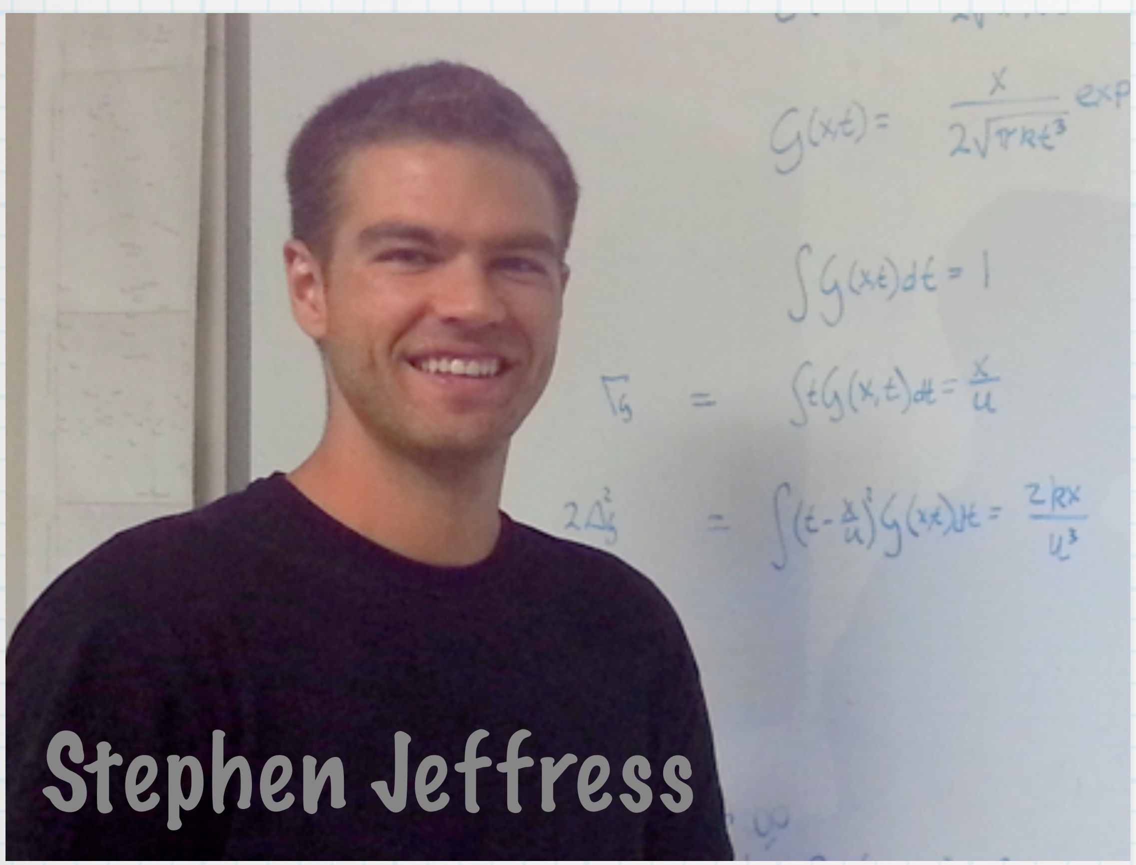
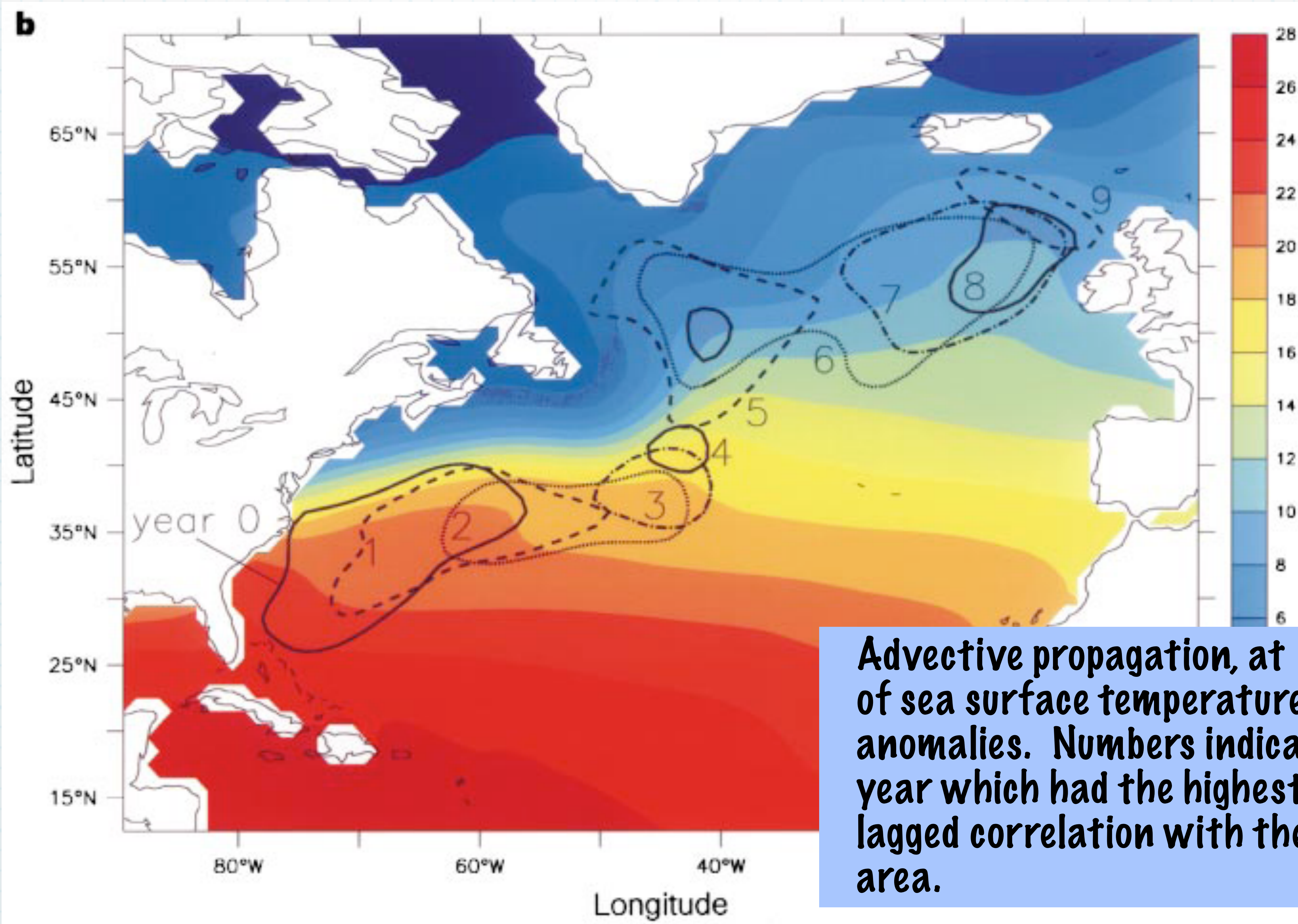


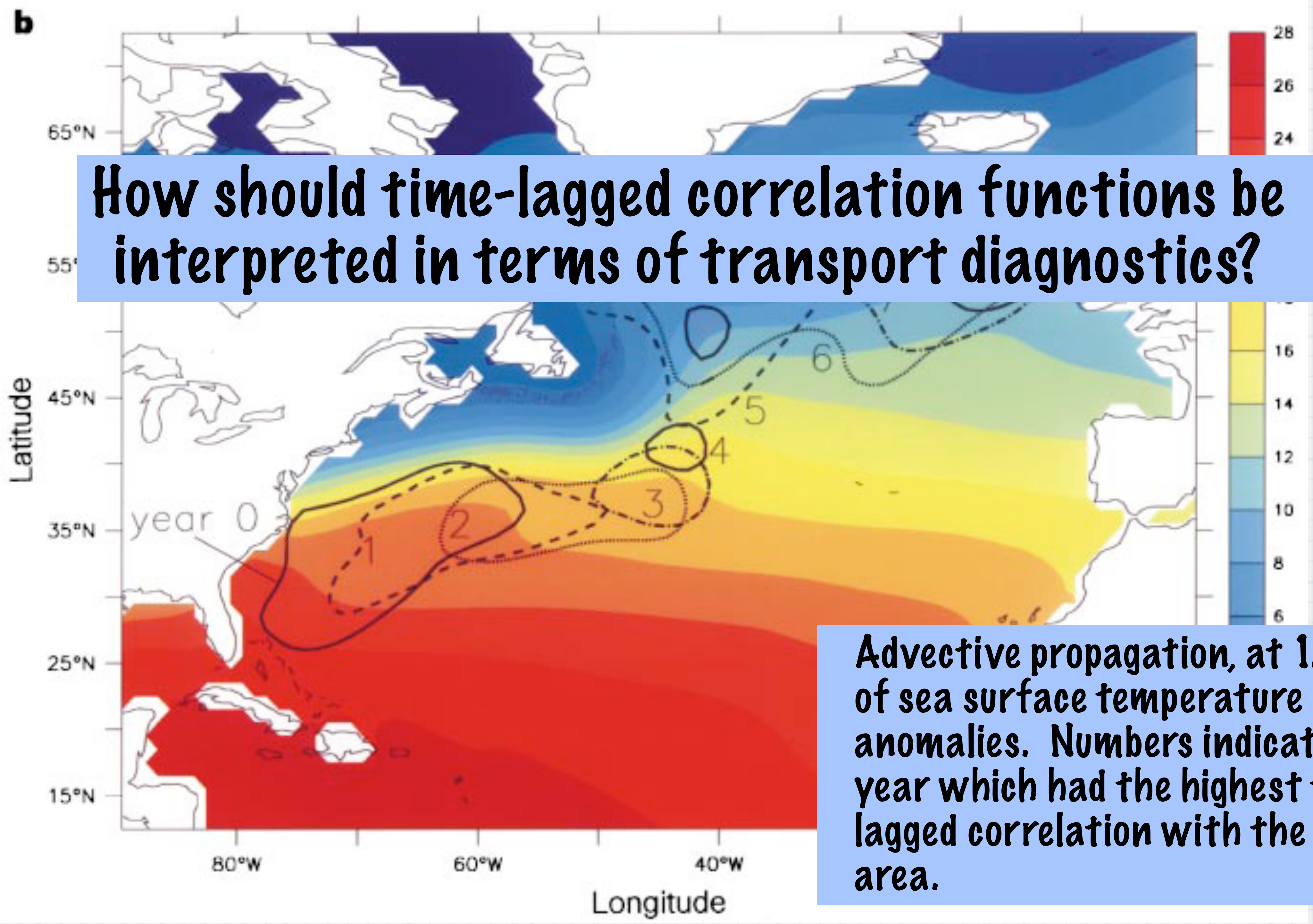
* S. A. Jeffress and T. W. N. Haine, Correlated signals and causal transport in ocean circulation, Q. J. R. Meteorol. Soc. (2014) 10.1002/qj.2313
* S. A. Jeffress and T. W. N. Haine, Estimating sea-surface temperature transport fields from stochastically-forced fluctuations, New Journal of Physics 16 (2014) 10.1088/1367-2630/16/10/105001
* S. A. Jeffress and T. W. N. Haine, The Transport of North Atlantic Sea Surface Temperature Anomalies from a Fluctuation-Dissipation Based Inverse Method, J. Climate, in prep.



Advective propagation, at 1.7cm/s, of sea surface temperature anomalies. Numbers indicate the year which had the highest time-lagged correlation with the year-0 area.

Sutton & Allen, Nature, 1997

How should time-lagged correlation functions be interpreted in terms of transport diagnostics?



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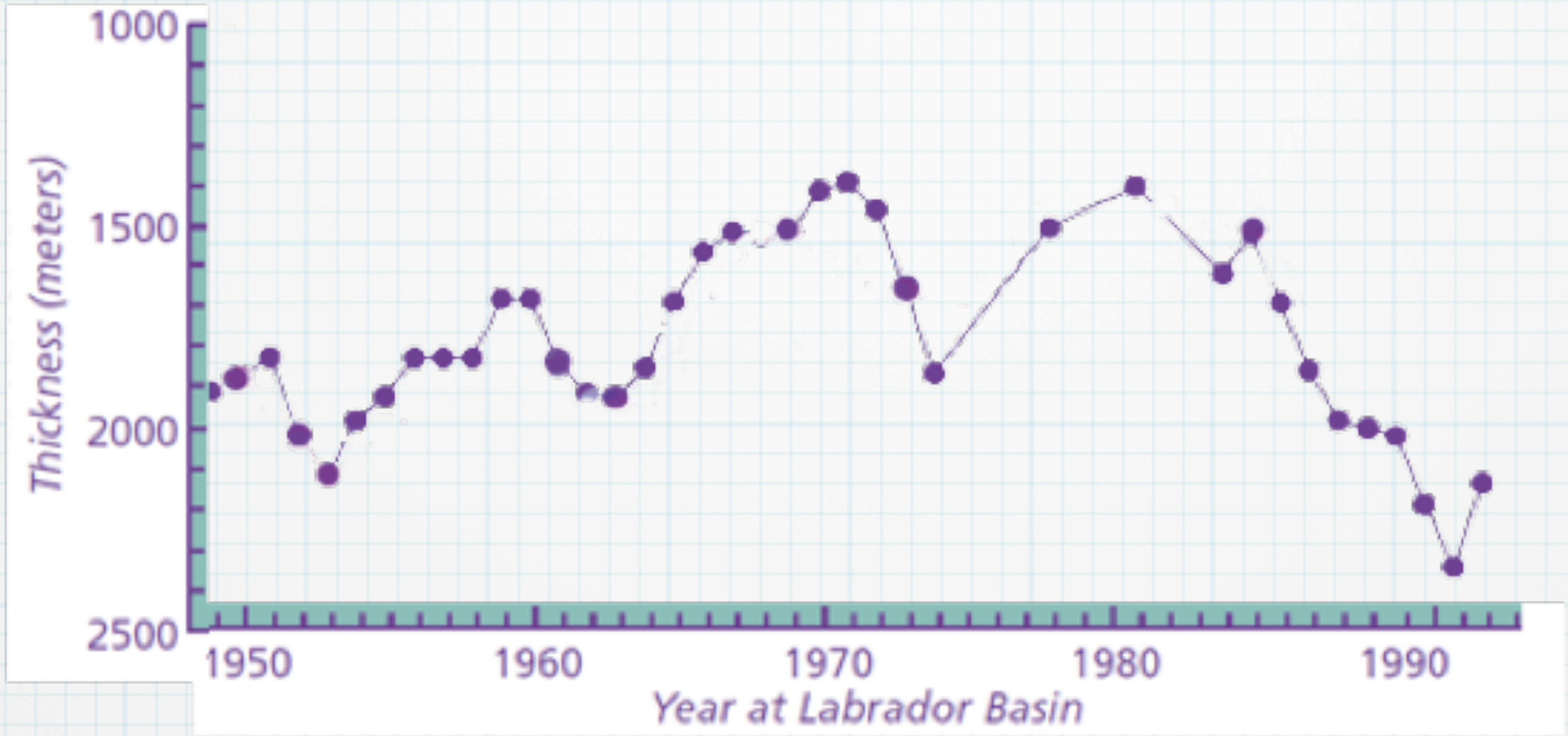
"Correlation Implies Transport"



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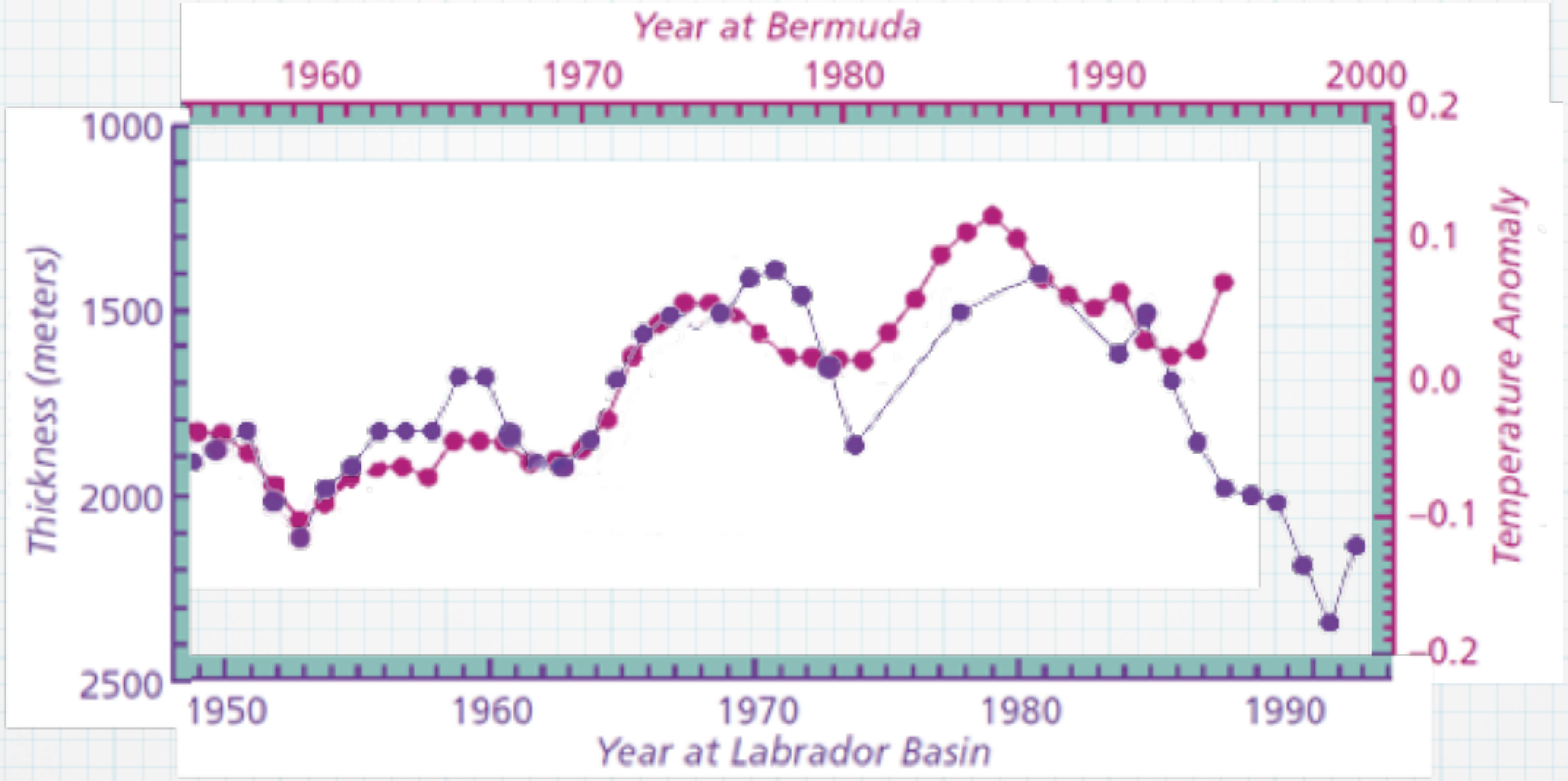

