Nonlinear Optimisation of Scalar Mixing in Plane Poiseuille Flow

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Or...
Be sure to listen to Tom,
Be sure to listen to Tom, Jean-Luc
Be sure to listen to Tom, Jean-Luc & Charlie!
• Mixing is of course a fundamental problem in fluid mechanics

• Mixing ultimately driven by diffusive processes, so need:

(I) Strong Gradients (high flux)

(II) “Filamentation” (high contact area so high transport)

• Classic picture of Eckart (1948)/Welander (1955): Stirring $\rightarrow$ (I), (II) $\rightarrow$ mixing

• Beautiful work on stirring: Rhines & Young, Aref, Ottino, Mezic, Doering, Thiffeault...

• But... prescribed flow (no Navier-Stokes) and often zero diffusion ($Pe = \frac{UL}{K} \rightarrow \infty$)

• Questions remain for real (finite Pe, coupled solutions of Navier-Stokes Eqns) flows:

• What is the “best” way to mix a freely evolving flow (Energy=mixing Aamo et al)?

• Is mixing really after stirring...what about Taylor dispersion...and how can we work it out?
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Optimising Mixing

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Flow control: new challenges for a new Renaissance

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Abstract

As traditional scientifi7c disciplines individually grow towards their maturity, many new opportunities for signifi7cant advances lie at their intersection. For example, remarkable developments in control theory in the last few decades have considerably expanded the selection of available tools which may be applied to regulate physical and electrical systems. These techniques hold great promise for several applications in fl7uid mechanics, including the delay of transition and the regulation of turbulence. Such applications of control theory require a very balanced perspective, in which one considers the relevant fl7ow physics when designing the control algorithms and, conversely, takes into account the requirements and limitations of control algorithms when designing both reduced-order fl7ow models and the fl7uid–mechanical systems to be controlled themselves. Such a balanced perspective is elusive, however, as both the research establishment in general and universities in particular are accustomed only to the dissemination and teaching of component technologies in isolated fields. To advance, we must not toss substantial new interdisciplinary questions over the fence for fear of them being “outside our area”; rather, we must break down these very fences that limit us, and attack these challenging new questions with a Renaissance approach. In this spirit, this paper surveys a few recent attempts at bridging the gaps between the several scientifi7c disciplines comprising the fl7ow control, in an attempt to clarify the author’s perspective on how recent advances in these constituent disciplines fl7t together in a manner that opens up signifi7cant new research opportunities. Published by Elsevier Science Ltd.

I'm beginning to understand...
• Taylor-(Aris) dispersion very good way to enhance diffusive transport in shear flows

• Is it possible for the fluid flow to work out what it is good for mixing all on its own?
Mixing in Plane Poiseuille Flow

- Test case: Bands of zero-mean scalar $\theta$ in 2D (stable) flow:

- Natural measure of mixing variance: only changes at finite $Pe$!

- Sobolev norms: Mathews et al. (2005)/Lin et al.: “mix-norm”/ $H^{-1}$ norm: $Mix$ $\theta = \frac{1}{V_\Omega} \| \nabla^{-1} \theta \|_2^2 = \frac{1}{V_\Omega} \int_\Omega [\nabla^{-1} \theta(x, t)]^2 \, d\Omega$

- Easy to understand for FTs: mix-norm favours large scales (cf. gradient): $|\nabla^{-1} \hat{\theta}|^2 = k^{-2} |\hat{\theta}|^2$; $|\nabla \hat{\theta}|^2 = k^2 |\hat{\theta}|^2$

- Mathews et al. proved usefulness for periodic domains...but are minimised mix-norms good proxies for real mixing?

- Classic hypothesis (see e.g. Aamo et al. 2003 etc) is that perturbation energy growth is the “best” way to mix?

