Nonlinear Optimisation of Scalar Mixing in Plane Poiseuille Flow

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IPAM Workshop on Mixing and Transport: October 2014

Or...

Be sure to listen to Tom,

Be sure to listen to Tom, Jean-Luc

Be sure to listen to Tom, Jean-Luc & Charlie!

IPAM Workshop on Mixing and Transport: October 2014

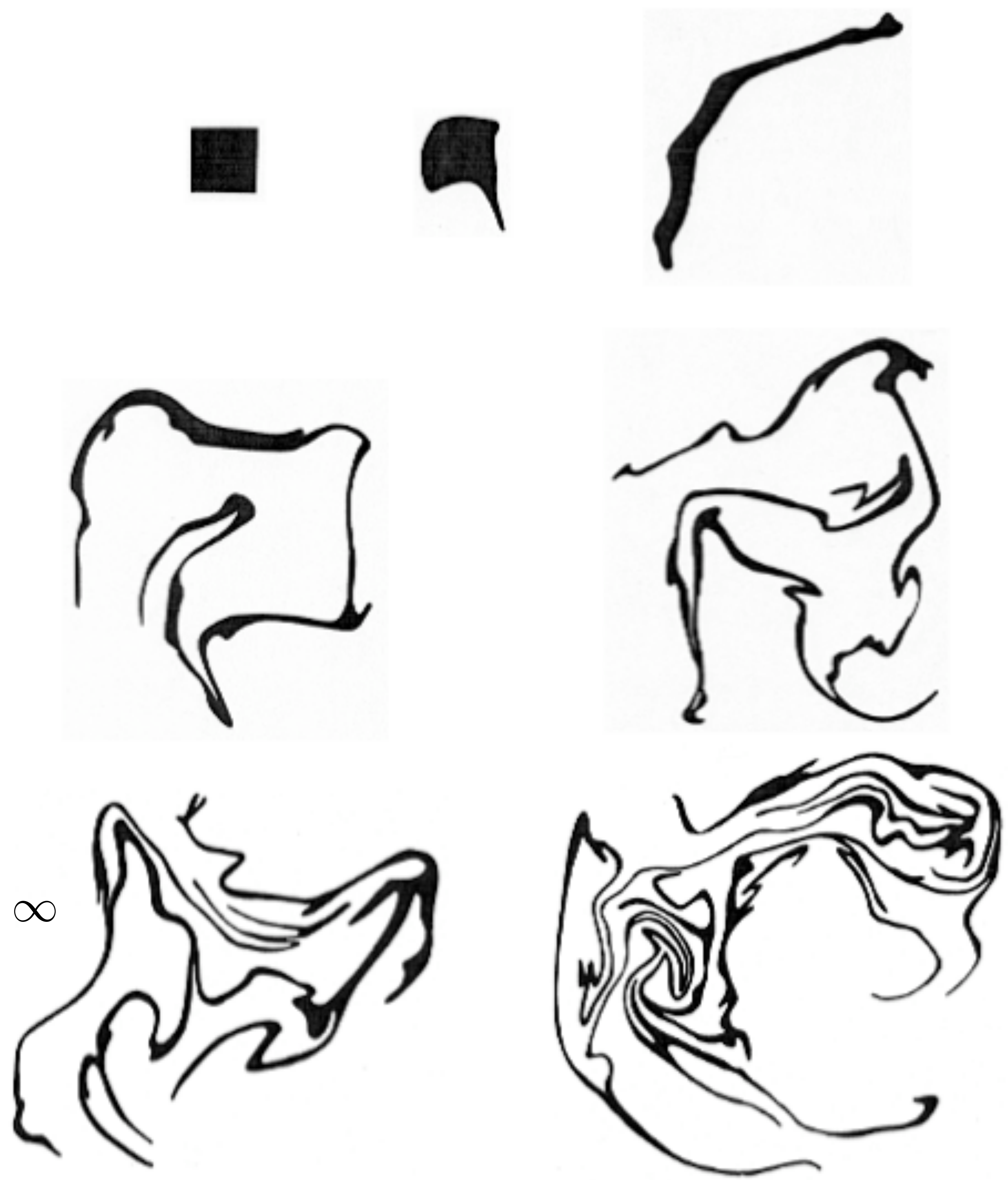
Optimising Mixing

- Mixing is of course a fundamental problem in fluid mechanics
- Mixing ultimately driven by diffusive processes, so need:

  (I) Strong Gradients (high flux)

  (II) "Filamentation" (high contact area so high transport)

- Classic picture of Eckart (1948)/Welander (1955): Stirring $\rightarrow (I), (II) \rightarrow$ mixing
  - Beautiful work on stirring: Rhines & Young, Aref, Ottino, Mezic, Doering, Thiffeault...
    - But... prescribed flow (no Navier-Stokes) and often zero diffusion ($Pe = \frac{UL}{\kappa} \rightarrow \infty$
- Questions remain for real (finite Pe, coupled solutions of Navier-Stokes Eqns) flows:
  - What is the "best" way to mix a freely evolving flow (Energy=mixing Aamo et al)?
  - Is mixing really after stirring...what about Taylor dispersion...and how can we work it out?