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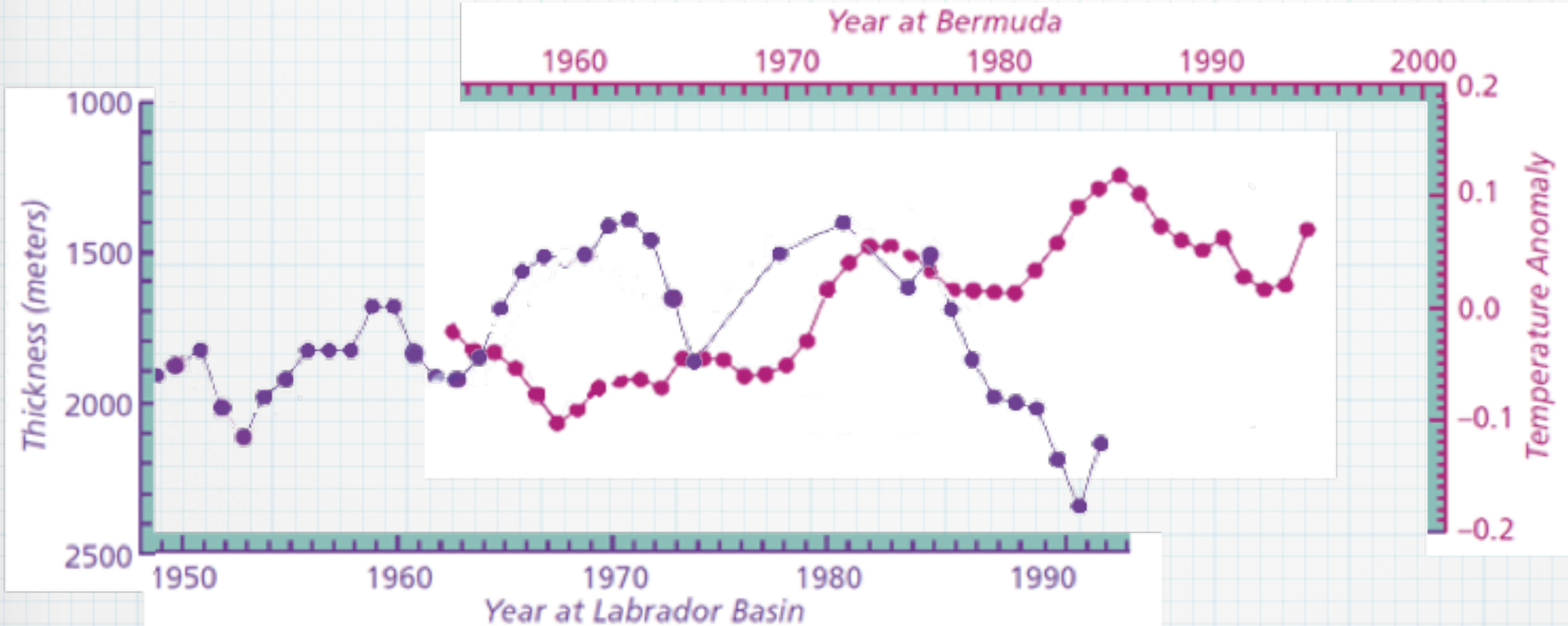

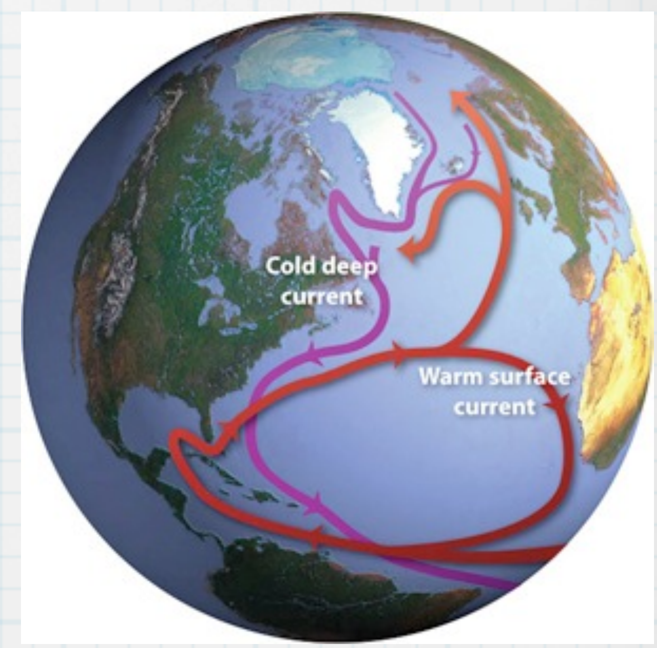

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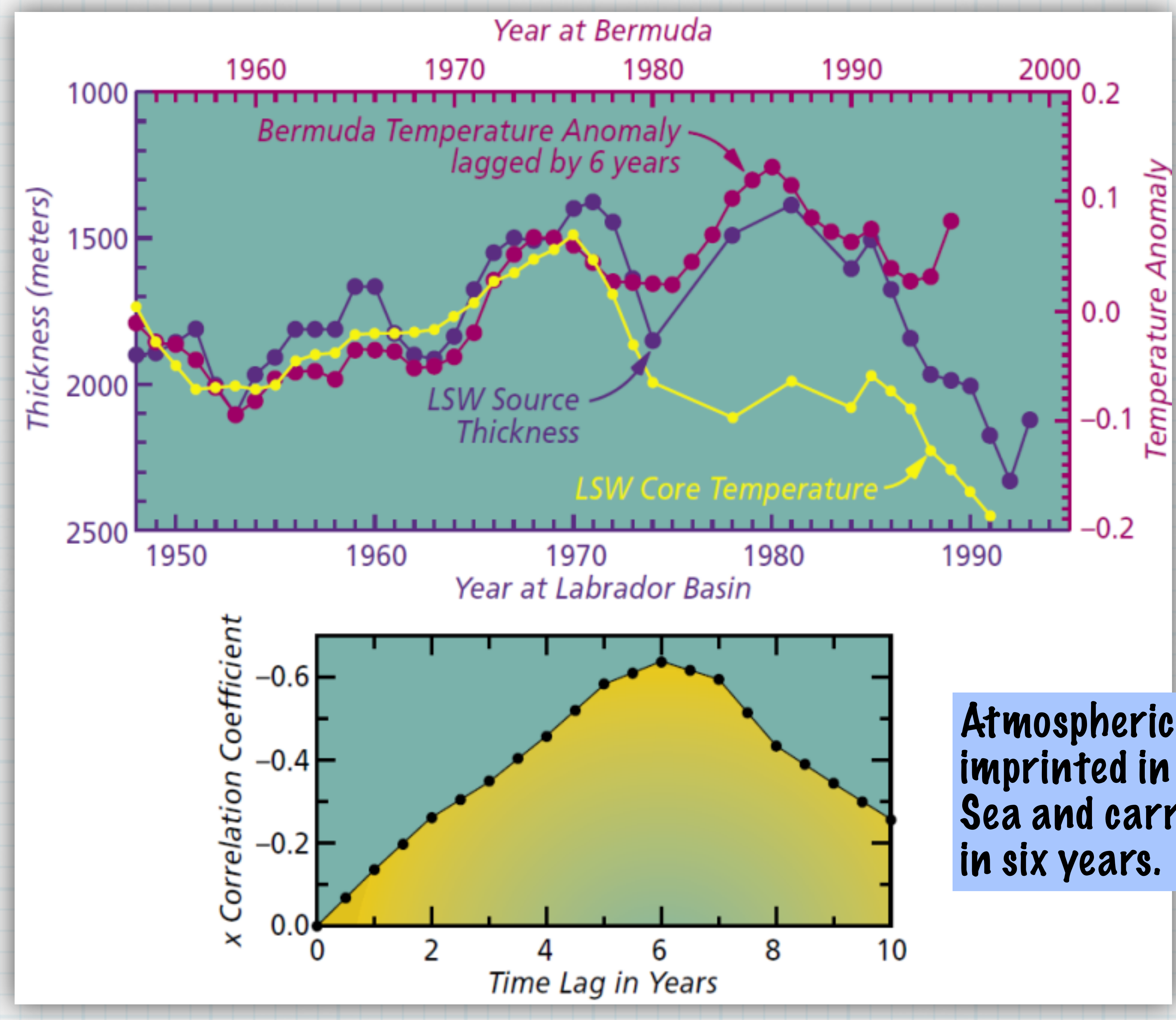


Atmospheric conditions are imprinted in the Labrador Sea and carried to Bermuda in six years.

Curry and McCartney, Nature, 1998

Outline

- When does correlation imply causation?
- How is SST transport information extracted from SST data?

Outline

✶ When does correlation imply causation?
✶ How is SST transport information extracted from SST data?

* Two-point lagged covariance analysis
* Global covariance information analysis
* Application to SST data
* Discussion/Conclusion

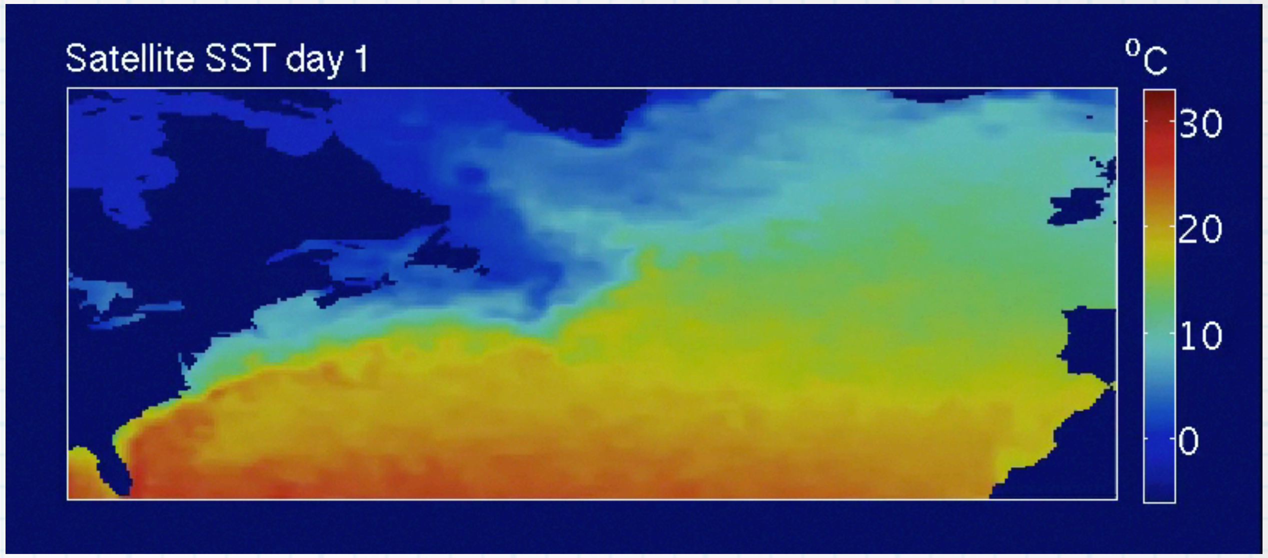
Governing equation:

$$\frac{d}{dt}\mathbf{c}(t) = \mathbf{A}(t)\mathbf{c}(t) + \mathbf{q}(t)$$

* Theory I
  Two-point lagged covariance

Governing equation:

$$\frac{d}{dt}\mathbf{c}(t) = \mathbf{A}(t)\mathbf{c}(t) + \mathbf{q}(t)$$



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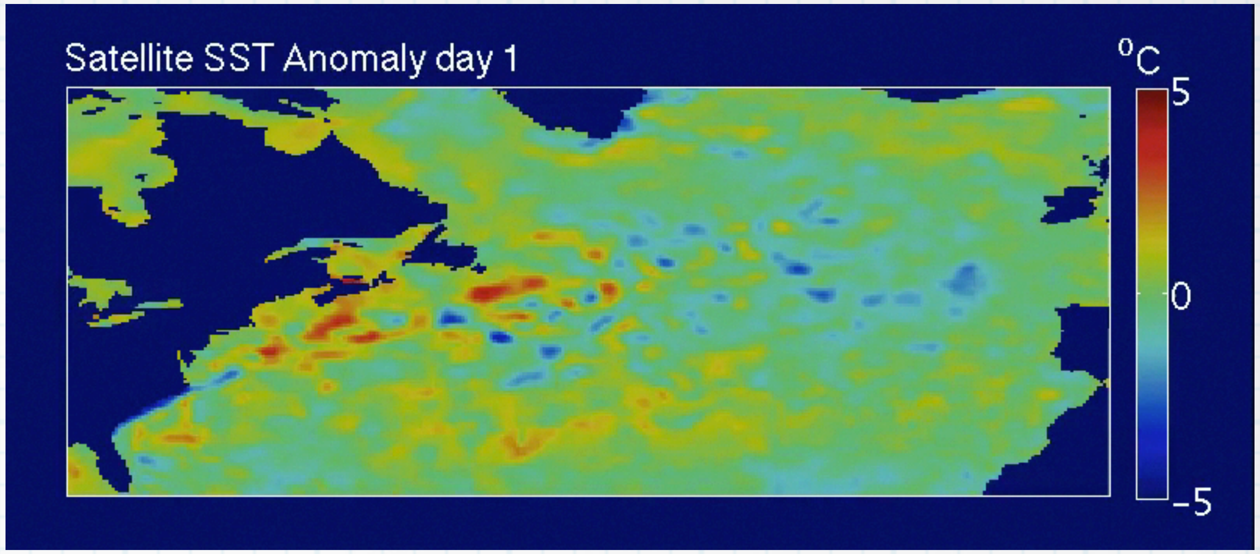

Time-averaged equation:

$$\frac{d}{dt}\mathbf{c}'(t) = \bar{\mathbf{A}}\mathbf{c}'(t) + \mathbf{f}'(t)$$

Governing equation:

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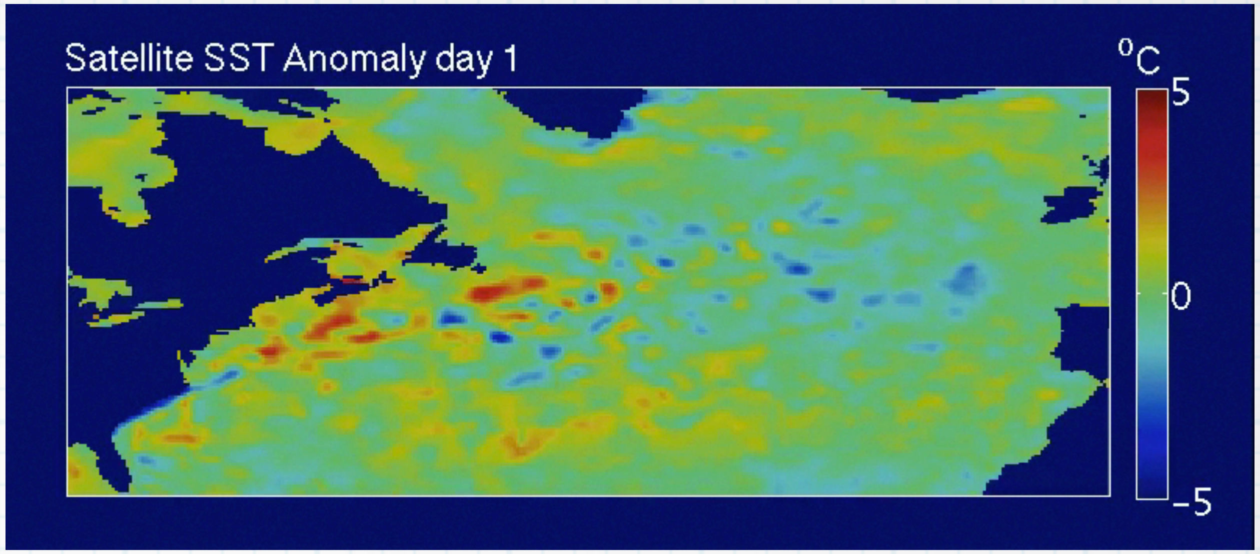

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Time averaging:

$$\mathbf{A}(t) = \bar{\mathbf{A}} + \mathbf{A}'(t)$$

$$\mathbf{f}'(t) = \mathbf{A}'(t)\mathbf{c}(t) - \overline{\mathbf{A}'(t)\mathbf{c}'(t)} + \mathbf{q}'(t)$$

$$\bar{\mathbf{A}} = \frac{1}{2\ell}\int_{-\ell}^{\ell} \mathbf{A}(t)dt$$

Governing equation:

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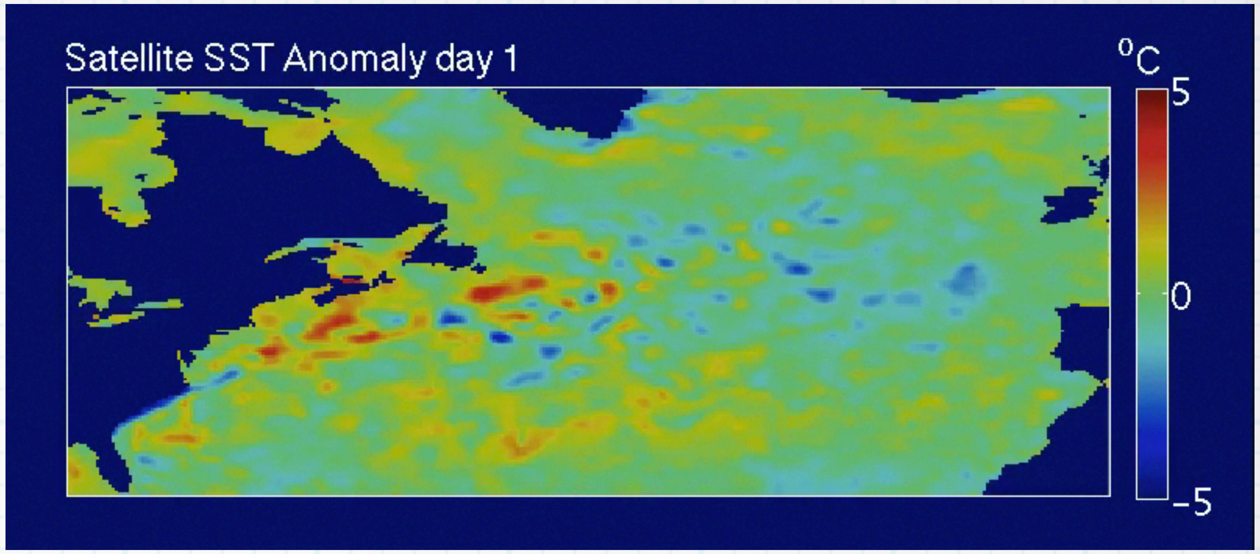