- Aim to determine “best” IC for mixing: $\mathcal{J} \{ \hat{u}, \theta \} = \frac{1 - \alpha}{2} \int_0^T \| \hat{u}(x, t) \|_2^2 \, dt + \frac{\alpha}{2} \| \nabla^{-\beta} \theta(x, T) \|_2^2$; $u = \overline{U} + \hat{u}$

- $\alpha = 0$: (average) energy gain; $\alpha = 1$, $\beta = 1$: mix norm; $\alpha = 1$, $\beta = 0$; variance (“real” mixing)
Direct-Adjoint Looping method

- Apply direct-adjoint looping (DAL) method (Hill, Luchini, Bottaro, Schmid etc) to mixing problem

- Derive adjoint equations to describe evolution of Lagrange multipliers which impose solution of Navier-Stokes (& theta) equations

\[ L = \mathcal{J} \{ \hat{u}, \theta \} - \left\langle u^\dagger, \partial_t \hat{u} + (\overline{U} + \hat{u}) \cdot \nabla \hat{u} + \hat{u} \cdot \nabla U + \nabla p - Re^{-1} \nabla^2 \hat{u} \right\rangle - \left\langle p^\dagger, \nabla \cdot \hat{u} \right\rangle \]

\[ - \left\langle \theta^\dagger, \partial_t \theta + (\overline{U} + \hat{u}) \cdot \nabla \theta - Pe^{-1} \nabla^2 \theta \right\rangle - \left( u_0^\dagger, \hat{u}(x, 0) - \hat{u}_0 \right) \]

\[ (b, a) = \frac{1}{VT} \int_D b^\dagger a dV \]

\[ \int_0^T \int_D b^\dagger a dV dt \]

Set ICS for u, theta at t=0

Integrate 0-T using NS + AD+ incompressibility

calculate u(T), theta(T) + J

\[ u^\dagger(x, T) = 0; \ \theta^\dagger(x, T) = (-1)^\beta \alpha \nabla^{-2\beta} \theta(x, T) \]

If not, update u(0), theta(0) using gradient info in E-L equations

Integrate T-0 using NSadj

Relate u(T), theta(T) to udag(T), thetadag(T) using E-L eqns

- Variations of L wrt direct variables:

\[ \partial_t u^\dagger + (\overline{U} + \hat{u}) \cdot \nabla u^\dagger - u^\dagger \cdot (\nabla [\overline{U} + \hat{u}])^T + \nabla p^\dagger + Re^{-1} \nabla^2 u^\dagger = \theta^\dagger \nabla \theta - (1 - \alpha) \hat{u} \]

\[ \partial_t \theta^\dagger + (\overline{U} + \hat{u}) \cdot \nabla \theta^\dagger + Pe^{-1} \nabla^2 \theta^\dagger = 0; \ \nabla . u^\dagger = 0; \ u = \overline{U} + \hat{u} \]

- Adjoint equations have a similar form, and “forced” by direct variables (and nonlinear in direct variables...nontrivial optimisation)
Normalisation, rotation and optimisation...

- Fix initial energy of perturbation to be finite amplitude but small: \( E_0 = 10^{-2} = \frac{1}{2} (\hat{u}_0, \hat{u}_0) = \frac{3}{80} \bar{E} \)

- Could impose as a further constraint, but introduces ambiguity...much better to enforce directly using \( \nabla \hat{u}_0 J \) when not parallel to \( \hat{u}_0 \)

- Project \( \nabla \hat{u}_n,0 J \) onto hyperplane orthogonal to \( \hat{u}_n,0 \) and tangent to hypersphere \( (\hat{u}_n,0, \hat{u}_n,0) = 2E_0 \)

- Restrict updated guess \( \hat{u}_{n+1,0} \) to rotations along intersection of hypersphere with hyperplane defined by scaled projected gradient and \( \hat{u}_{n,0} \)

\[
X \equiv \hat{u}_{n,0} \\
\tilde{X} \equiv \nabla \hat{u}_{n,0} J
\]
Results for $T=2,5,10,20,30$

- Mix-norm and variance similar/small with early gain...very different from energy, converge to same behaviour & most mixing done by $T=10$