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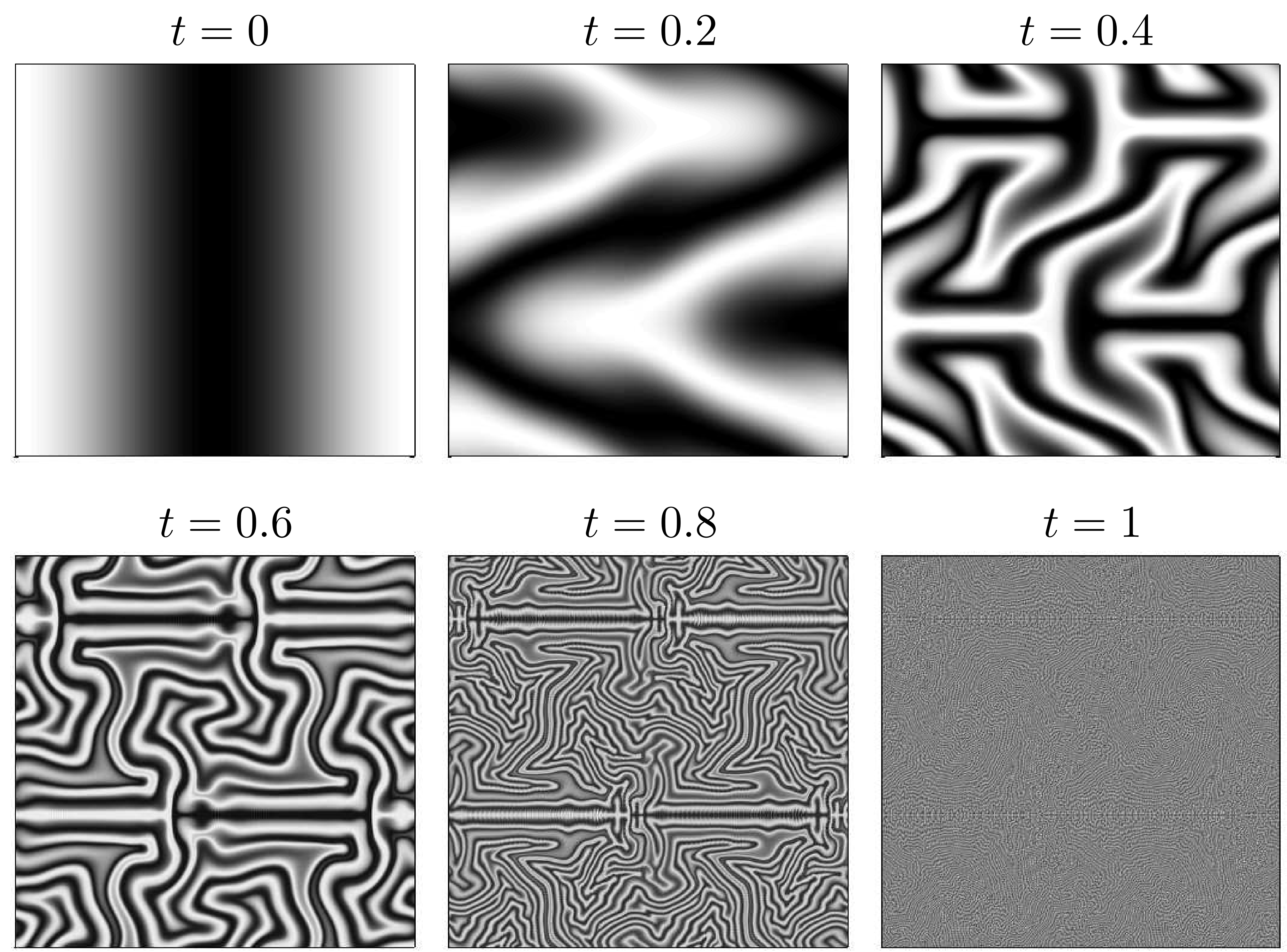

Lin, Thiffeault & Doering 2011...
But prescribed flow
(and here zero diffusion)

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Progress in Aerospace Sciences 37 (2001) 21–58

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Flow control: new challenges for a new Renaissance

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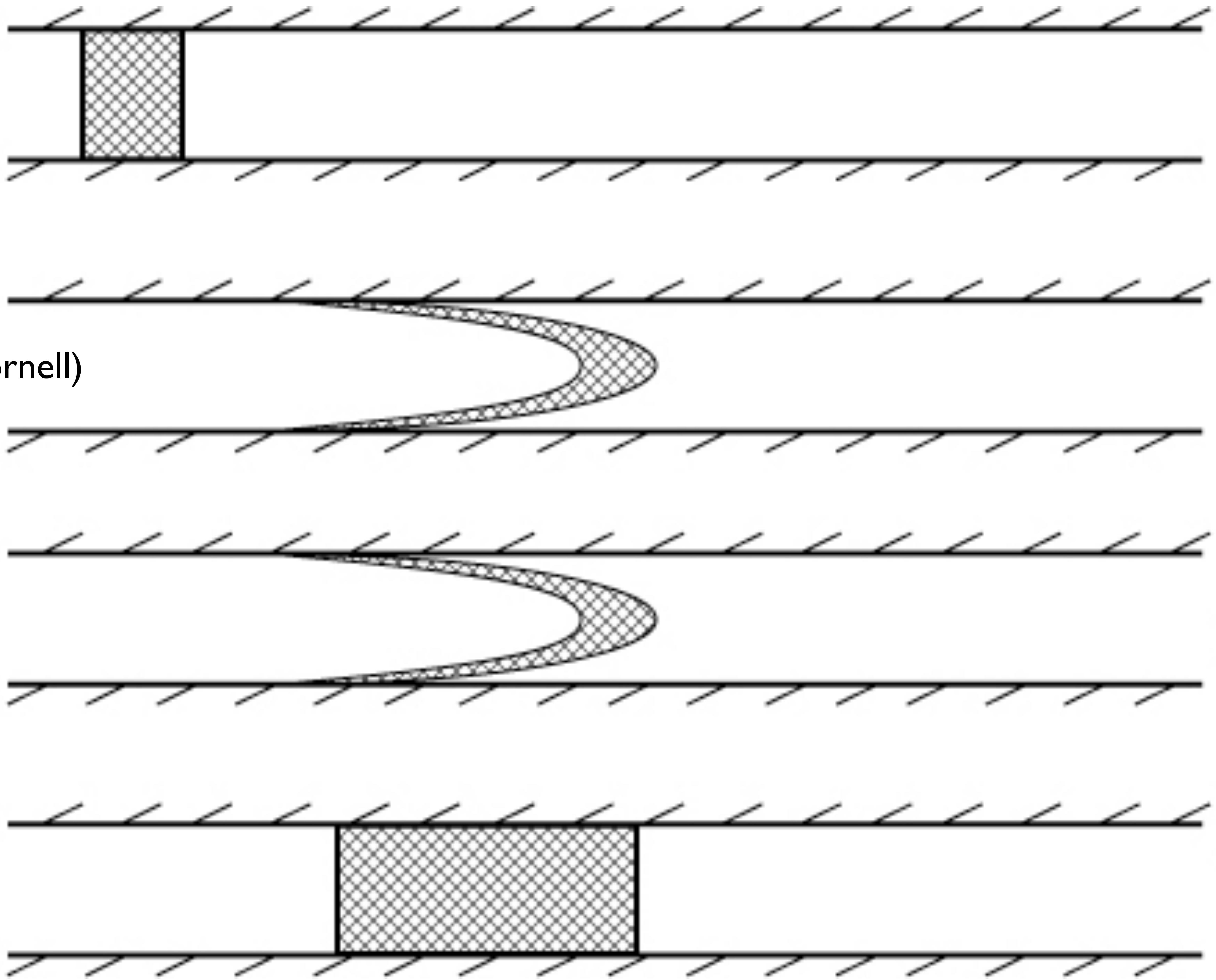
Abstract