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Solution with Green's fn:

$$\mathbf{c}'(t) = \int_0^t \mathbf{G}(t-t')\mathbf{f}'(t')dt'$$

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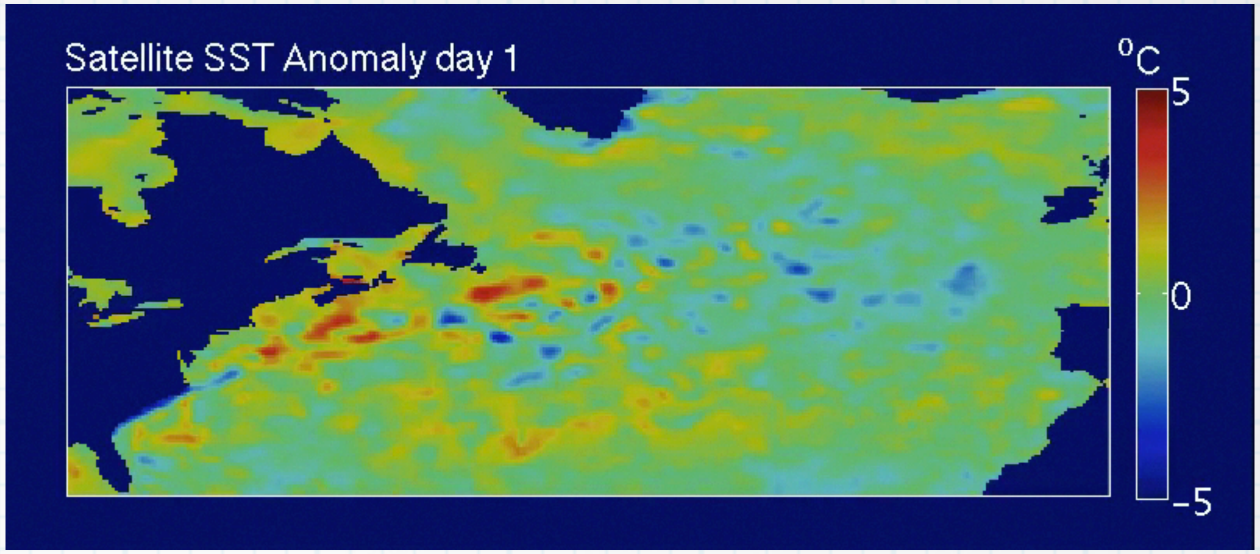

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Green's fn equation:

$$\frac{d}{dt}\mathbf{G}(t) = \bar{\mathbf{A}}\mathbf{G}(t) + \delta(t)$$

$$\mathbf{G}(\tau) = e^{\bar{\mathbf{A}}\ \tau}\ H(\tau)$$

$$\tau = t - t'$$

Forcing covariance:

$$\mathbf{S}_{\mathbf{ff}}(\tau) = \left\langle \mathbf{f}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \right\rangle$$

Forcing covariance:

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Field/forcing covariance:

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Propagation:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \int \mathbf{G}(\tau - t')\mathbf{S}_{\mathbf{ff}}(t')dt'$$

Forcing covariance:

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Propagation:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \int \mathbf{G}(\tau - t')\mathbf{S}_{\mathbf{ff}}(t')dt'$$

$$[S_{cf}(\tau)]_{ij} = \sum_k \int_{-\infty}^{\infty} G_{ik}(\tau - t') \, [S_{ff}(t')]_{kj} \, dt'$$

Forcing covariance:

$$\mathbf{S}_{\mathbf{ff}}(\tau) = \langle \mathbf{f}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \rangle$$

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field point i

forcing point j

Forcing covariance:

$$\mathbf{S}_{\mathbf{ff}}(\tau) = \langle \mathbf{f}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \rangle$$

Field/forcing covariance:

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Propagation:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \int \mathbf{G}(\tau - t')\mathbf{S}_{\mathbf{ff}}(t')dt'$$

Field point i/forcing point j
time-lagged covariance fn.

$$[S_{cf}(\tau)]_{ij} = \sum_k \int_{-\infty}^{\infty} G_{ik}(\tau - t')\,[S_{ff}(t')]_{kj}\,dt'$$

field point i

forcing point j

G_{ij} smoothed over space/time
forcing covariance
at source point j

Forcing covariance:

$$\mathbf{S}_{\mathbf{ff}}(\tau) = \langle \mathbf{f}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \rangle$$

Field point i/forcing point j
time-lagged covariance fn.

Field/forcing covariance:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \langle \mathbf{c}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \rangle$$

Propagation:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \int \mathbf{G}(\tau - t')\mathbf{S}_{\mathbf{ff}}(t')dt'$$

$$[S_{cf}(\tau)]_{ij} = \qquad G_{ij}(\tau)$$ For white noise forcing

field point i

forcing point j

Forcing covariance:

$$\mathbf{S}_{\mathbf{ff}}(\tau) = \left\langle \mathbf{f}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \right\rangle$$

Field/forcing covariance:

$$\mathbf{S}_{\mathbf{cf}}(\tau) = \left\langle \mathbf{c}'(\tau)\mathbf{f}'(0)^{\mathrm{T}} \right\rangle$$

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Field/forcing covariance:

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Field covariance:

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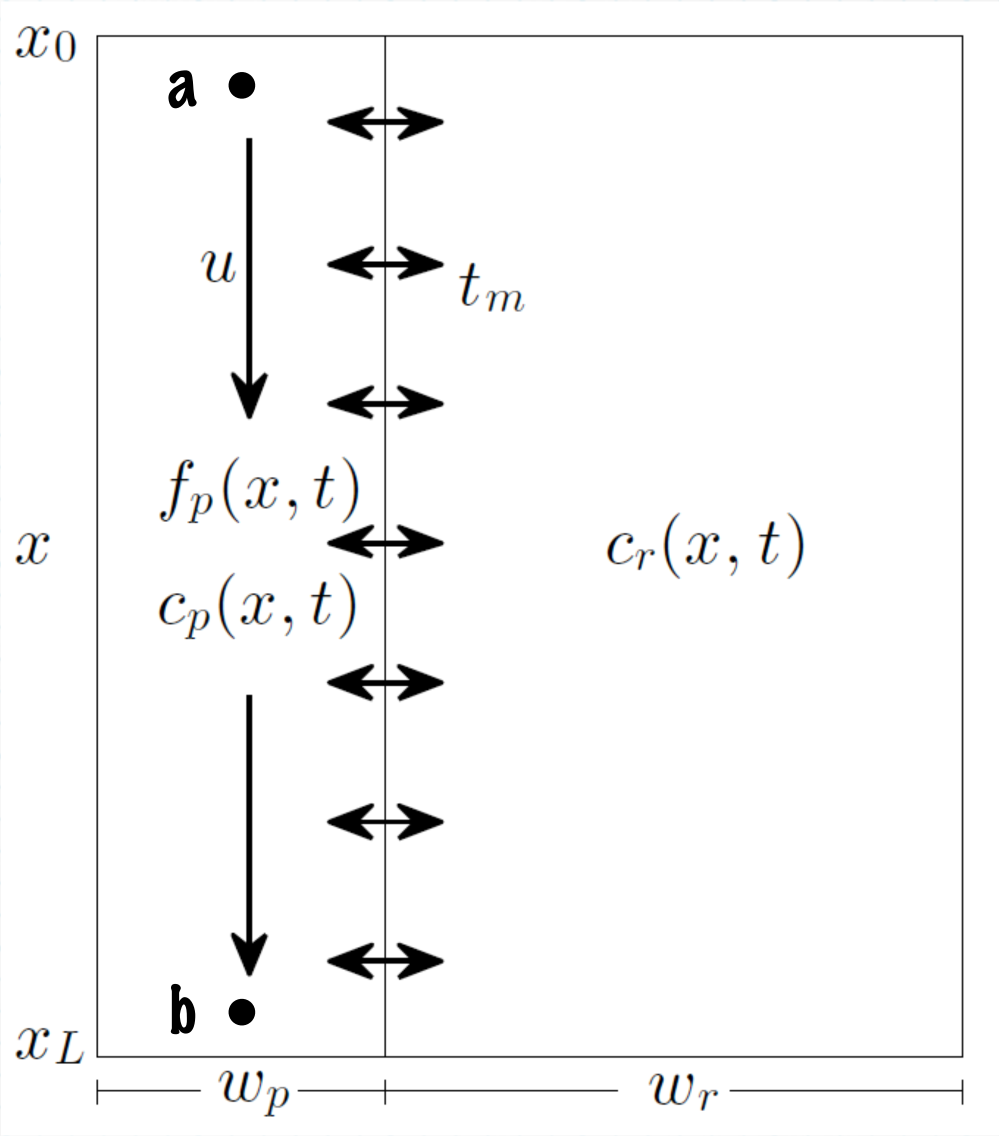
Propagation:

$$\mathbf{S}_{\mathbf{cc}}(\tau) = \iint \mathbf{G}(t')\mathbf{S}_{\mathbf{ff}}(\tau - t' + t'')\mathbf{G}^{\mathrm{T}}(t'')dt'dt''$$

Illustrative Example

Leaky pipe model of advective flow exchanging with stagnant reservoir:

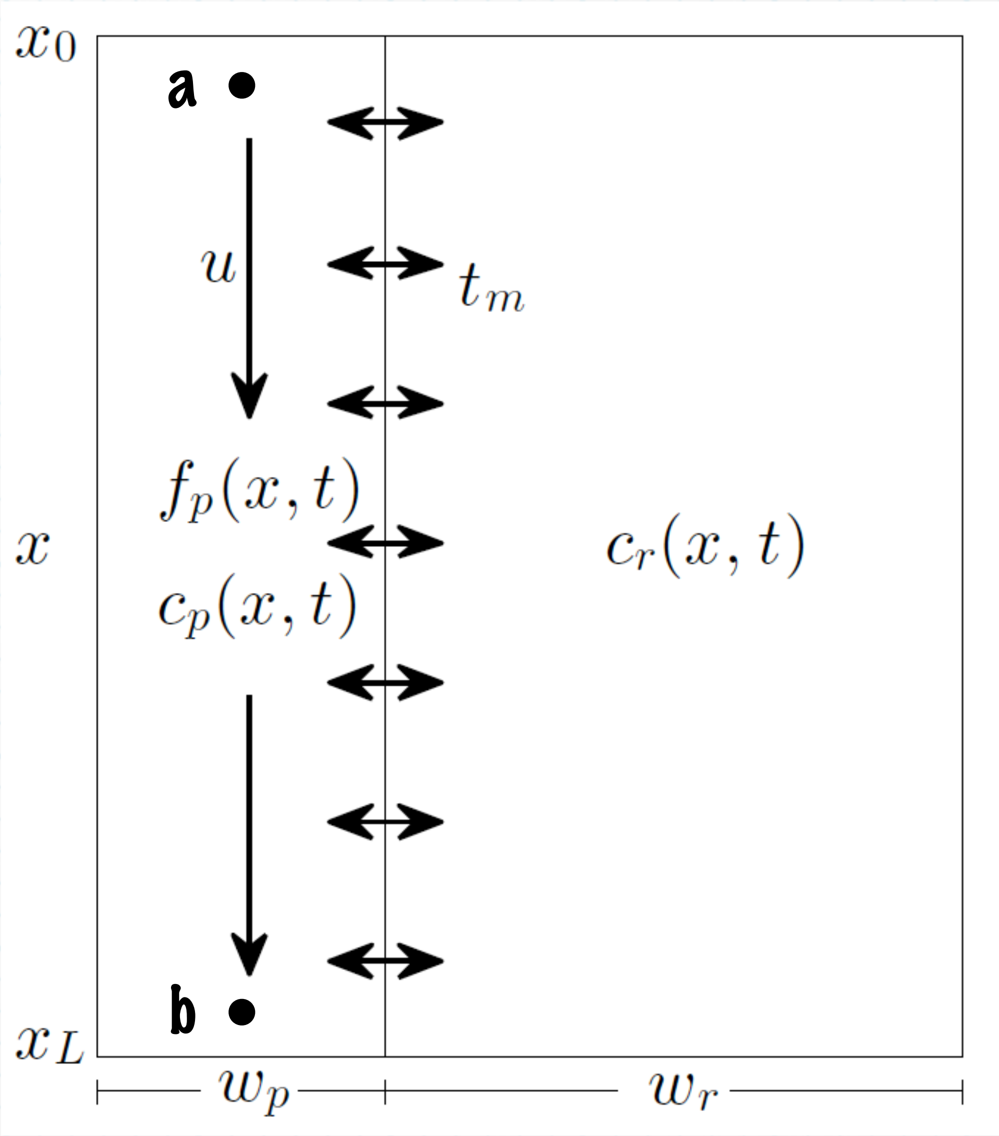


Illustrative Example

Leaky pipe model of advective flow exchanging with stagnant reservoir:

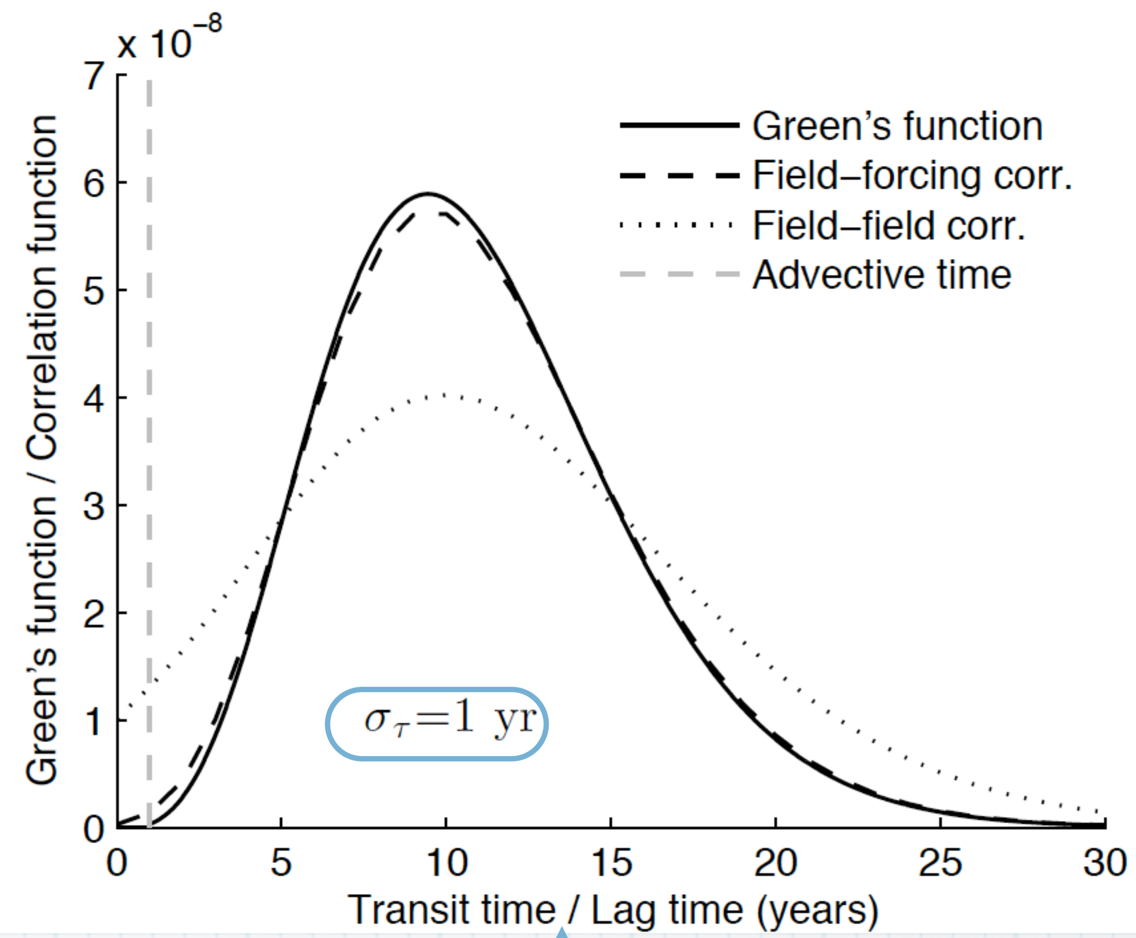
$$S_{ff}(x, x', \tau) = \frac{1}{\sigma_\tau \sqrt{2\pi}} \exp\left(-\frac{\tau^2}{2\sigma_\tau^2}\right) \delta(x - x')$$

Forcing covariance is white in space, Gaussian in time with scale σ_τ



Illustrative Example

- Time-lagged field/forcing covariance fn. is an unbiased estimate of the Green's fn. smoothed by the forcing covariance.



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Forcing covariance is white in space, Gaussian in time with scale σ_τ

Response at b from impulsive source at a, for perfect covariance estimates

* Illustrative Example

- Time-lagged field/forcing covariance fn. is an unbiased estimate of the Green's fn. smoothed by the forcing covariance.
- Not true for field/field covariance in general.

$$S_{ff}(x, x', \tau) = \frac{1}{\sigma_\tau \sqrt{2\pi}} \exp\left(-\frac{\tau^2}{2\sigma_\tau^2}\right) \delta(x - x')$$