- $G(t) = \frac{1}{t} \int_0^t \|\mathbf{u}\|_2^2(\tau) d\tau$

- $M(t) = \frac{\|\nabla^{-1}\theta\|_2^2}{\|\nabla^{-1}\theta_d\|_2^2}$

- $V(t) = \frac{\|\theta\|_2^2}{\|\theta_d\|_2^2}$

- Optimal variance $V(T)$
- Optimal mix-norm $M(T)$
- Optimal energy $G(T)$

Figure 7. Time-evolution of the several considered measures $G(t)$ (black), $M(t)$ (blue) and $V(t)$ (red) corresponding to the variance identified optimals (as defined in (5.1), (5.2) and (5.2)). The first, second and third columns correspond respectively to the optimisation of the time-averaged energy $G(T)$, the mix-norm $M(T)$ and the variance $V(T)$. For each series of optimisations, 5 different horizon times were considered: $T \in [2; 5; 10; 20; 30]$. The dots on each curve indicate the optimisation times. The figures located on the diagonal correspond to the time-evolution of the optimised quantities. Therefore, we can define from these diagonal plots, the optimal envelopes $G_{\text{opt}}(T)$, $M_{\text{opt}}(T)$ and $V_{\text{opt}}(T)$ (obtained by cubic interpolation of the computed optima). These optimal envelopes are plotted with dashed lines on the plots of each row.
Results for T=5: Energy: Vorticity & Scalar

\[ t = 0.000000 \]
Results for T=5: Mix-norm: Vorticity & Scalar

\[ t = 0.000000 \]
Results for $T=5$: Variance: Vorticity & Scalar
Results for T=30: Energy: Vorticity & Scalar

$t = 0.000000$
Results for $T=30$: Mix-norm: Vorticity & Scalar

$t = 0.000000$
Results for $T=30$: Variance: Vorticity & Scalar
Results for $T=30$

- Energy much larger scale
- Strong vortices perturb interface
- Effect dies out with little mixing...
- ... unlike Mix-norm & variance:
  - very similar evolution
  - much smaller scale and 3 stage:
    1. Transient perturbation **Transport**
    2. (Mean flow) Taylor **Dispersion**
    3. **Relaxation** by diffusion
- No clear separation of stirring & mixing
Scalar Gradients: $T=30$ Variance

Scalar gradients evolution involves $\hat{u}$

\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta \|^2 = - \int_{\Omega} \nabla \theta \cdot \nabla (\hat{u} + \bar{U}) \cdot \nabla d\Omega - Pe^{-1} \| \nabla^2 \theta \|^2
\]

\[
P_{\hat{u}}(y, t) = -\frac{1}{L} \int_{0}^{L} \nabla \theta \cdot \nabla \hat{u} \cdot \nabla \theta \, dx
\]

\[
P_{\bar{U}}(y, t) = -\frac{1}{L} \int_{0}^{L} \nabla \theta \cdot \nabla \bar{U} \cdot \nabla \theta \, dx \quad \bar{U} = (1 - y^2) e_x
\]

\[
P_d(y, t) = -\frac{1}{PeL} \int_{0}^{L} |\nabla \theta|^2 \, dx
\]

\[
P = P_{\bar{U}} + P_{\hat{u}} + P_d
\]

Due to Taylor dispersion, “mixing” occurs while the mean flow is “stirring”...
Conclusions

- Nonlinear Direct-Adjoint-Looping method works for mixing
- Mix-norm is an excellent proxy for (real) mixing
- Energy gain not related to “best” mixing...but maybe @ higher Re?
- Not stirring THEN mixing for this flow but timing is crucial:

  Transport - Dispersion - Relaxation

- DAL Optimisation technique very general:
  - Also used to optimise wall-forcing (Foures et al 2014 JFM 748)
  - 3D/other flows straightforward but...
- What happens when flow is:
  - unstable?
  - turbulent...a minimal mixing seed?
Forcing to $T=20$

Horizontal forcing at the boundaries
Symmetry between walls
Use $p$-norm to constrain maximum value
Uses much more energy
Still exploit Taylor dispersion
Three stage mixing still evident...

$$\int_0^T \int_0^L |u_\pm|^{20} \, dx \, dt = 1$$

$$\int_0^T \int_0^L |u_\pm|^{20} \, dx \, dt = 2$$
Forcing to $T=20$

Horizontal forcing at the boundaries
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Use $p$-norm to constrain maximum value
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\[
\int_0^T \int_0^L |u_\pm|^20 \, dx \, dt = 1
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3D Results for $T=2,5,10,20,30$

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$M(t) = \frac{||\nabla^{-1} \theta||_2}{||\nabla^{-1} \theta_d||_2} \frac{2}{2}$

$V(t) = \frac{||\theta||_2}{||\theta_d||_2} \frac{2}{2}$

- Late times ~5% better than 2D...quite different initial structure, and finite scale in 3D as time increases...
Results for $T=2,5,10,20,30$

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3D: Variance T=10

T = 0.0

X-Y plane

X-Z plane
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