As traditional scientific disciplines individually grow towards their maturity, many new opportunities for significant advances lie at their intersection. For example, remarkable developments in control theory in the last few decades have considerably expanded the selection of available tools which may be applied to regulate physical and electrical systems. These techniques hold great promise for several applications in fluid mechanics, including the delay of transition and the regulation of turbulence. Such applications of control theory require a very balanced perspective, in which one considers the relevant flow physics when designing the control algorithms and, conversely, takes into account the requirements and limitations of control algorithms when designing both reduced-order flow models and the fluid–mechanical systems to be controlled themselves. Such a balanced perspective is elusive, however, as both the research establishment in general and universities in particular are accustomed only to the dissemination and teaching of component technologies in isolated fields. To advance, we must not toss substantial new interdisciplinary questions over the fence for fear of them being "outside our area"; rather, we must break down these very fences that limit us, and attack these challenging new questions with a Renaissance approach. In this spirit, this paper surveys a few recent attempts at bridging the gaps between the several scientific disciplines comprising the field of flow control, in an attempt to clarify the author's perspective on how recent advances in these constituent disciplines fit together in a manner that opens up significant new research opportunities. Published by Elsevier Science Ltd.

I'm beginning to understand...

WWGID?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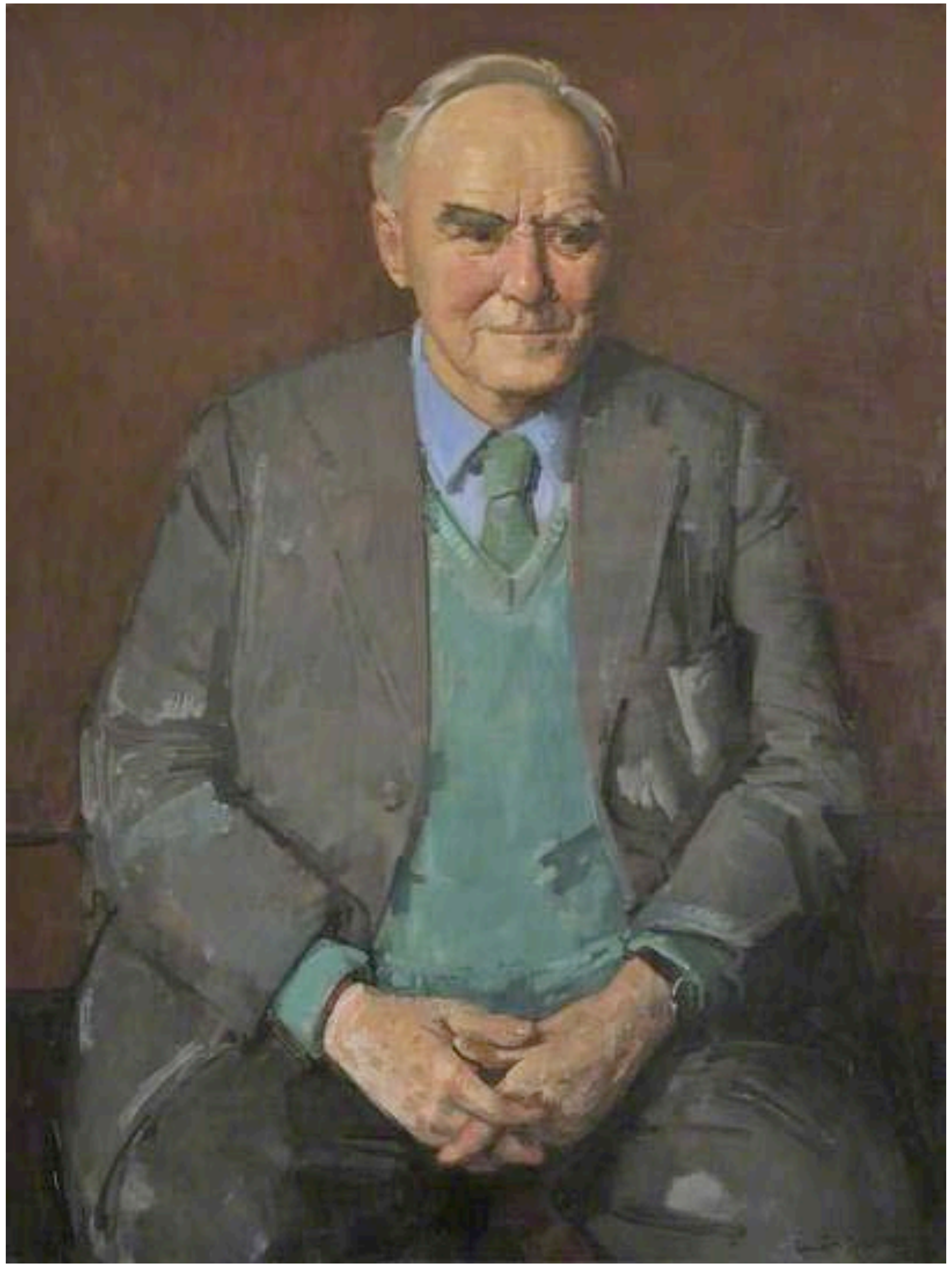
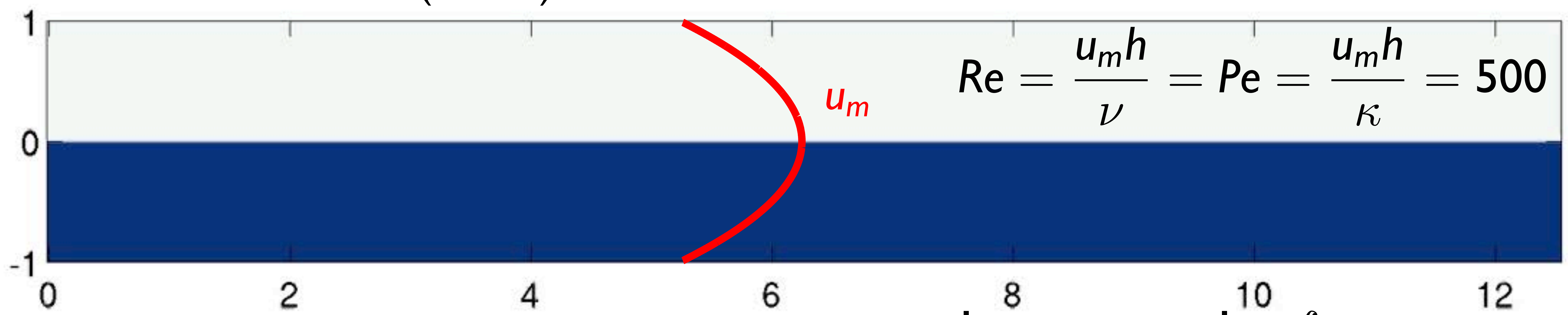


- Taylor-(Aris) dispersion very good way to enhance diffusive transport in shear flows
- Is it possible for the fluid flow to work out what it is good for mixing all on its own?