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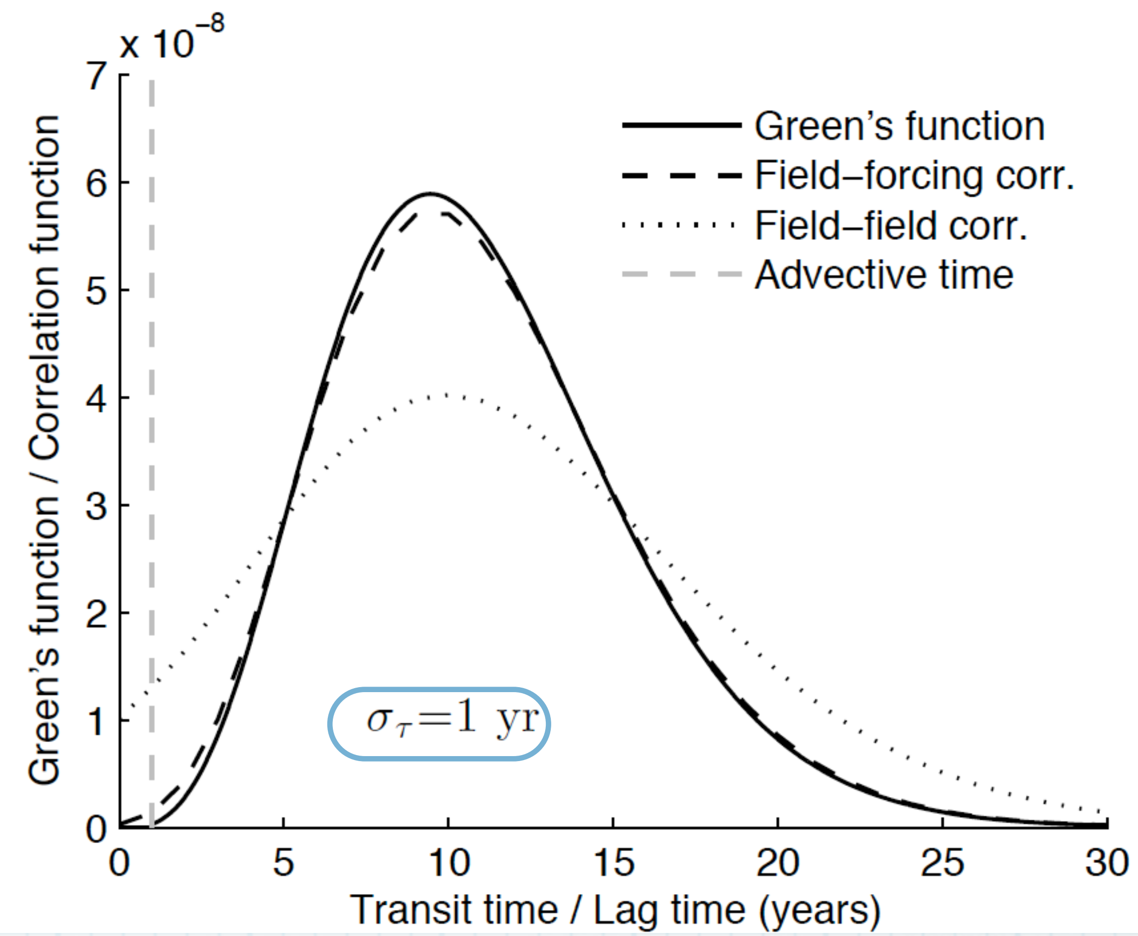
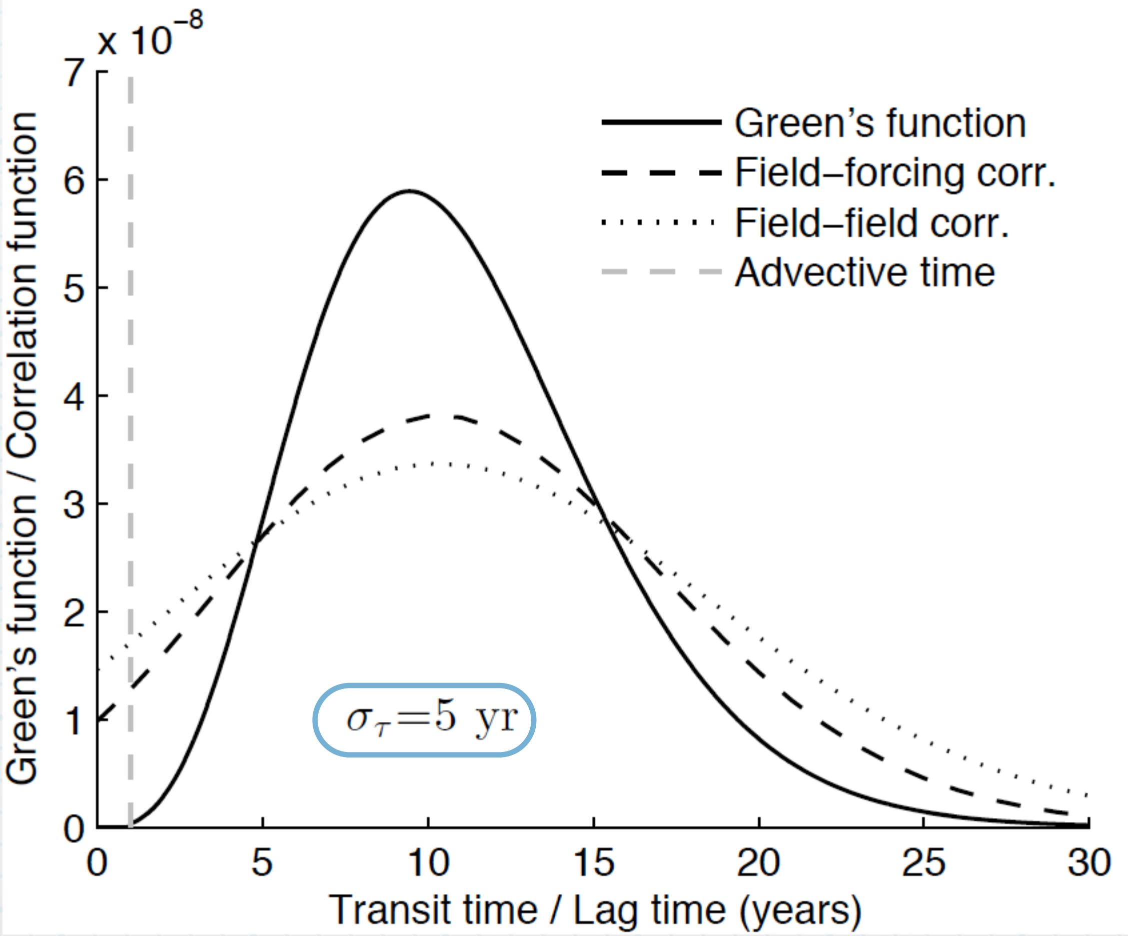




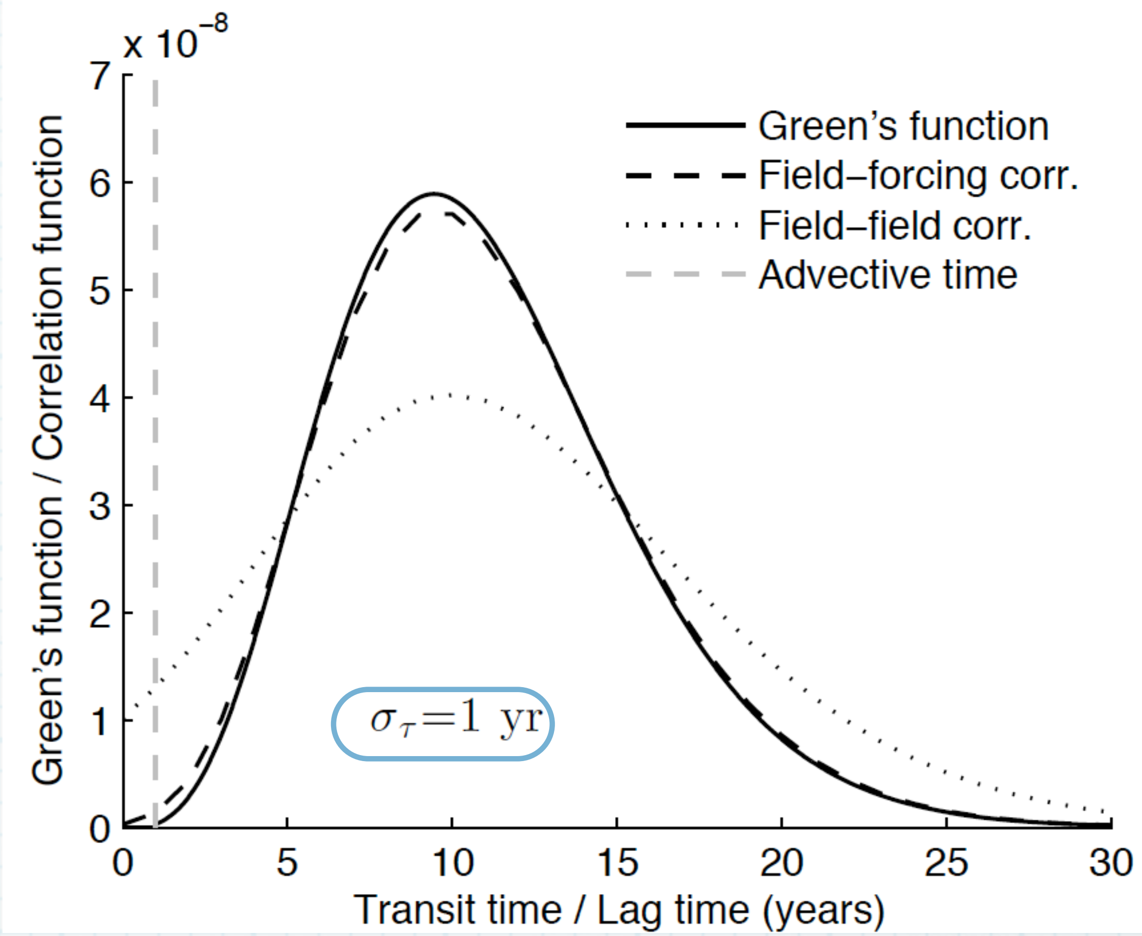
* Illustrative Example

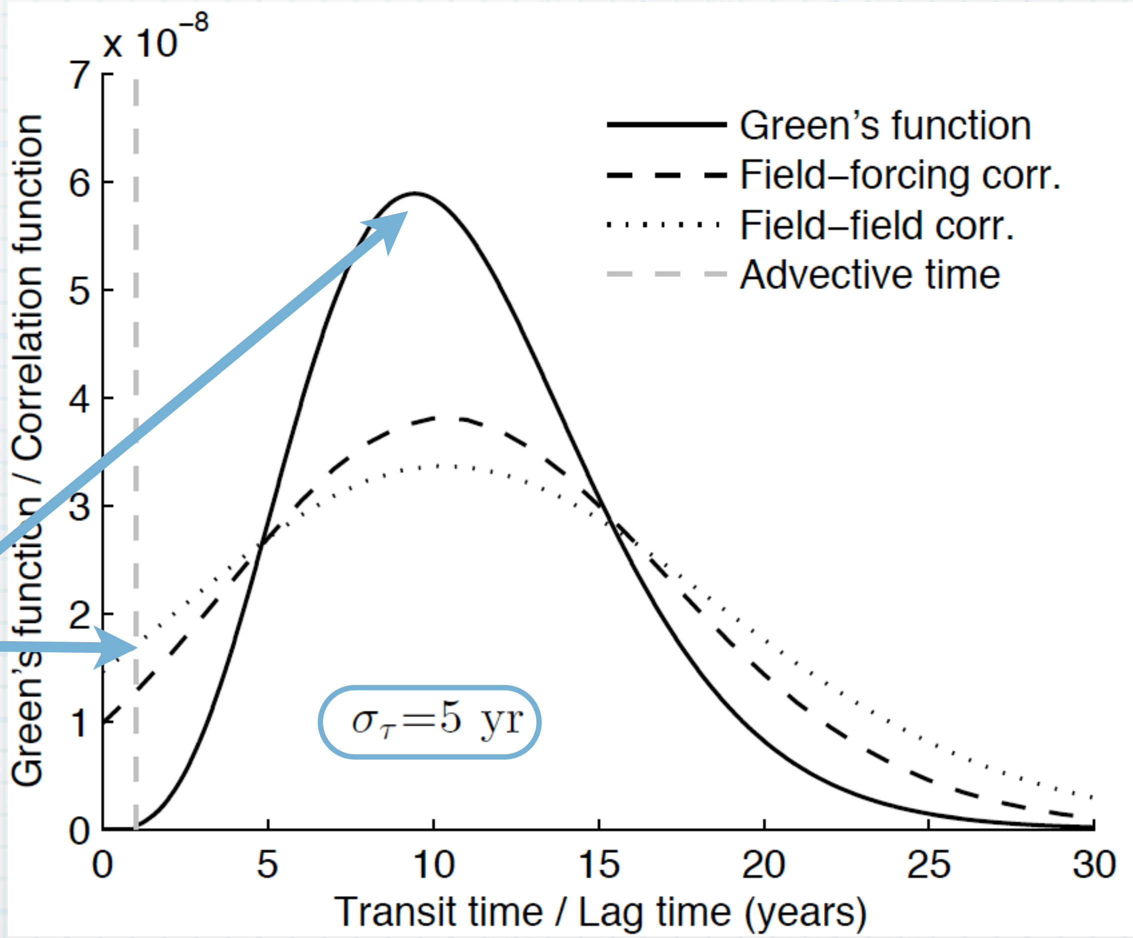
- Time-lagged field/forcing covariance fn. is an unbiased estimate of the Green's fn. smoothed by the forcing covariance.
- Not true for field/field covariance in general.
- Time-lagged field/forcing covariance fn. contains useful information for all lags: width reflects pathway mixing.

$$S_{ff}(x,x',\tau) = \frac{1}{\sigma_\tau\sqrt{2\pi}}\exp\left(-\frac{\tau^2}{2\sigma_\tau^2}\right)\delta(x-x')$$

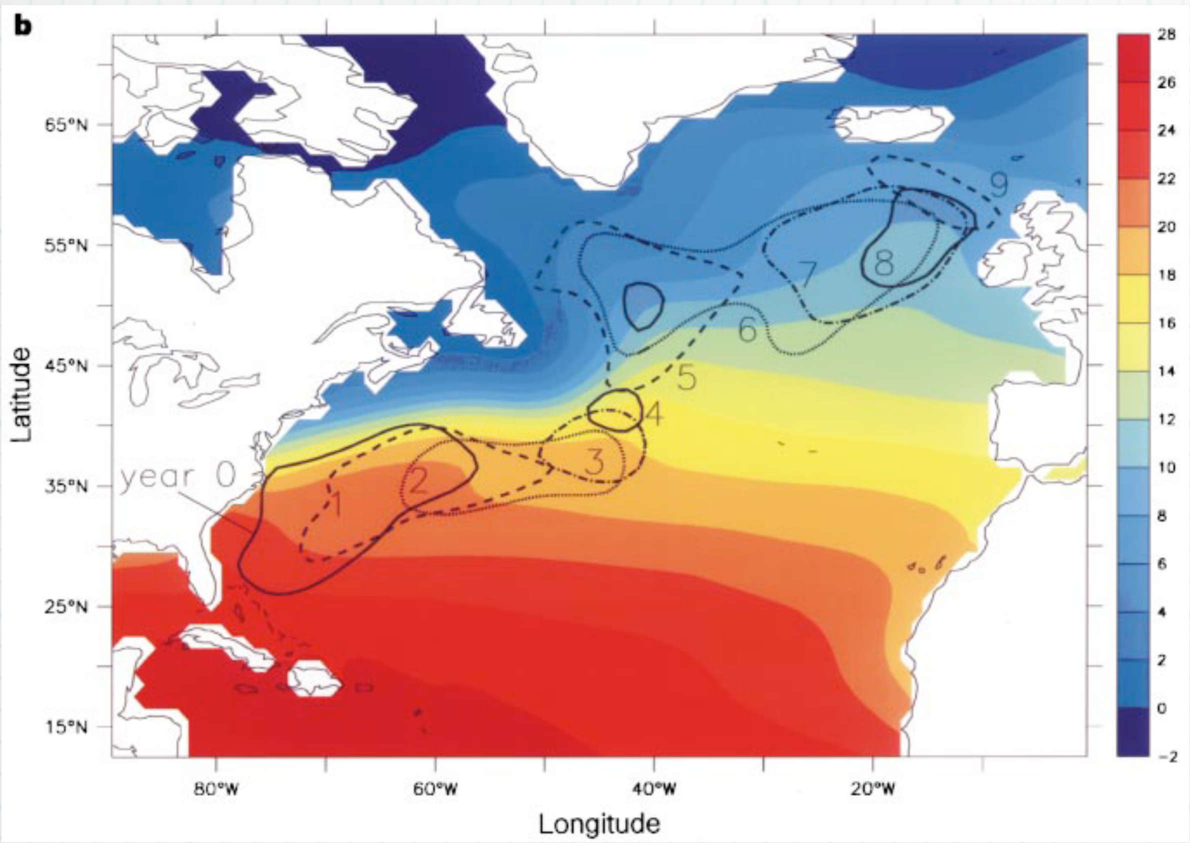
Forcing covariance is white in space, Gaussian in time with scale σ_τ

Advective time is 10x shorter than modal time





How does this apply to SST data?



Unbiased

Field/forcing covariance

$$\mathbf{S_{cf}}(\tau) = \int \mathbf{G}(\tau - t')\mathbf{S_{ff}}(t')dt'$$

But Unavailable

Biased

Field/field covariance

$$\mathbf{S_{cc}}(\tau) = \iint \mathbf{G}(t')\mathbf{S_{ff}}(\tau - t' + t'')\mathbf{G}^{\mathrm{T}}(t'')dt'dt''$$

Available!

Time-averaged equation:

$$\frac{d}{dt}\mathbf{c}'(t) = \bar{\mathbf{A}}\mathbf{c}'(t) + \mathbf{f}'(t)$$

Time-averaged equation:

$$\frac{d}{dt}\mathbf{c}'(t) = \bar{\mathbf{A}}\mathbf{c}'(t) + \mathbf{f}'(t)$$

Transport estimate:

forcing decorrelation time

$$\hat{\bar{\mathbf{A}}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle^{-1}$$

$$\left\langle \mathbf{f}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle = 0$$

$$\hat{\mathbf{G}}(t) = e^{\hat{\bar{\mathbf{A}}}t}$$

Assumes f'(t) forcing is stochastic

Time-averaged equation:

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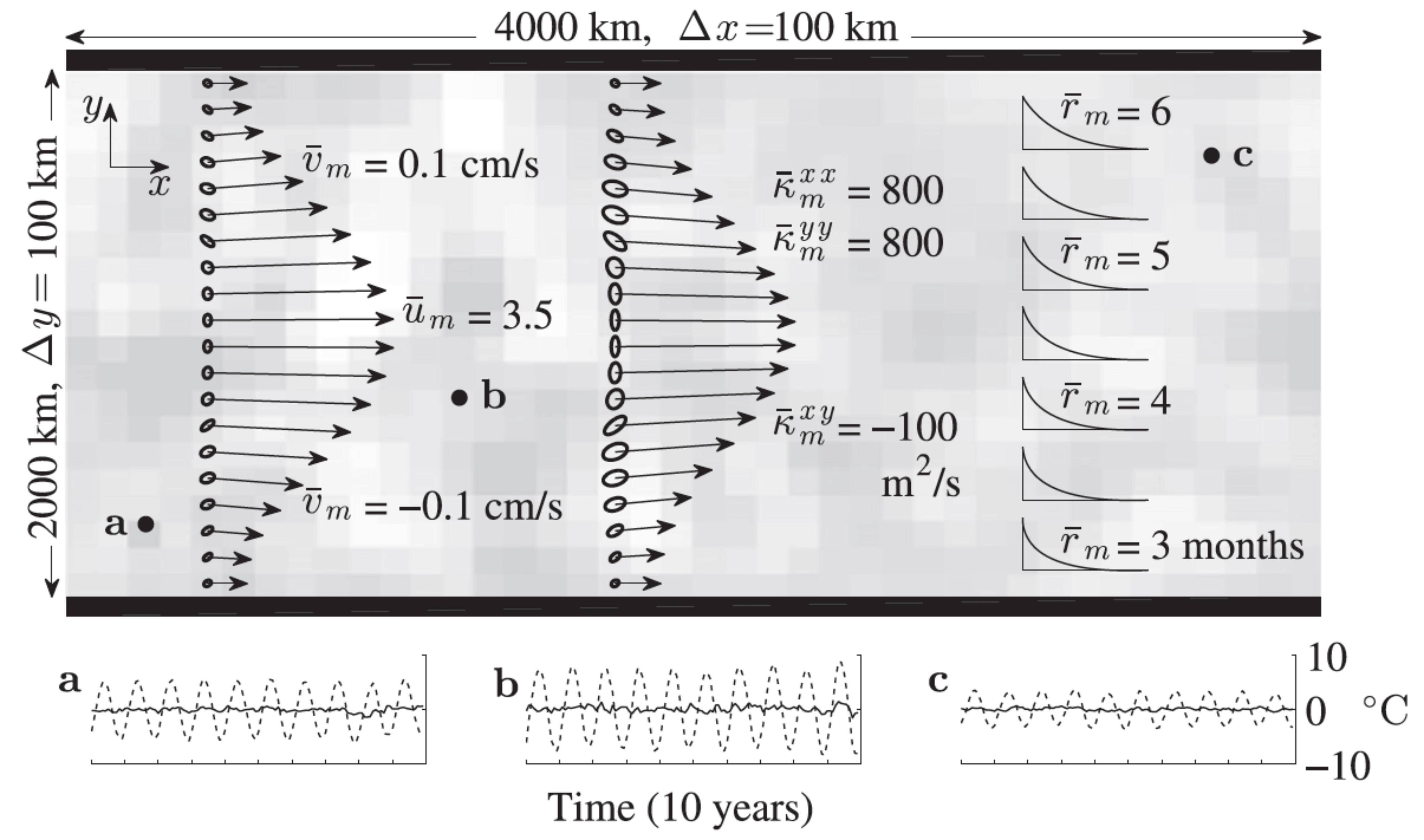
$$\hat{\mathbf{G}}(t) = e^{\hat{\mathbf{A}}t}$$

Assumes f'(t) forcing is stochastic

Can also split response function into anti-symmetric, symmetric, and diagonal parts

advective diffusive relaxing

Illustrative Example



* Illustrative Example

$$\hat{\mathbf{A}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f)\right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f)\right\rangle^{-1}$$

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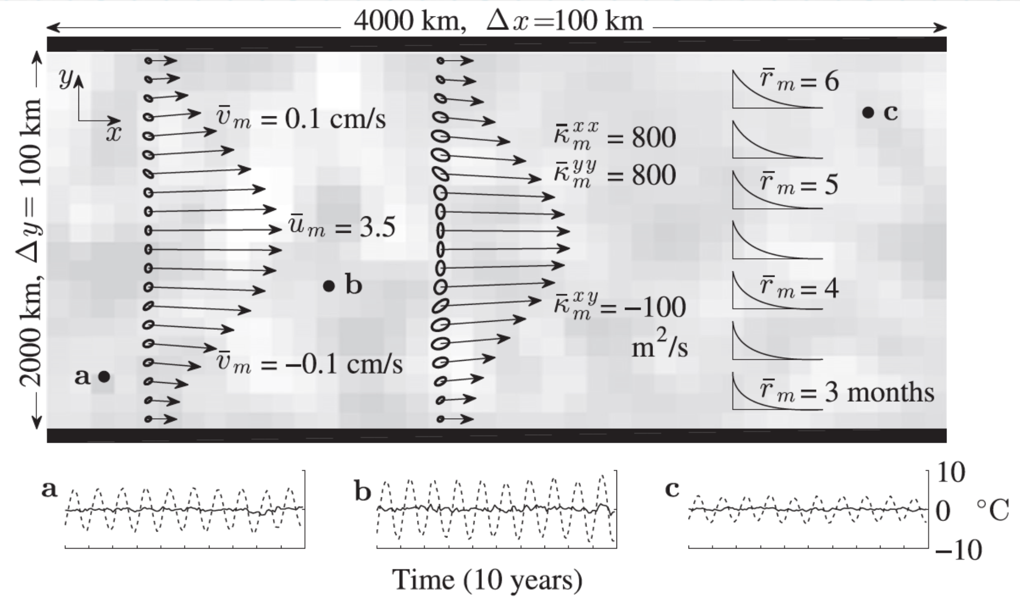

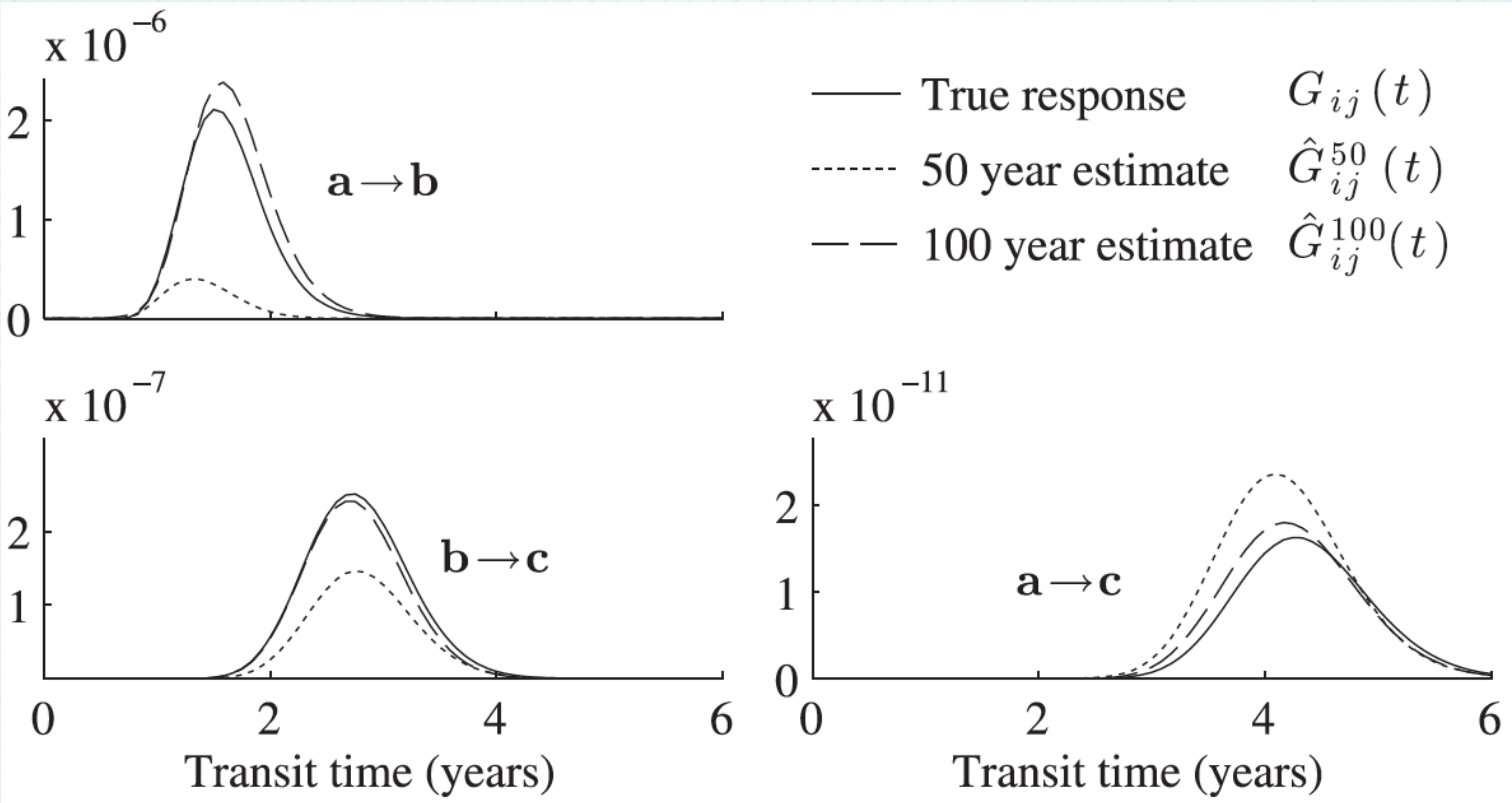

Response estimates for imperfect covariance estimates: no bias.

* Illustrative Example

$$\hat{\mathbf{A}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle^{-1}$$

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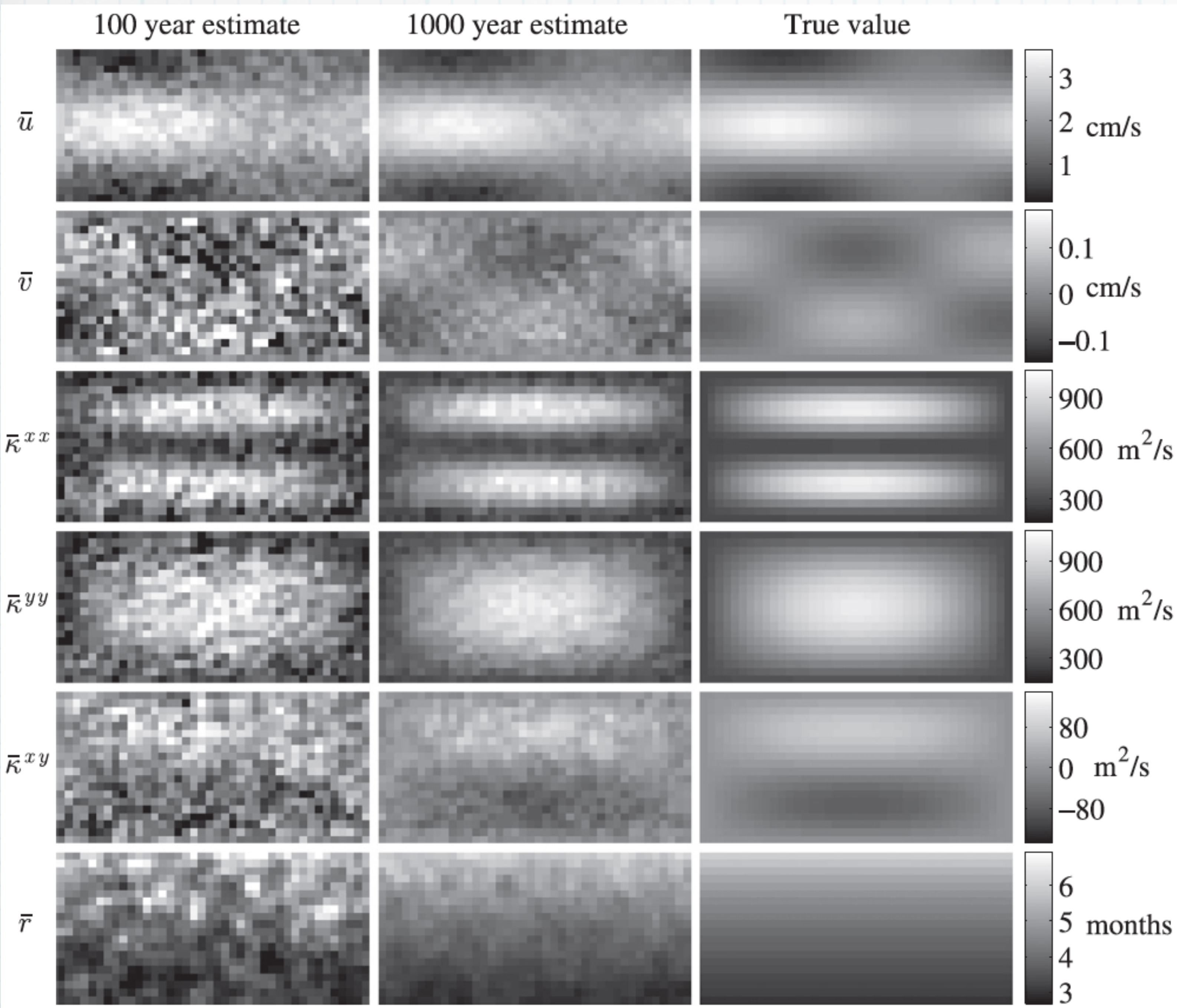

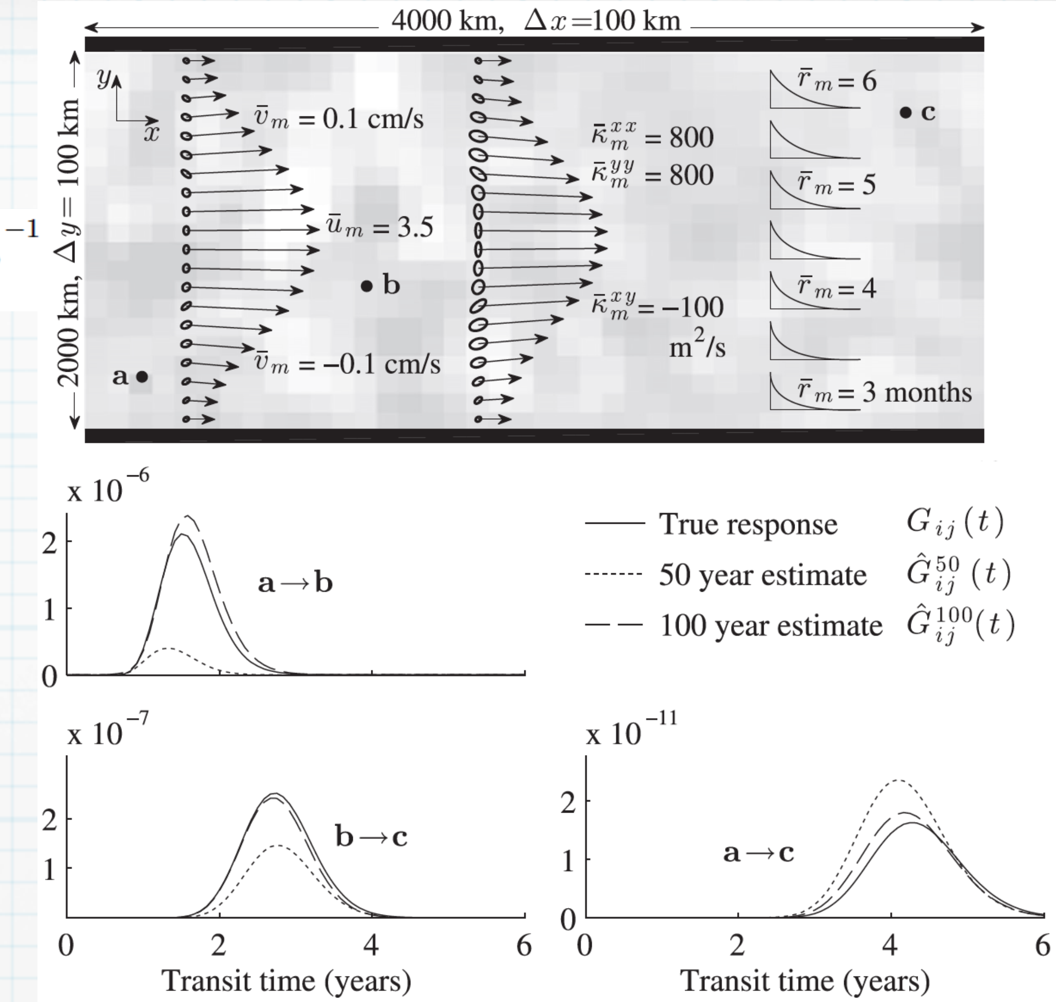

Decomposition into advective, diffusive, and relaxing components: no bias

* Illustrative Example

$$\hat{\mathbf{A}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle^{-1}$$

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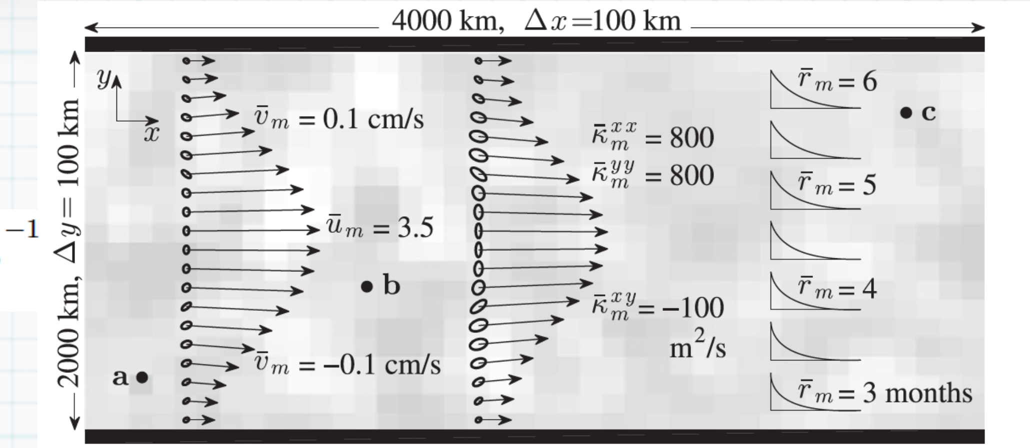