Mixing in Plane Poiseuille Flow

- Test case: Bands of zero-mean scalar θ in 2D (stable) flow:



- Natural measure of mixing variance: only changes at finite Pe! $Var\ \theta = \frac{1}{V_\Omega}\|\theta\|_2^{\ 2} = \frac{1}{V_\Omega}\int_\Omega \theta(\mathbf{x},t)^2 d\Omega \leftrightarrow \frac{1}{2}\frac{d}{dt}\|\theta\|_2^{\ 2} = -Pe^{-1}\|\nabla\theta\|_2^{\ 2}$
- Sobolev norms: Mathews et al. (2005)/Lin et al.:"mix-norm"/ H^{-1} norm: $Mix\ \theta = \frac{1}{V_\Omega}\left\|\nabla^{-1}\theta\right\|_2^{\ 2} = \frac{1}{V_\Omega}\int_\Omega \left[\nabla^{-1}\theta(\mathbf{x},t)\right]^2 d\Omega$
- Easy to understand for FTs: mix-norm favours large scales (cf. gradient): $|\nabla^{-1}\hat{\theta}|^2 = k^{-2}|\hat{\theta}|^2;\ |\nabla\hat{\theta}|^2 = k^2|\hat{\theta}|^2$
- Mathews et al proved usefulness for periodic domains...but are minimised mix-norms good proxies for real mixing?
- Classic hypothesis (see e.g. Aamo et al 2003 etc) is that perturbation energy growth is the "best" way to mix?
- Aim to determine "best" IC for mixing: $\mathcal{J}\left\{\hat{\mathbf{u}},\theta\right\} = \frac{1-\alpha}{2}\int_0^T \|\hat{\mathbf{u}}(\mathbf{x},t)\|_2^{\ 2}\,dt + \frac{\alpha}{2}\left\|\nabla^{-\beta}\theta(\mathbf{x},T)\right\|_2^{\ 2};\ \mathbf{u} = \overline{\mathbf{U}} + \hat{\mathbf{u}}$
- $\alpha = 0$: (average) energy gain; $\alpha = 1, \beta = 1$: mix norm; $\alpha = 1, \beta = 0$; variance ("real" mixing)

Direct-Adjoint Looping method

- Apply direct-adjoint looping (DAL) method (Hill, Luchini, Bottaro, Schmid etc) to mixing problem

- Derive adjoint equations to describe evolution of Lagrange multipliers which impose solution of Navier-Stokes (& theta) equations

$$\mathcal{L} = \mathcal{J}\left\{\hat{\mathbf{u}},\theta\right\} - \left\langle \mathbf{u}^\dagger, \partial_t\hat{\mathbf{u}} + (\overline{\mathbf{U}}+\hat{\mathbf{u}})\cdot\nabla\hat{\mathbf{u}} + \hat{\mathbf{u}}\cdot\nabla\overline{\mathbf{U}} + \nabla p - Re^{-1}\nabla^2\hat{\mathbf{u}}\right\rangle - \left\langle p^\dagger, \nabla.\hat{\mathbf{u}}\right\rangle$$
$$- \left\langle \theta^\dagger, \partial_t\theta + (\overline{\mathbf{U}}+\hat{\mathbf{u}})\cdot\nabla\theta - Pe^{-1}\nabla^2\theta\right\rangle - \left(\mathbf{u}_0^\dagger, \hat{\mathbf{u}}(\mathbf{x},0) - \hat{\mathbf{u}}_0\right)$$

$$(\mathbf{b},\mathbf{a}) = \frac{1}{V}\int_{\mathcal{D}} \mathbf{b}^\dagger\mathbf{a}\,\mathrm{d}V$$
$$\langle\mathbf{b},\mathbf{a}\rangle = \frac{1}{VT}\int_0^T\int_{\mathcal{D}} \mathbf{b}^\dagger\mathbf{a}\,\mathrm{d}V\mathrm{d}t$$

Set ICS for u, theta at t=0

Integrate 0-T using NS + AD+ incompressibility

calculate u(T), theta(T)+ J

$$\mathbf{u}^\dagger(\mathbf{x},T) = 0;\ \theta^\dagger(\mathbf{x},T) = (-1)^\beta\alpha\nabla^{-2\beta}\theta(\mathbf{x},T)$$

Relate u(T), theta(T) to udag(T), thetadag(T) using E-L eqns

Integrate T-0 using NSadj

Calculate udag(0)...
do they satisfy E-L eqn: udag=u?

If not, update u(0), theta(0) using gradient info in E-L equations

$$\mathbf{u}_0^\dagger = \hat{\mathbf{u}}(\mathbf{x},0) \equiv \nabla_{\hat{\mathbf{u}}_0}\mathcal{J}$$

- Variations of L wrt direct variables: $\partial_t\mathbf{u}^\dagger + (\overline{\mathbf{U}}+\hat{\mathbf{u}})\cdot\nabla\mathbf{u}^\dagger - \mathbf{u}^\dagger\cdot\left(\nabla\left[\overline{\mathbf{U}}+\hat{\mathbf{u}}\right]\right)^\top + \nabla p^\dagger + Re^{-1}\nabla^2\mathbf{u}^\dagger = \theta^\dagger\nabla\theta - (1-\alpha)\,\hat{\mathbf{u}}$

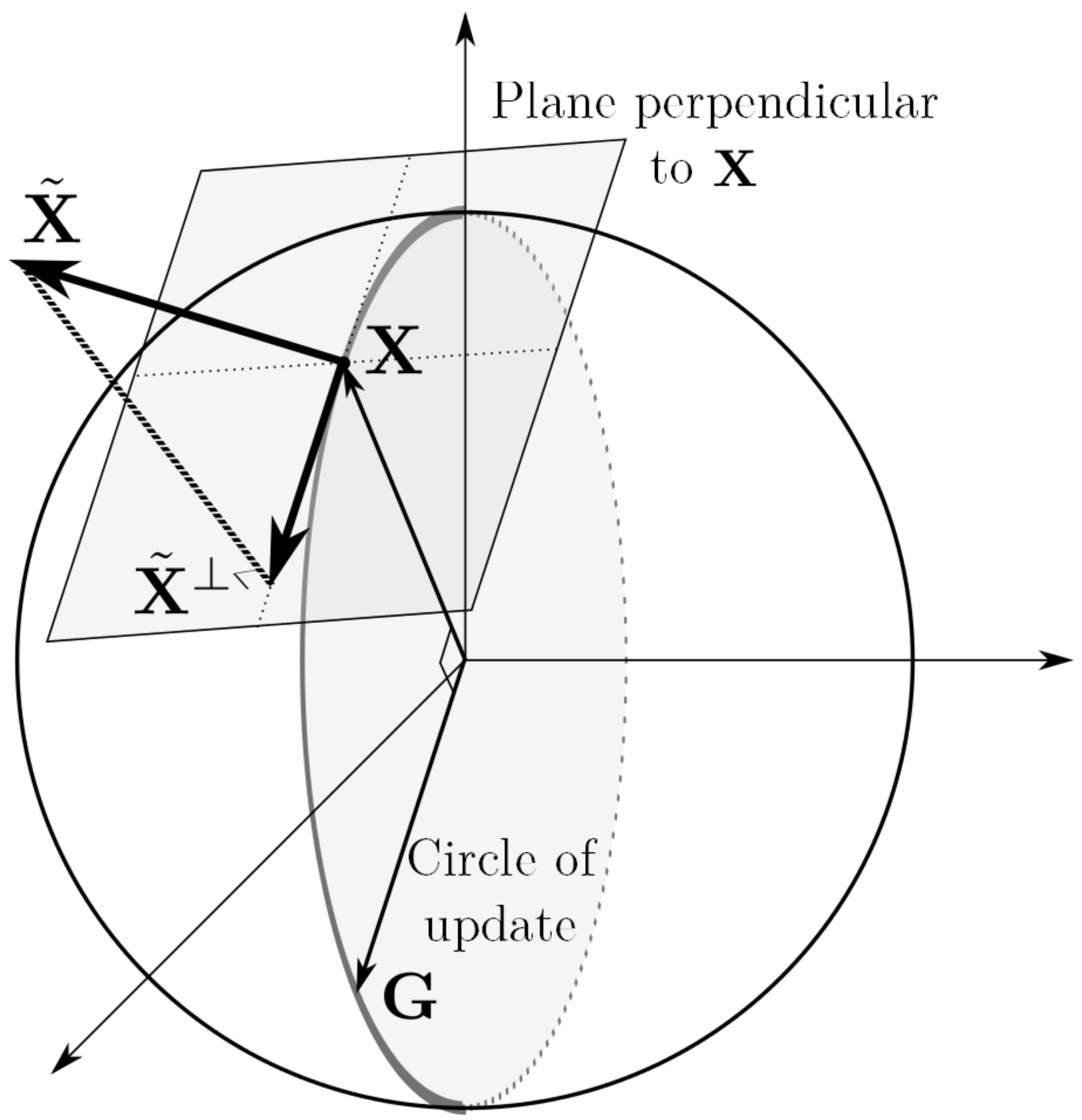
$$\partial_t\theta^\dagger + (\overline{\mathbf{U}}+\hat{\mathbf{u}})\cdot\nabla\theta^\dagger + Pe^{-1}\nabla^2\theta^\dagger = 0;\ \nabla.\mathbf{u}^\dagger = 0;\ \mathbf{u} = \overline{\mathbf{U}}+\hat{\mathbf{u}}$$

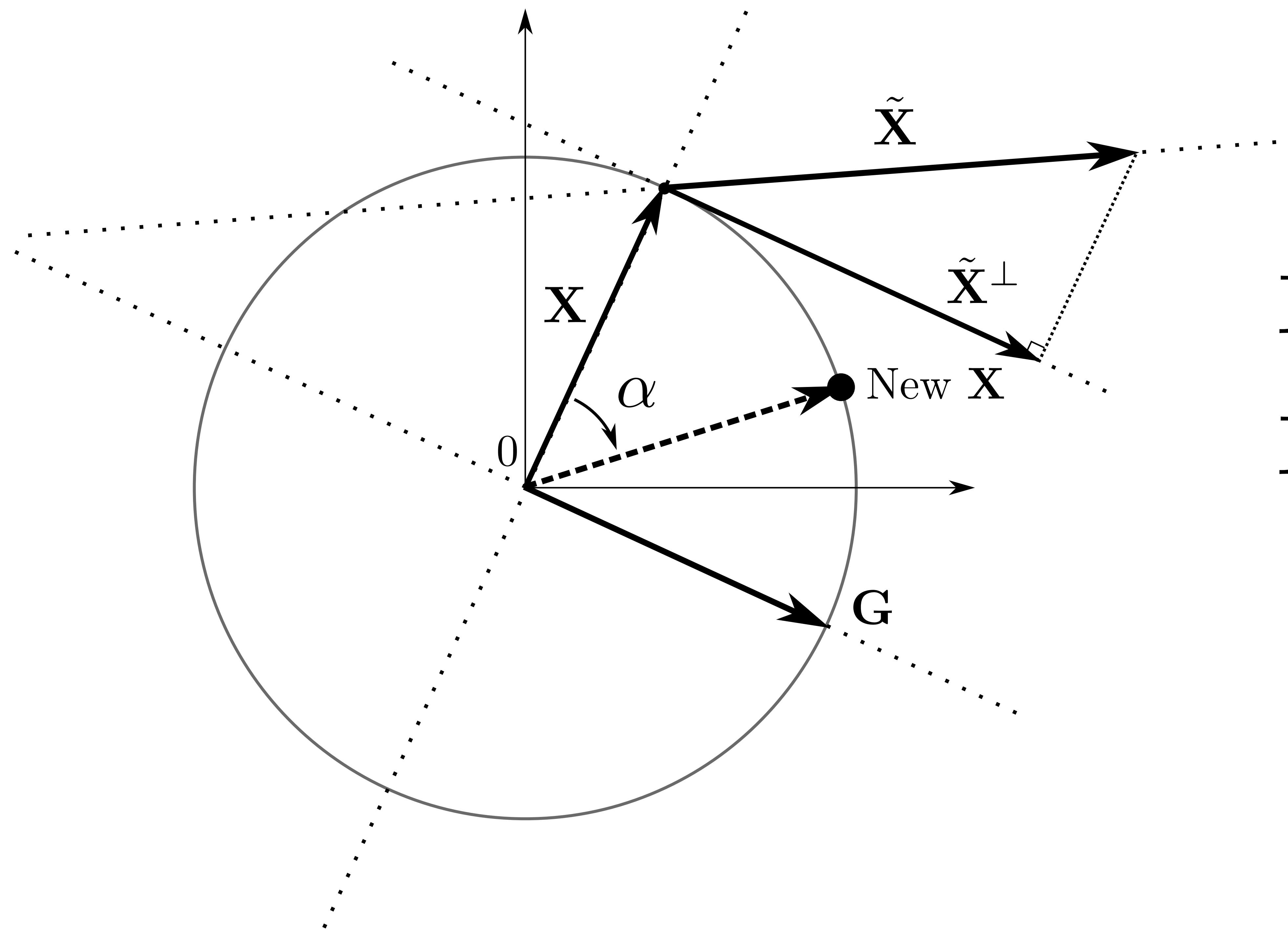
- Adjoint equations have a similar form, and “forced” by direct variables (and nonlinear in direct variables...nontrivial optimisation)

Normalisation, rotation and optimisation...