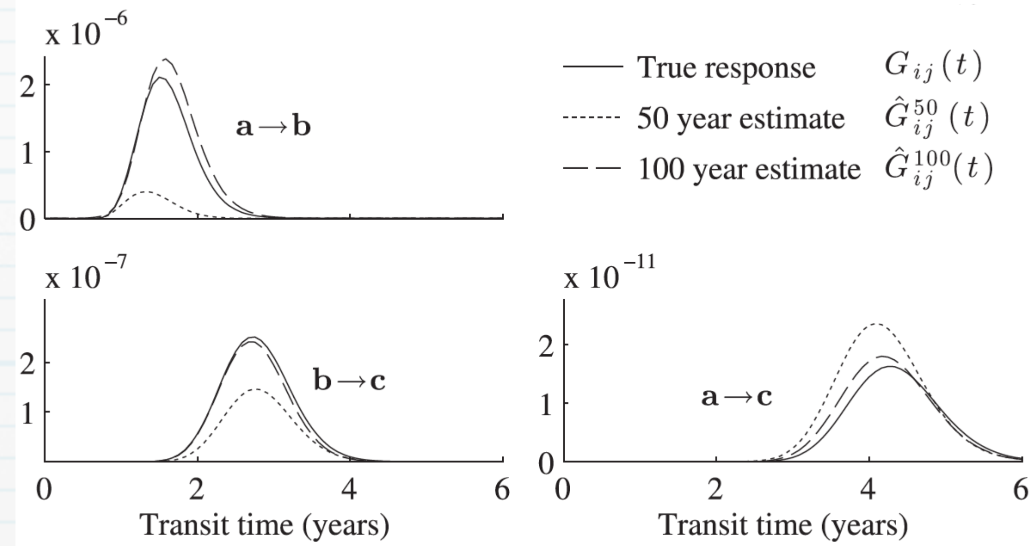

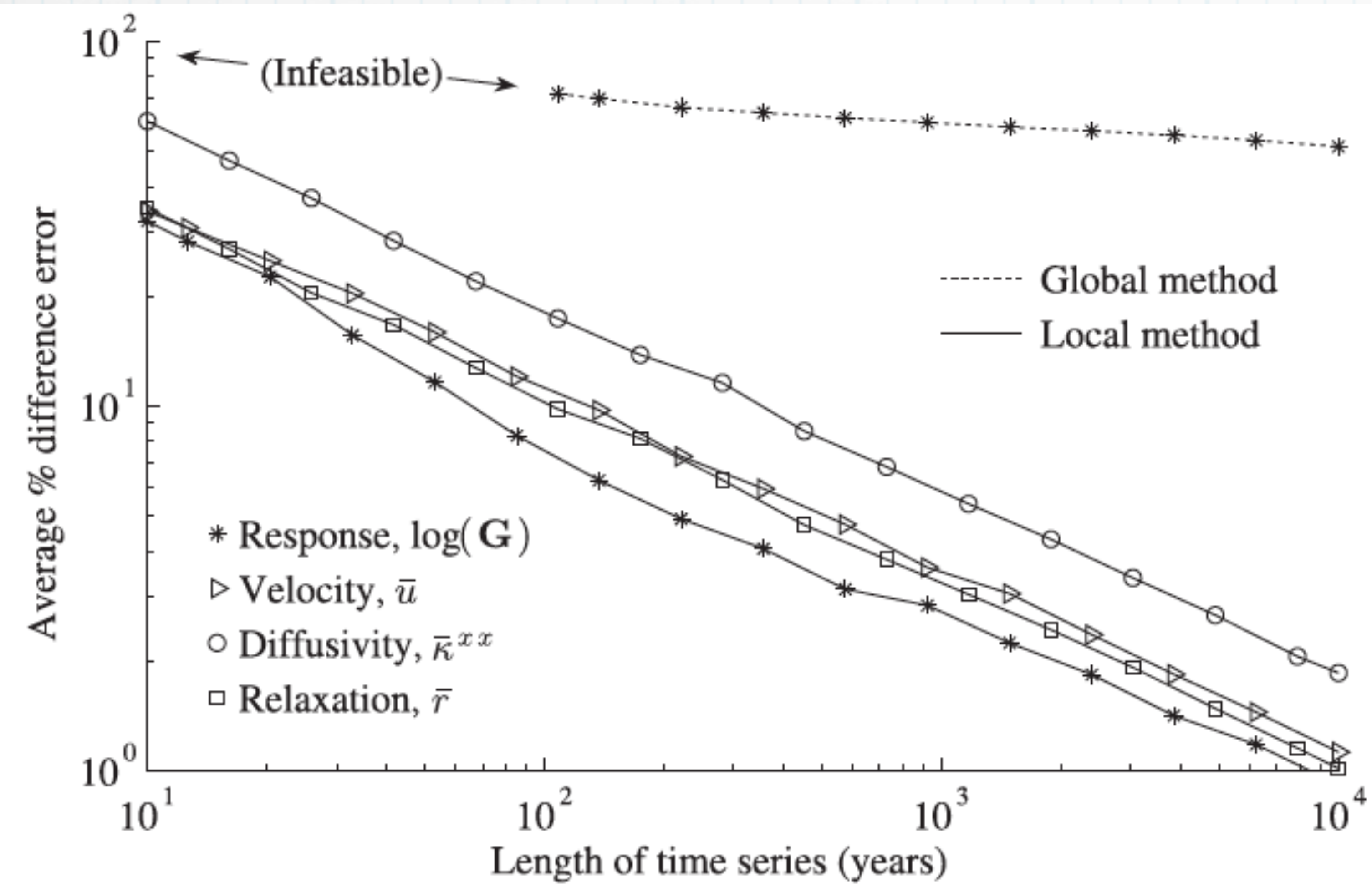

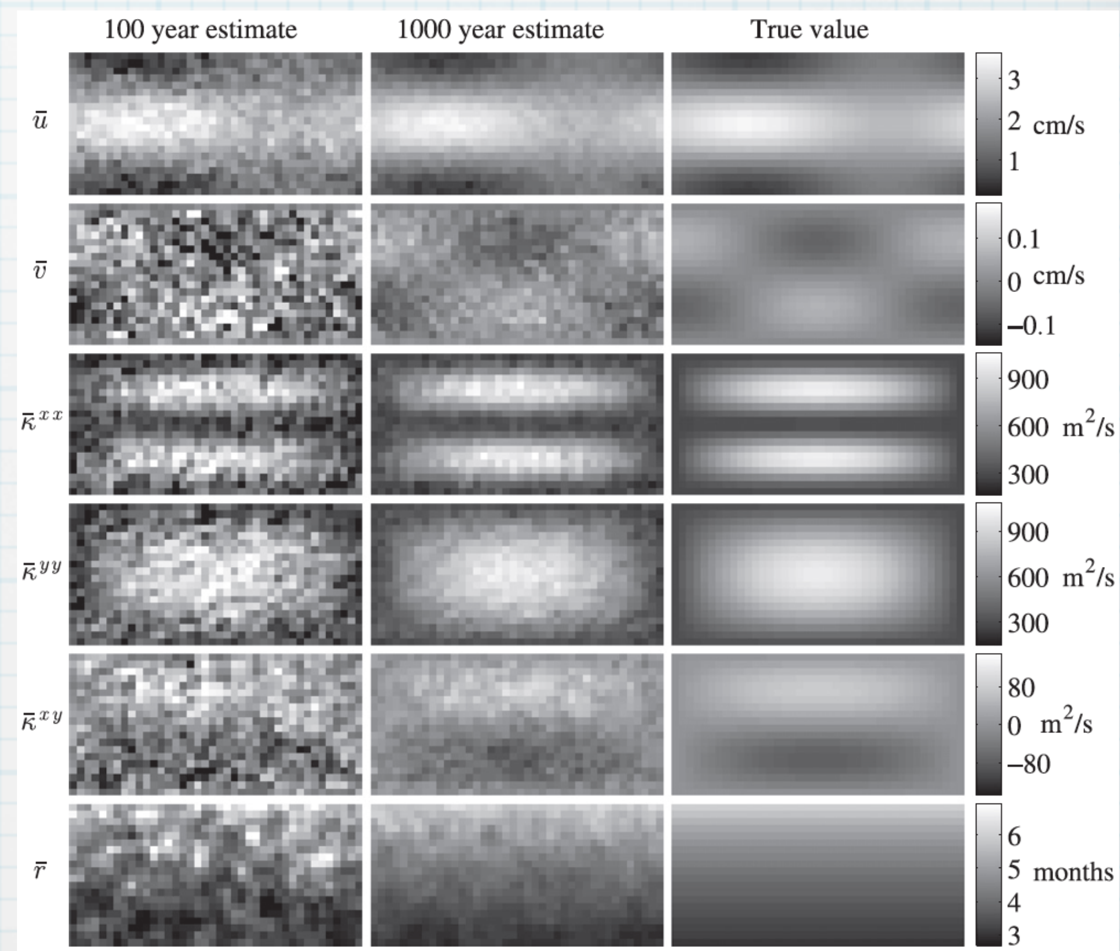

- Exploiting sparsity in $\hat{\mathbf{A}}$ is essential (local method but global data).
- Response function error is smallest.

* Illustrative Example

$$\hat{\mathbf{A}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f)\right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f)\right\rangle^{-1}$$

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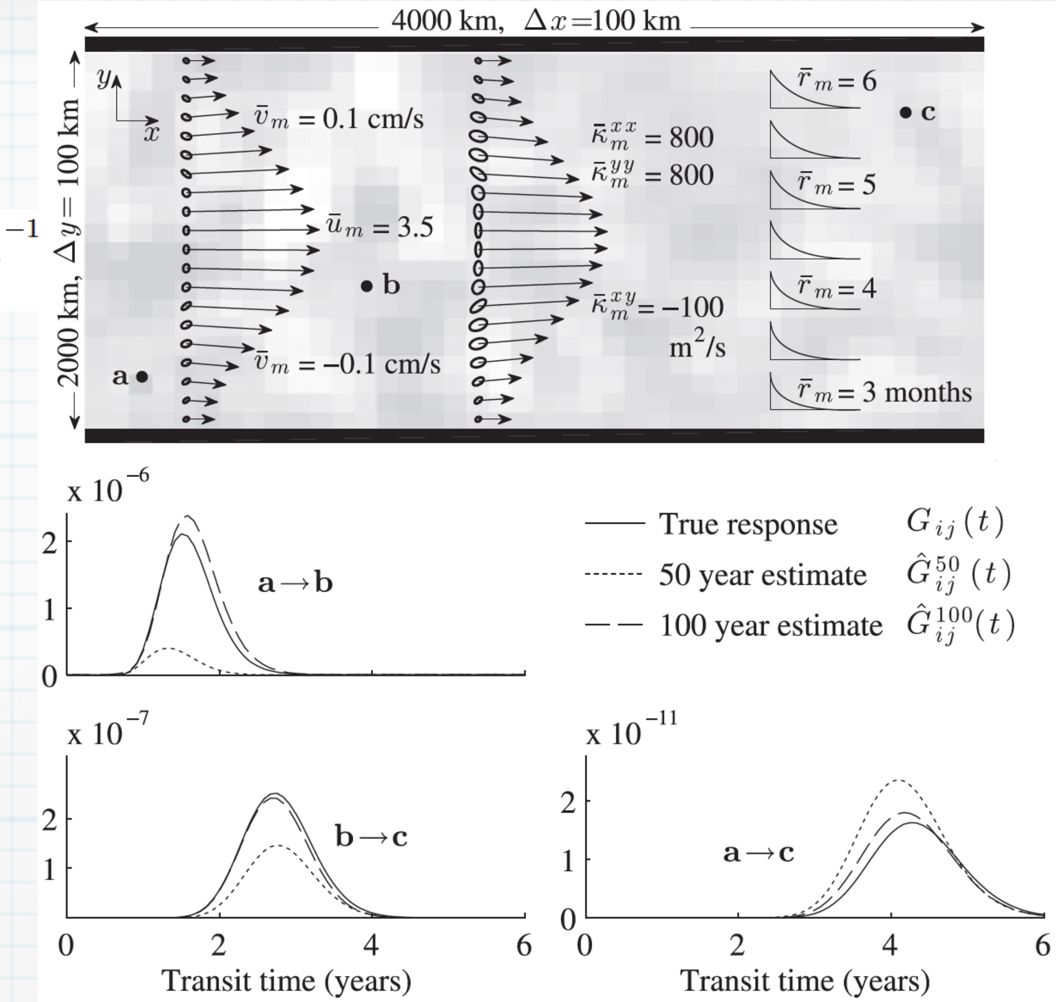
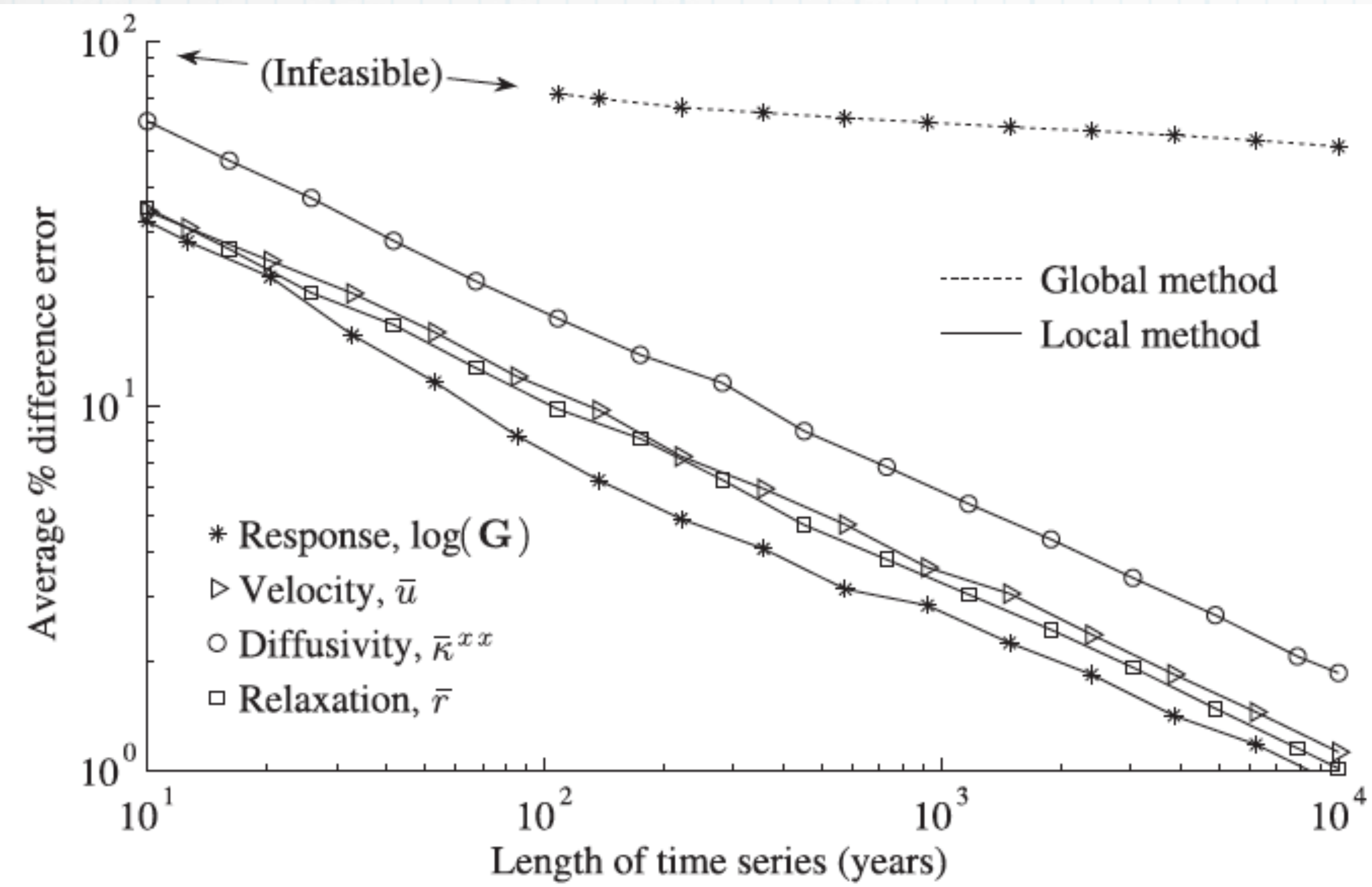
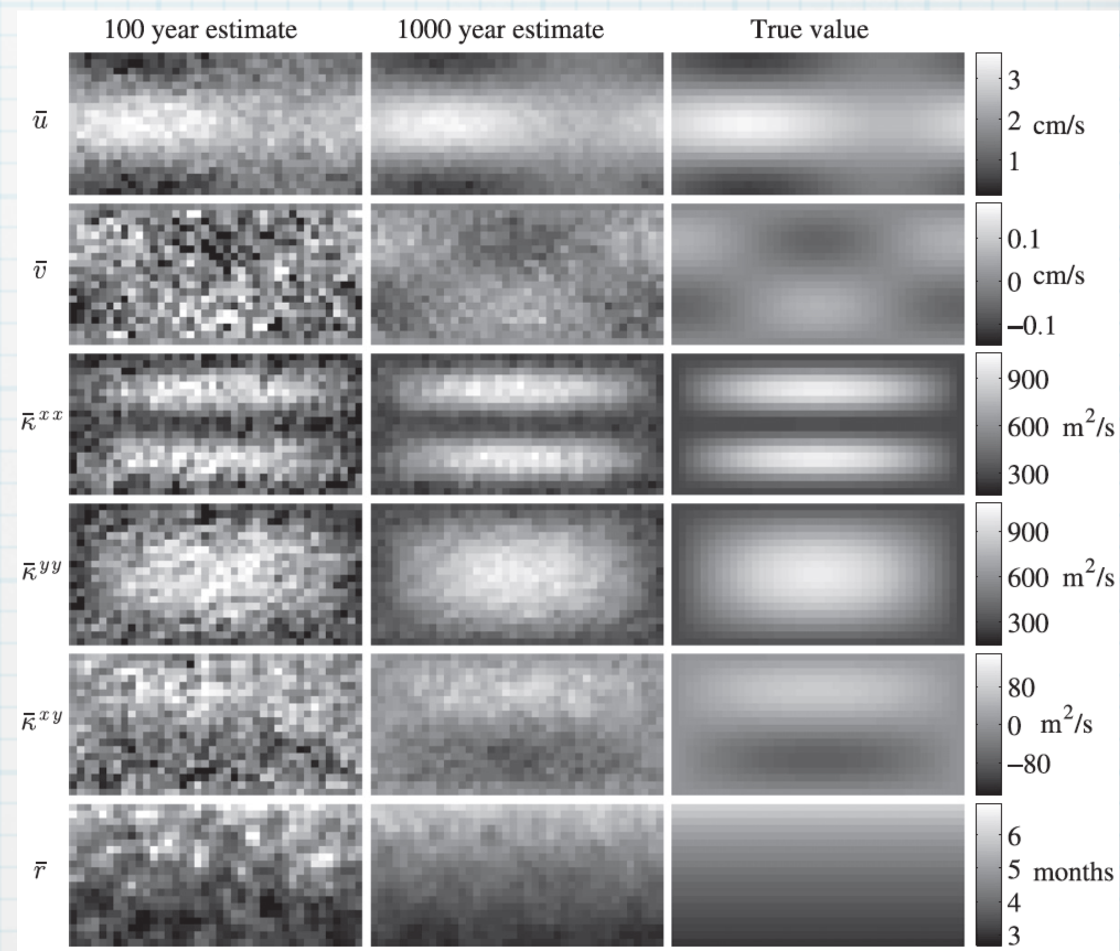




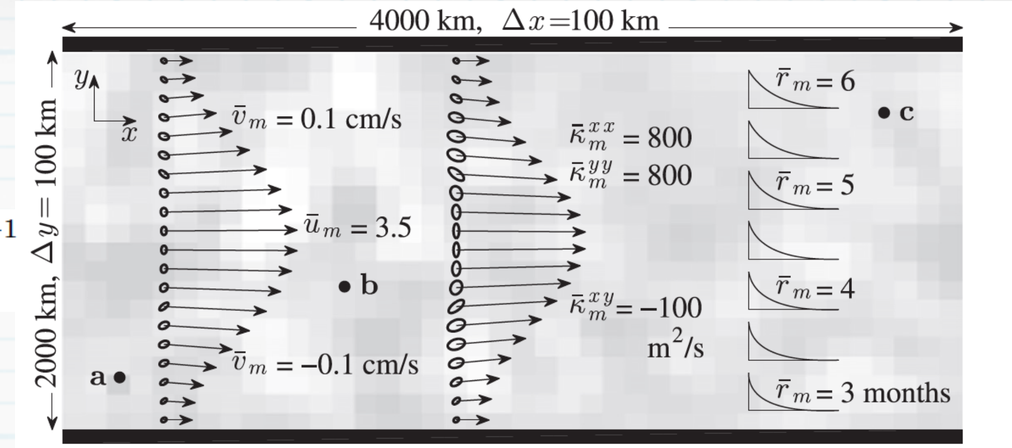


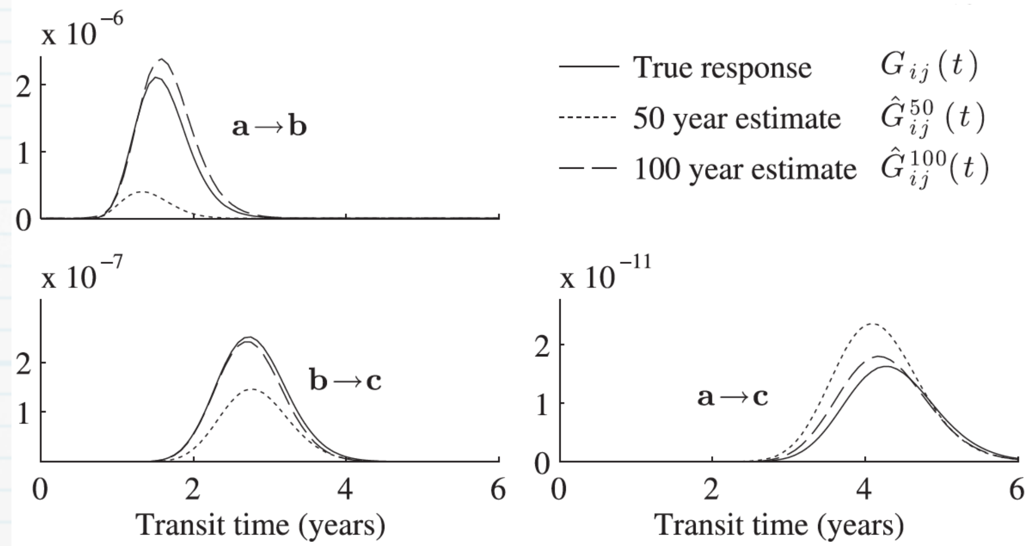
- Exploiting sparsity in $\hat{\mathbf{A}}$ is essential (local method but global data).
- Response function error is smallest.
- This example is with monthly (synthetic) SST anomalies
  100yrs <=> 1200 samples