- Fix initial energy of perturbation to be finite amplitude but small: $E_0 = 10^{-2} = \frac{1}{2}(\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0) = \frac{3}{80}\overline{E}$

- Could impose as a further constraint, but introduces ambiguity...much better to enforce directly using $\nabla_{\hat{\mathbf{u}}_0}\mathcal{J}$ when not parallel to $\hat{\mathbf{u}}_0$



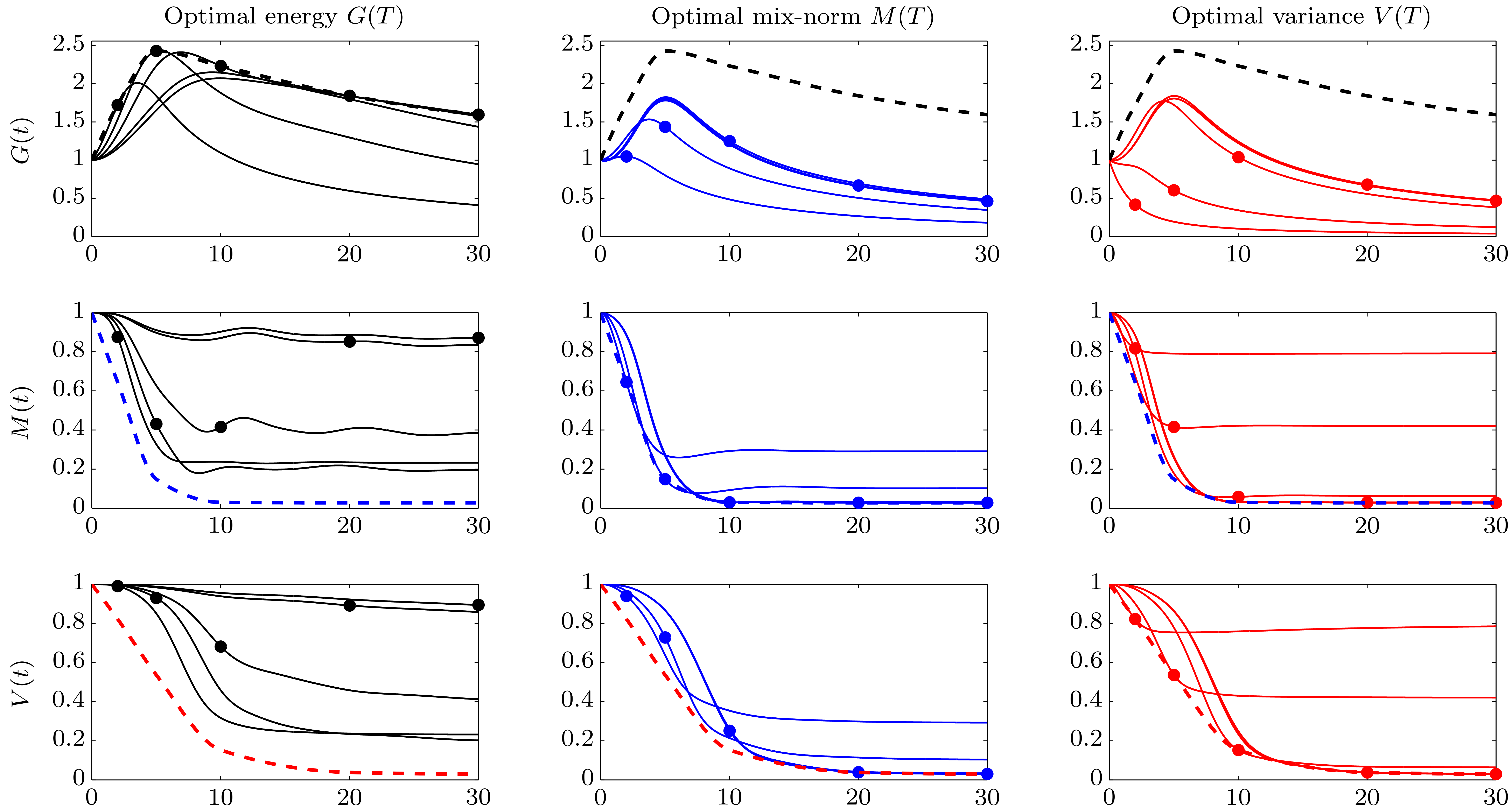


$$\mathbf{X} \equiv \hat{\mathbf{u}}_{n,0}$$
$$\tilde{\mathbf{X}} \equiv \nabla_{\hat{\mathbf{u}}_{n,0}}\mathcal{J}$$

- Project $\nabla_{\hat{\mathbf{u}}_{n,0}}\mathcal{J}$ onto hyperplane orthogonal to $\hat{\mathbf{u}}_{n,0}$ and tangent to hypersphere $(\hat{\mathbf{u}}_{n,0}, \hat{\mathbf{u}}_{n,0}) = 2E_0$

- Restrict updated guess $\hat{\mathbf{u}}_{n+1,0}$ to rotations along intersection of hypersphere with hyperplane defined by scaled projected gradient and $\hat{\mathbf{u}}_{n,0}$

Results for T=2,5,10,20,30



$$G(t) = \frac{\frac{1}{t}\int_0^t \|\mathbf{u}\|_2^{\,2}(\tau)\,d\tau}{\|\mathbf{u}_0\|_2^{\,2}}$$

$$M(t) = \frac{\|\nabla^{-1}\theta\|_2^{\,2}}{\|\nabla^{-1}\theta_d\|_2^{\,2}}$$

$$V(t) = \frac{\|\theta\|_2^{\,2}}{\|\theta_d\|_2^{\,2}}$$

- Mix-norm and variance similar/small with early gain...very different from energy, converge to same behaviour & most mixing done by T=10

Results for T=5: Energy: Vorticity & Scalar

Results for T=5: Mix-norm: Vorticity & Scalar

Results for T=5: Variance: Vorticity & Scalar

Results for T=30: Energy: Vorticity & Scalar

Results for T=30: Mix-norm: Vorticity & Scalar

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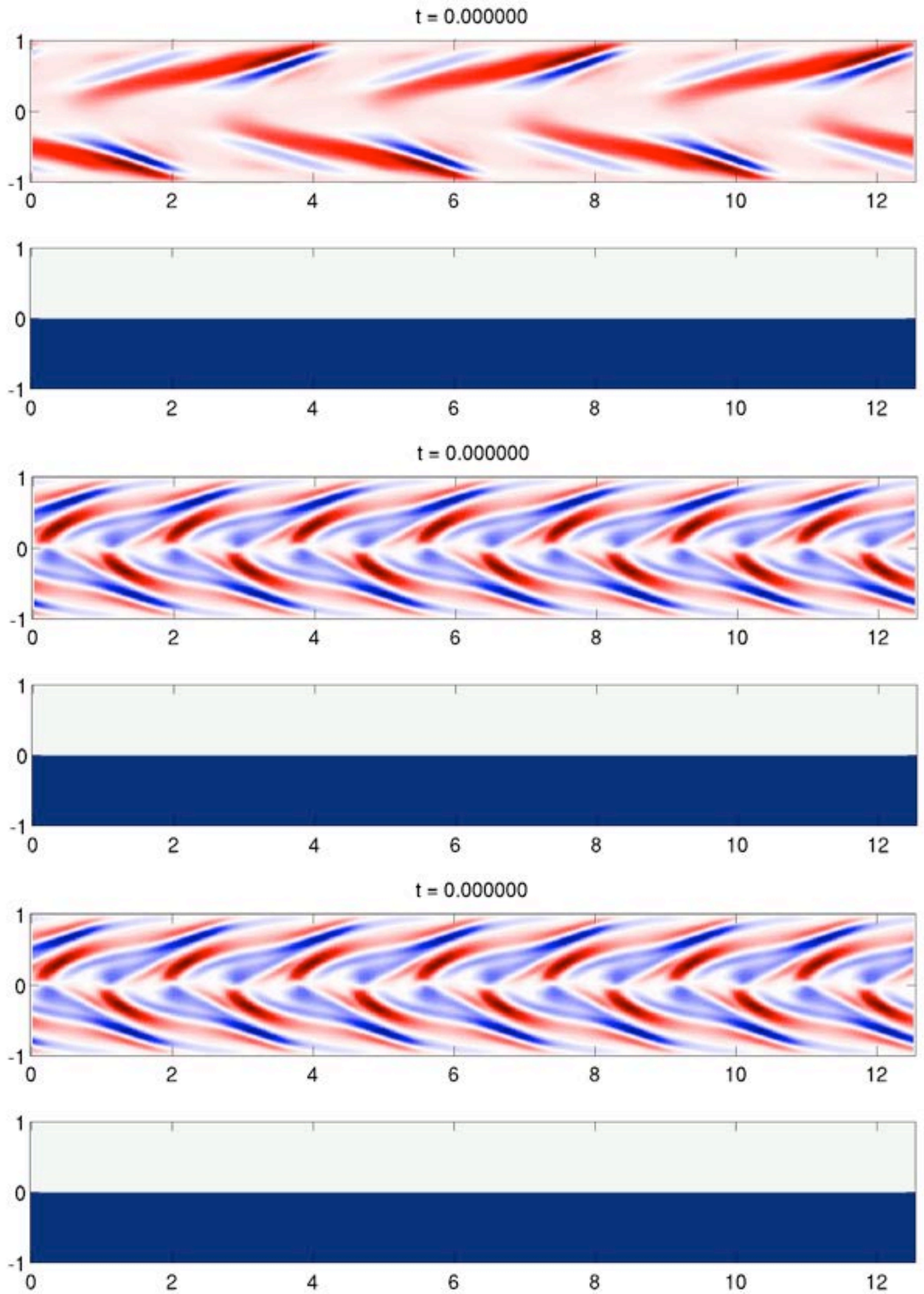
Results for T=30