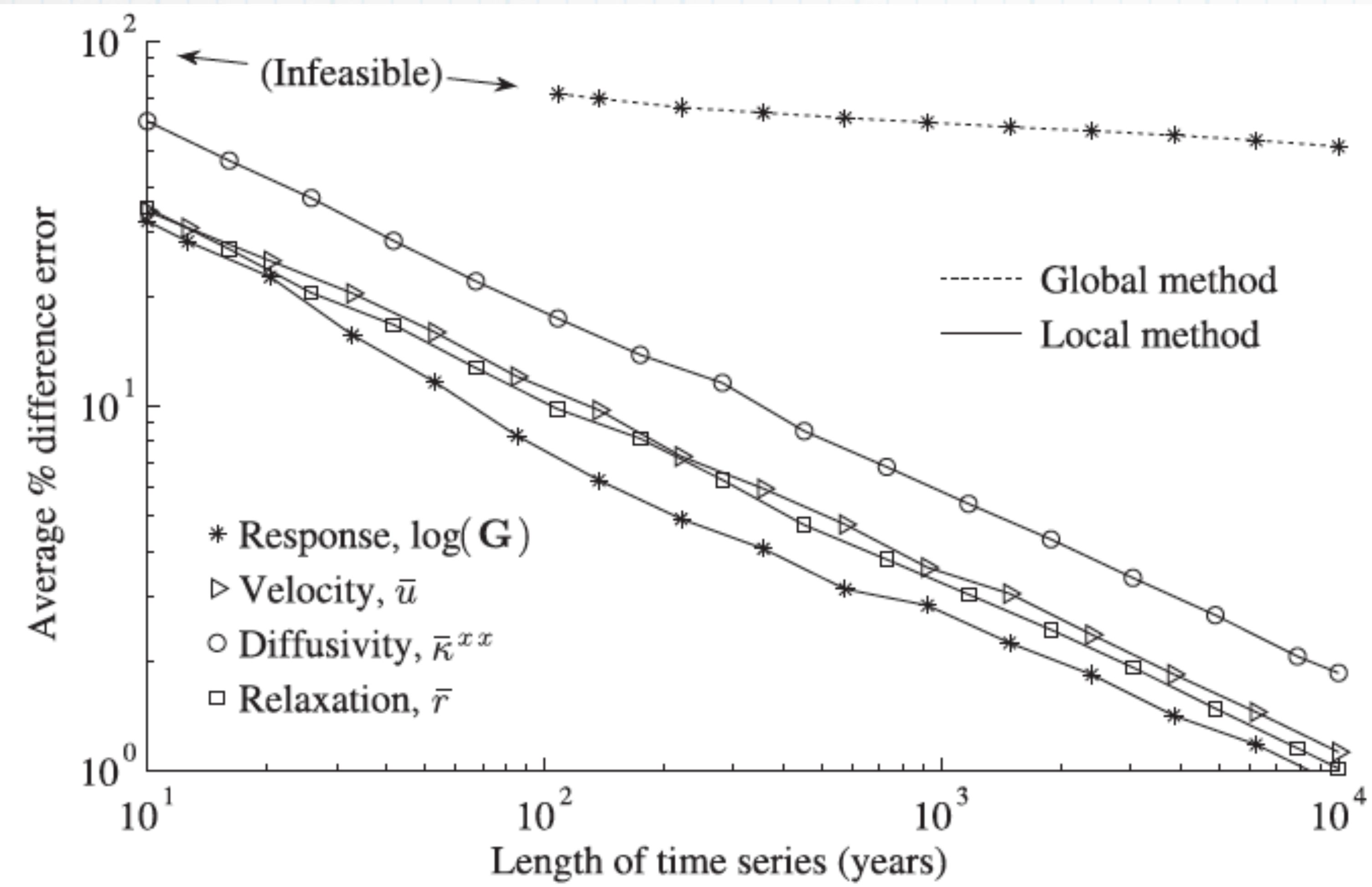
* Illustrative Example

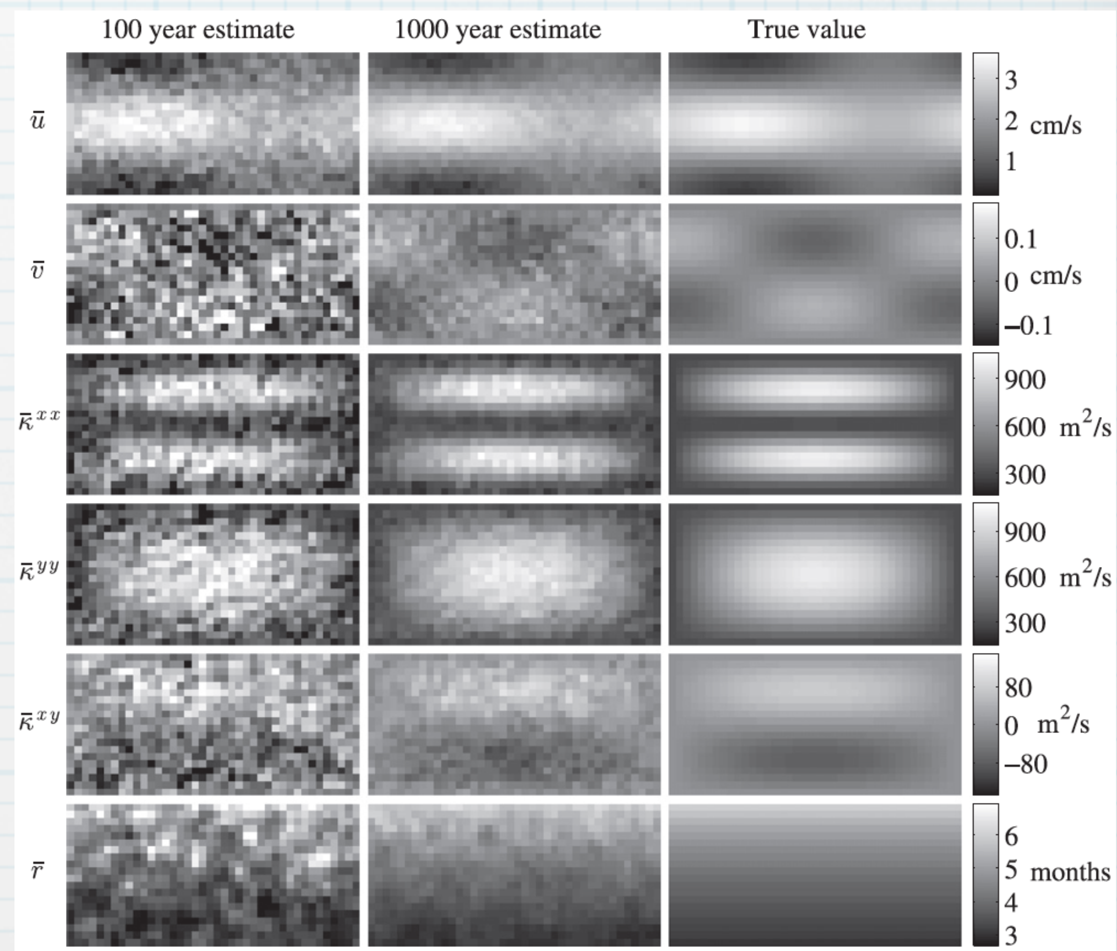
$$\hat{\mathbf{A}} = \left\langle \frac{d}{dt}\mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle \left\langle \mathbf{c}'(t)\mathbf{c}'^T(t-\tau_f) \right\rangle^{-1}$$

$$\hat{\mathbf{G}}(t) = e^{\hat{\mathbf{A}}t}$$









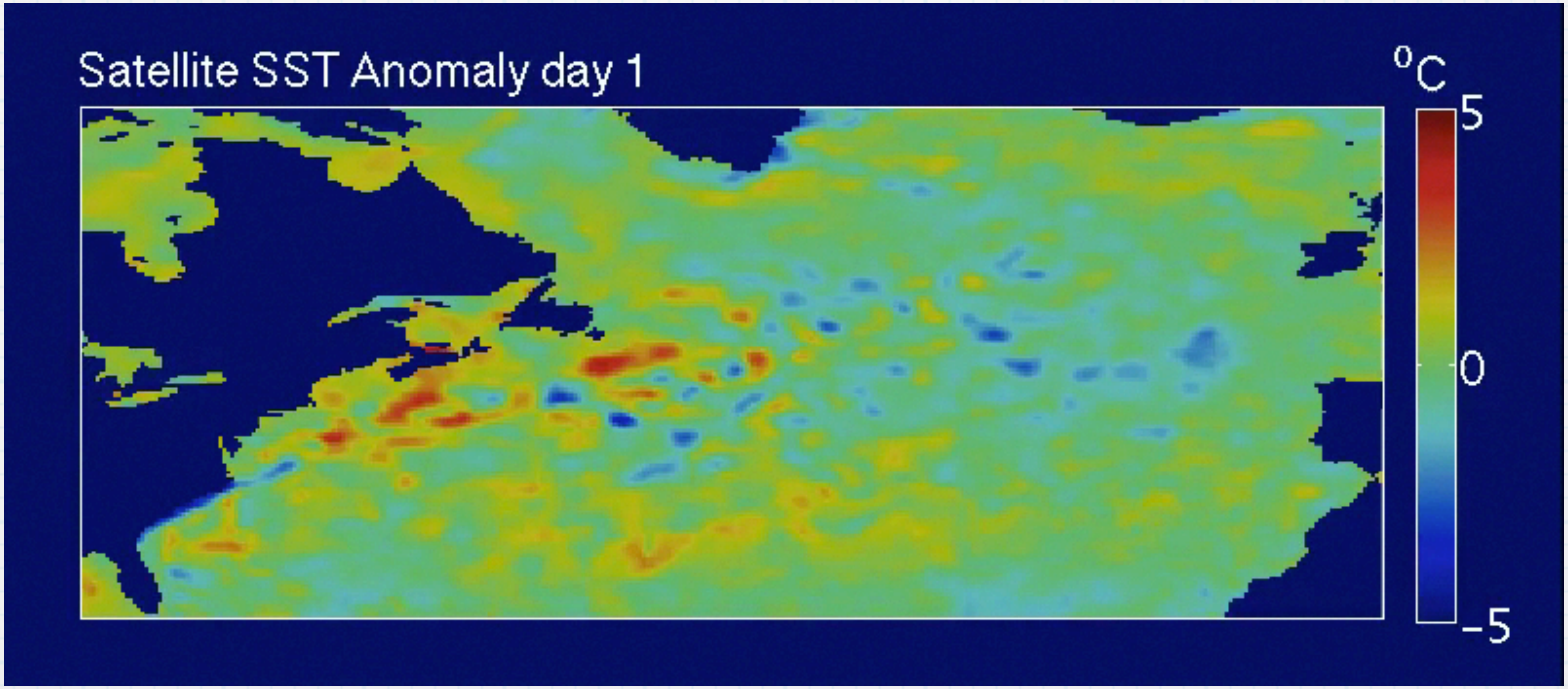
- Exploiting sparsity in $\hat{\mathbf{A}}$ is essential (local method but global data).
- Response function error is smallest.
- This example is with monthly (synthetic) SST anomalies

100yrs <=> 1200 samples

30 yrs daily SST fields <=> 1200 samples

* Application to SST data

- **NOAA Optimum Interpolation 1/4 Degree Daily SST Analysis (Reynolds et al., 2007)**
- 1 degreeHadISST product (Rayner et al., 2003)
- 1 degree GFDL-CM2.1 coupled climate model (Delworth et al., 2006)
- 3 degree GFDL-ESM2Mc global coupled climate model (also SSS, SSH, color...)



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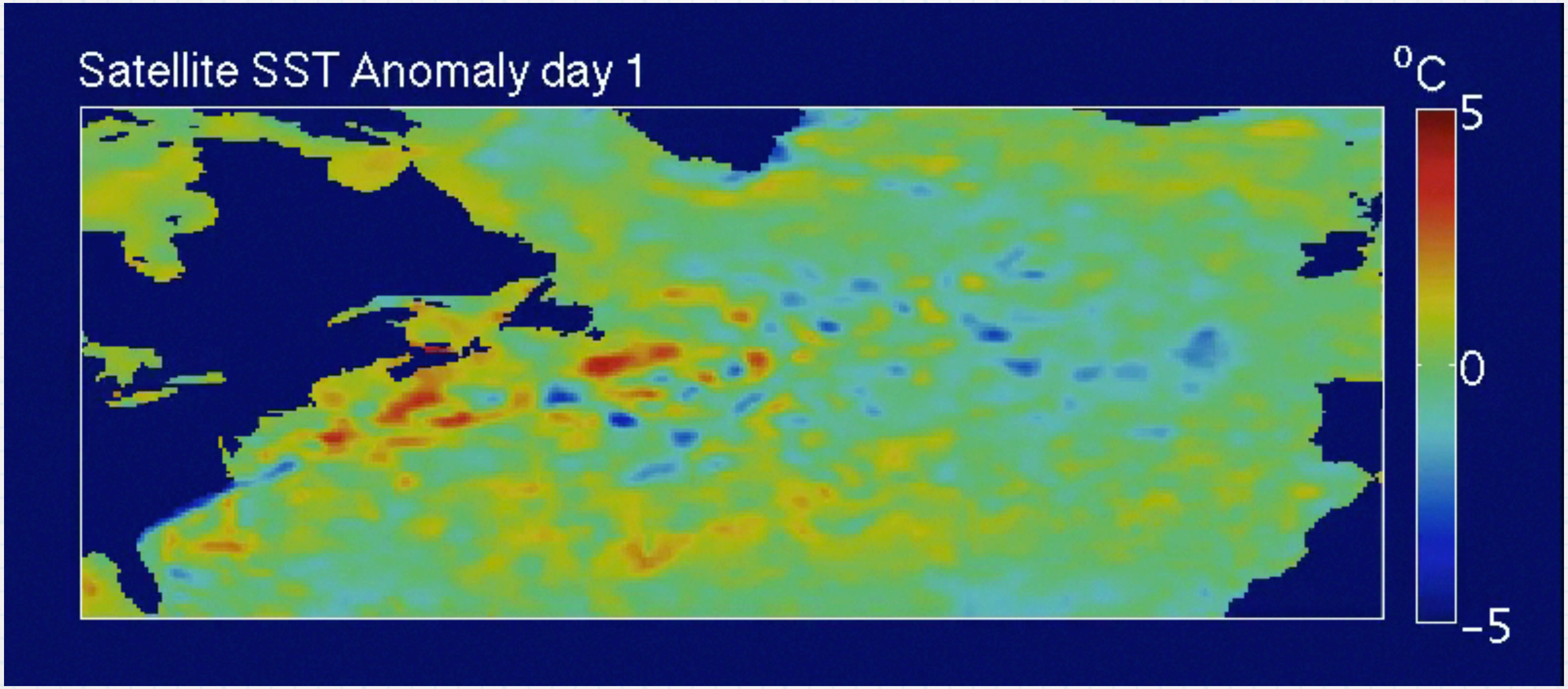
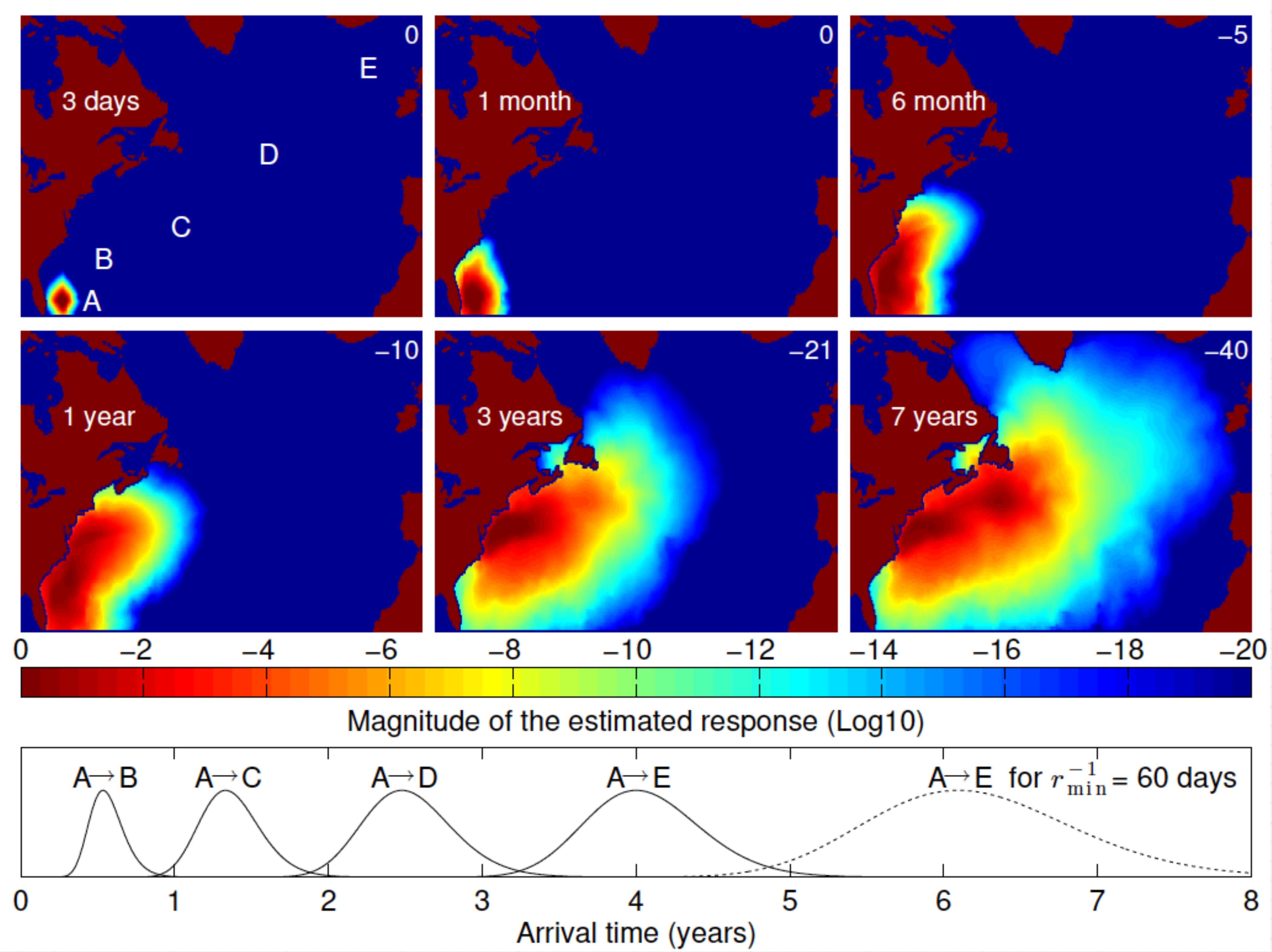
* Local inversion algorithm is quick, parallel, and scales linearly with data size.



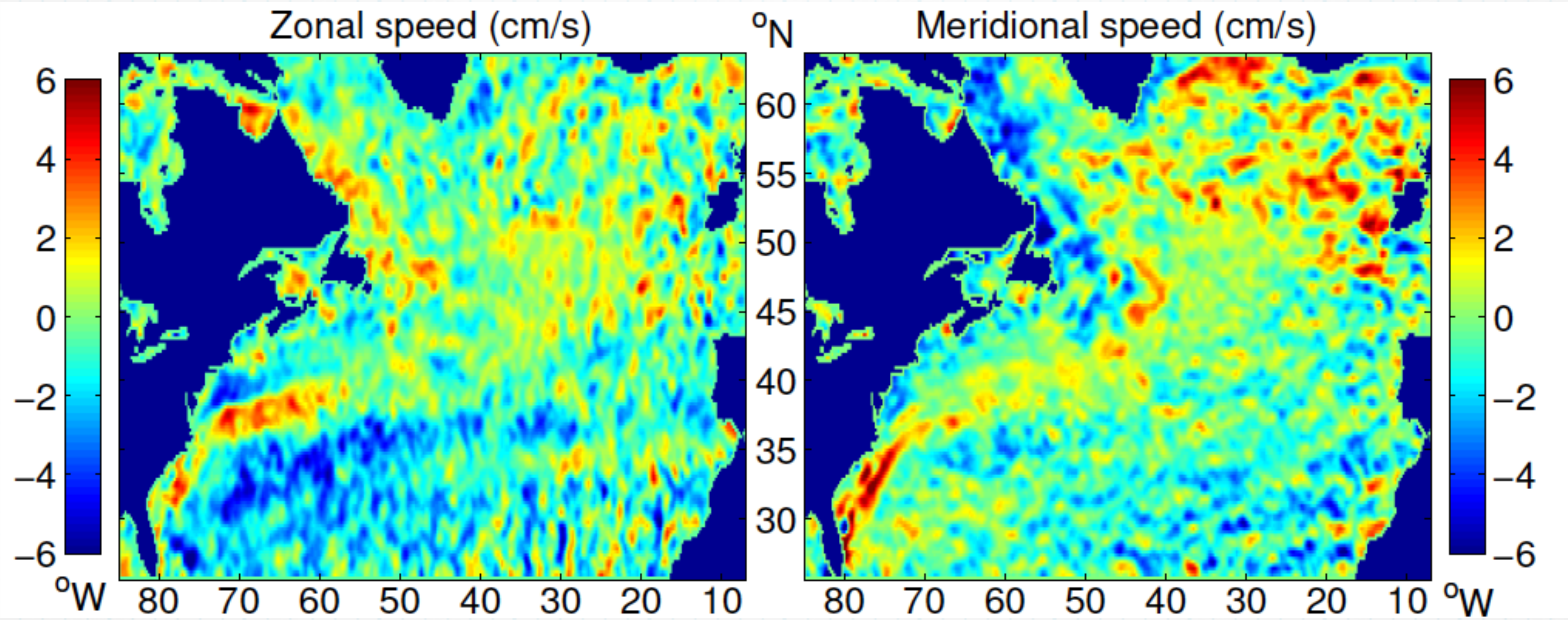
* Application to SST data

Sample response function



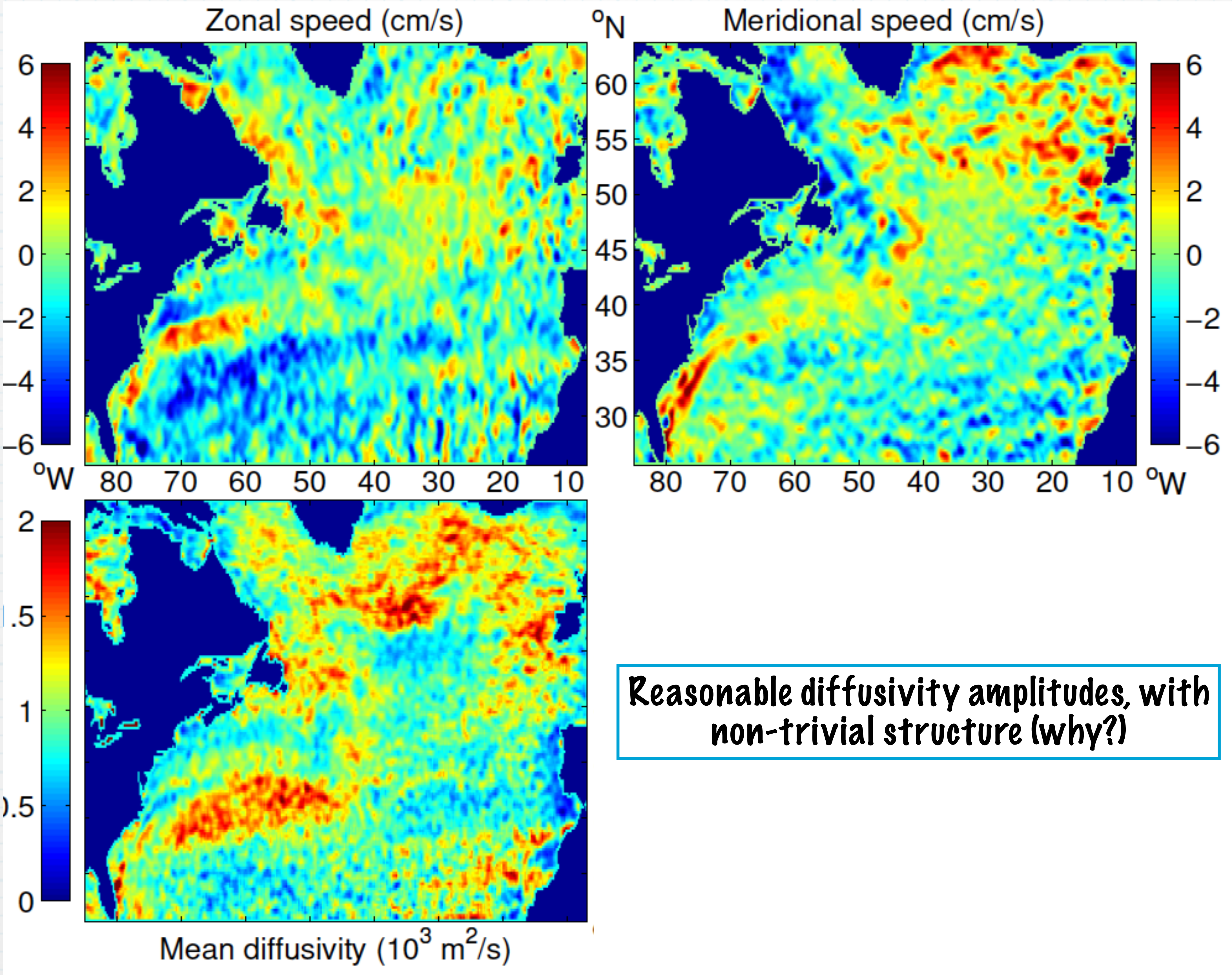
Relaxation dominates and truncates diffusive response function tails

* Application to SST data



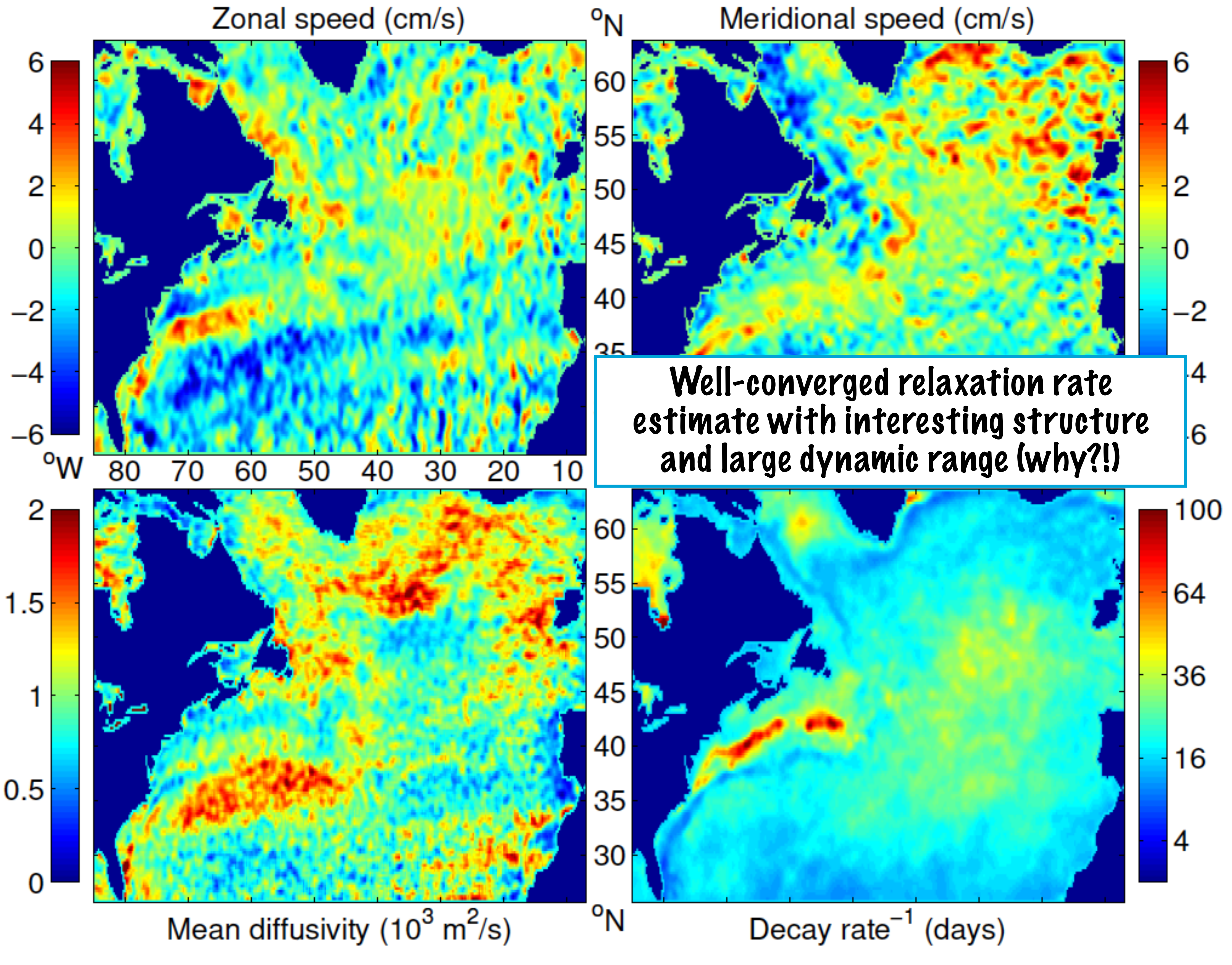
Familiar velocity fields, but small amplitudes (why?)

* Application to SST data



Reasonable diffusivity amplitudes, with non-trivial structure (why?)

* Application to SST data



Well-converged relaxation rate estimate with interesting structure and large dynamic range (why?!)

Discussion

- Global anomaly inversion is an example of fluctuation-dissipation theorem
- Framework is generic and applies to any linear system
- Numerical models with weak stochastic forcing (or hardware) give the response fn. (and adjoint) for free!
- Response fn. quantifies causality in linear systems: correlation implies causation under <u>certain</u> circumstances

Discussion

Imaging with ambient noise

Roel Snieder and Kees Wapenaar

feature article

Whether noise is a nuisance or a signal depends on how it's processed. By cross-correlating noise recorded at two sensors, researchers can retrieve the waves that propagate between them and extract details about the intervening medium.

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r Wave Phenomena at the
echnology in Delft, the

the location and nature
ransmitted through the
eismic velocity. Perhaps
an be determined from
simple processing step:
e of the waveforms' sim-
as a function of the time

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rbitrary points in space,
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the two points and a re-
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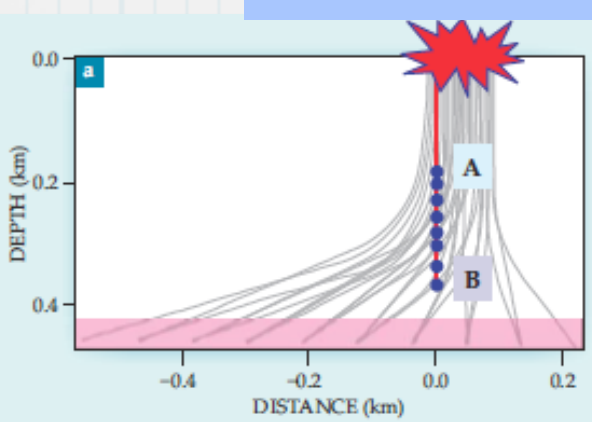

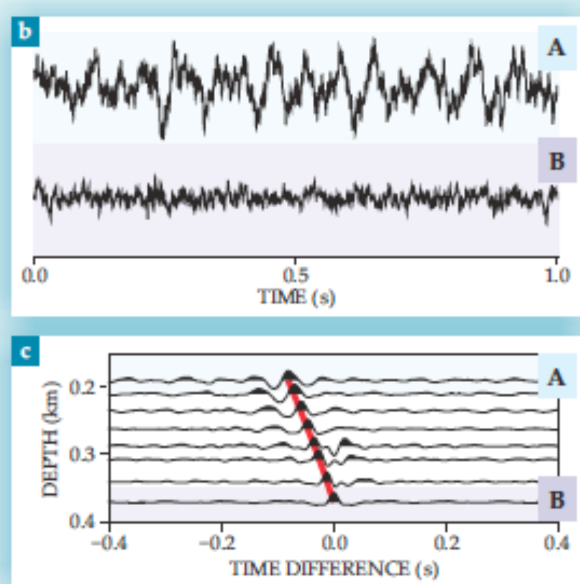

Figure 1. (a) At an oil-production facility in Canada, a layer of heavy oil (pink) is liquefied by the injection of steam through a series of underground wells (gray), as depicted in the schematic. Noise (red star) from industrial pumps and other equipment is generated at the surface and recorded along a vertical array of geophones (blue dots). **(b)** As the noise signal propagates down the array, geophones A and B record the wave motion at the shallowest and deepest sites. **(c)** Each of the eight traces is the result of cross-correlating one day of noise recorded by geophone B with noise recorded by another geophone in the series. The projection of the red line along the time axis gives the travel time of a compressive wave propagating from A to B. (Adapted from ref. 8.)

Retrieving the Green function

Imagine a closed system that vibrates in response to random noise sources. Given a set of normal modes $u_n(\mathbf{x})$, the Green function that describes the impulsive response can be written

$$G(\mathbf{x},\mathbf{x}',t) = \sum_n u_n(\mathbf{x})u_n(\mathbf{x}')\cos(\omega_n t)H(t)\,, \qquad (1)$$

where $H(t)$ is the Heaviside function, zero for negative time and 1 for positive time, and ω_n is the angular frequency of mode n.

We outline Oleg Lobkis and Richard Weaver's derivation of such a Green function,[5] starting with a state of motion in which the time derivative of pressure fluctuations is given by

$$v(\mathbf{x},t) = \sum_n (a_n \sin(\omega_n t) + b_n \cos(\omega_n t))u_n(\mathbf{x})\,, \qquad (2)$$

where the modal coefficients a_n and b_n are random numbers with zero mean. The modes are assumed to be excited with equal energy and have uncorrelated excitations. That is,

$$\langle a_n a_m\rangle = \langle b_n b_m\rangle = S\delta_{nm} \quad \text{and} \quad \langle a_n b_m\rangle = 0\,, \qquad (3)$$

where $\langle\ \rangle$ denotes the expectation value and S is the modes' excitation energy.

Next, consider the time-averaged cross-correlation of the field at two locations $\mathbf{x}_A$ and $\mathbf{x}_B$,

$$C_{AB}(\tau) = \frac{1}{T}\int_0^T \langle v(\mathbf{x}_A, t+\tau)v(\mathbf{x}_B,t)\rangle dt\,. \qquad (4)$$

The length of the time integration is denoted by T, and τ denotes the lag time used in the correlation. Inserting the normal-mode expansion (2) in that integral gives a double sum over modes. After taking the expectation value, the double sum reduces to the following single sum by virtue of the expectation values of equation (3):

$$C_{AB}(\tau) = \sum_n S\, u_n(\mathbf{x}_A)u_n(\mathbf{x}_B)\frac{1}{T}\int_0^T \{\cos[\omega_n(t+\tau)]\cos(\omega_n t) + \sin[\omega_n(t+\tau)]\sin(\omega_n t)\}\, dt \qquad (5)$$
$$= \sum_n S\, u_n(\mathbf{x}_A)u_n(\mathbf{x}_B)\cos(\omega_n \tau)\,.$$

A comparison of this equation with the general Green function (1) shows that when $\tau > 0$, the last term is equal to $SG(\mathbf{x}_A,\mathbf{x}_B,\tau)$, and when $\tau < 0$, it is equal to $SG(\mathbf{x}_A,\mathbf{x}_B,-\tau)$. Hence,

$$C_{AB}(\tau) = S[G(\mathbf{x}_A, \mathbf{x}_B, \tau) + G(\mathbf{x}_A, \mathbf{x}_B, -\tau)]\,. \qquad (6)$$

The expectation value of the cross-correlation thus gives the superposition of the Green function and its time-reversed counterpart.

Conclusions

- Time-lagged two-point field/forcing covariance fn. is an unbiased estimate of the response fn. smoothed by the forcing covariance.
- Not true for two-point field/field covariance (in general).
- Global field anomaly data can be (locally) inverted to estimate transport operator and response function.
- Transport operator can be decomposed into advective, diffusive, and relaxation components.
- Physical interpretation of SST transport operator is just beginning...

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- Time-lagged two-point field/forcing covariance fn. is an unbiased estimate of the response fn. smoothed by the forcing covariance.
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- Transport operator can be decomposed into advective, diffusive, and relaxation components.
- Physical interpretation of SST transport operator is just beginning...

✳ Much more information can be extracted from the covariance than just the modal transit-time.
✳ But you have to look at it the right way to understand what it means.

We quantify the transport of sea surface temperature (SST) from SST fluctuations. Previous studies have estimated the advective transport of SST from time-lag correlation of SST anomalies. However, this approach does not consider diffusive SST transport or relaxation to atmospheric temperatures. To quantify the transport more completely we use a response function (Green's function) which solves the SST continuity equation for an impulsive forcing. The response function is estimated from SST anomalies using a fluctuation-dissipation approach. Decomposing the linear operator into symmetric, anti-symmetric, and divergent operators enables estimates of the model's spatially dependent velocity vector, diffusivity tensor, and relaxation rate.