Vorticity
Energy
Scalar

Vorticity
Mix
Scalar

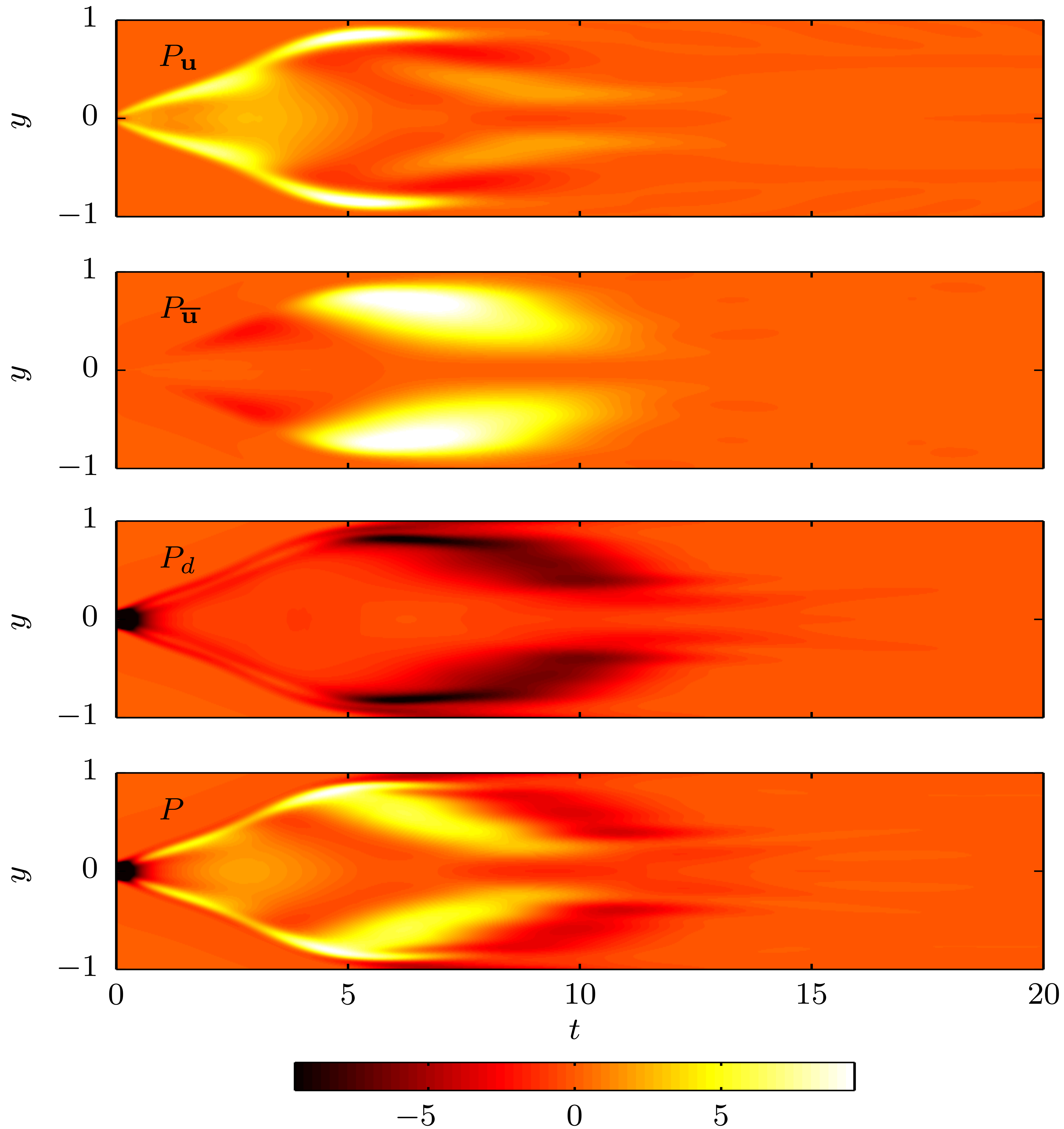
Vorticity
Variance
Scalar

- Energy much larger scale
  - Strong vortices perturb interface
  - Effect dies out with little mixing...
- ... unlike Mix-norm & variance:
  - very similar evolution
  - much smaller scale and **3 stage**:
    1. Transient perturbation **Transport**
    2. (Mean flow) Taylor **Dispersion**
    3. **Relaxation** by diffusion
- No clear separation of stirring & mixing

Scalar Gradients: T=30 Variance

- Scalar gradients evolution involves $\hat{\mathbf{u}}$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2{}^2 = -\int_\Omega \nabla\theta\cdot\nabla\left(\hat{\mathbf{u}}+\overline{\mathbf{U}}\right)\cdot\nabla\theta d\Omega - Pe^{-1}\left\|\nabla^2\theta\right\|_2^2$$



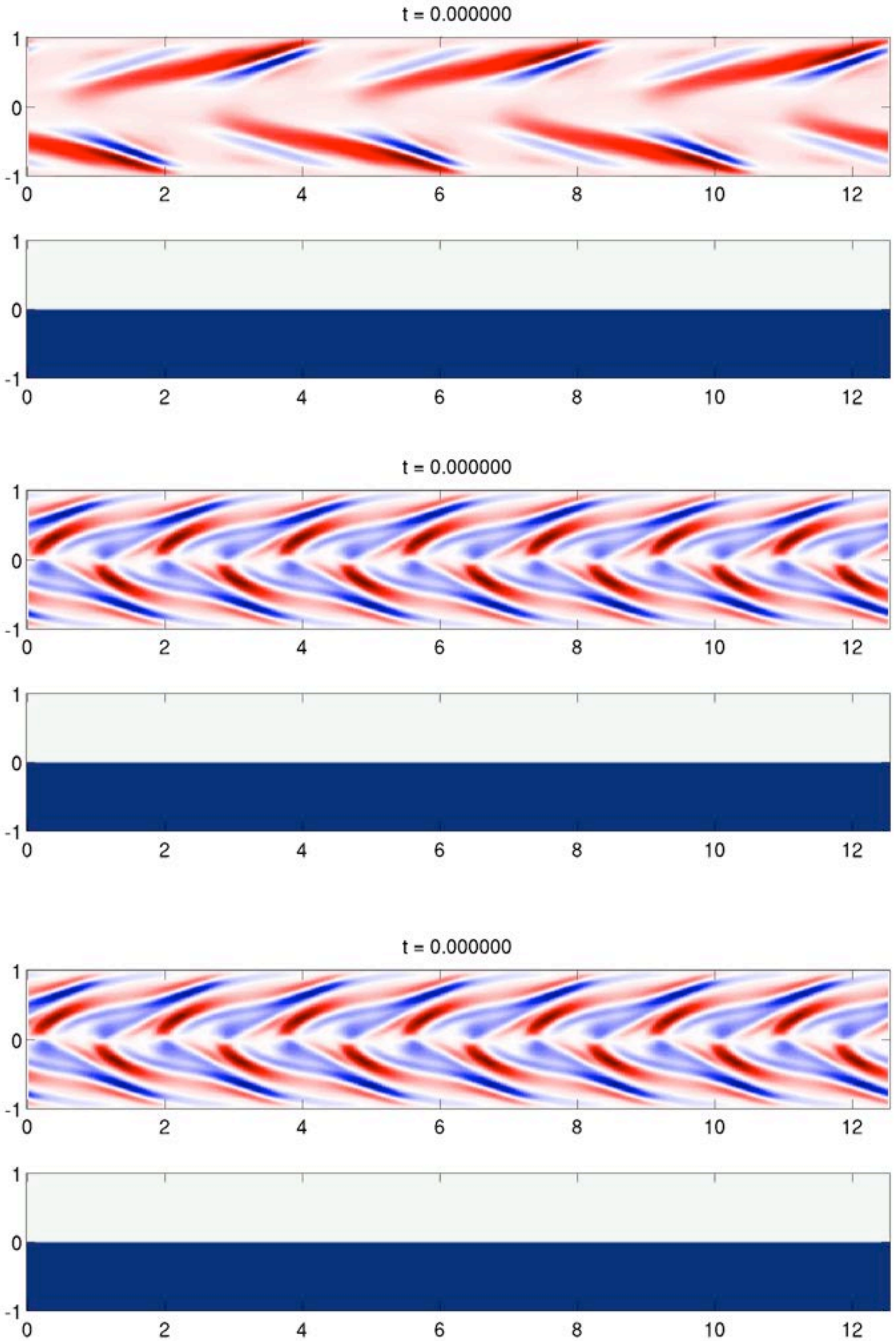
$$P_{\hat{\mathbf{u}}}(y,t) = -\frac{1}{L}\int_0^L \nabla\theta\cdot\nabla\hat{\mathbf{u}}\cdot\nabla\theta dx$$

$$P_{\overline{\mathbf{U}}}(y,t) = -\frac{1}{L}\int_0^L \nabla\theta\cdot\nabla\overline{\mathbf{U}}\cdot\nabla\theta dx \qquad \overline{\mathbf{U}} = (1-y^2)\mathbf{e}_x$$

$$P_d(y,t) = -\frac{1}{PeL}\int_0^L \left|\nabla\theta\right|^2 dx$$

$$P = P_{\overline{\mathbf{U}}} + P_{\hat{\mathbf{u}}} + P_d$$

Due to Taylor dispersion, "mixing" occurs while the mean flow is "stirring"...



Conclusions

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Transport - Dispersion - Relaxation

- DAL Optimisation technique very general:
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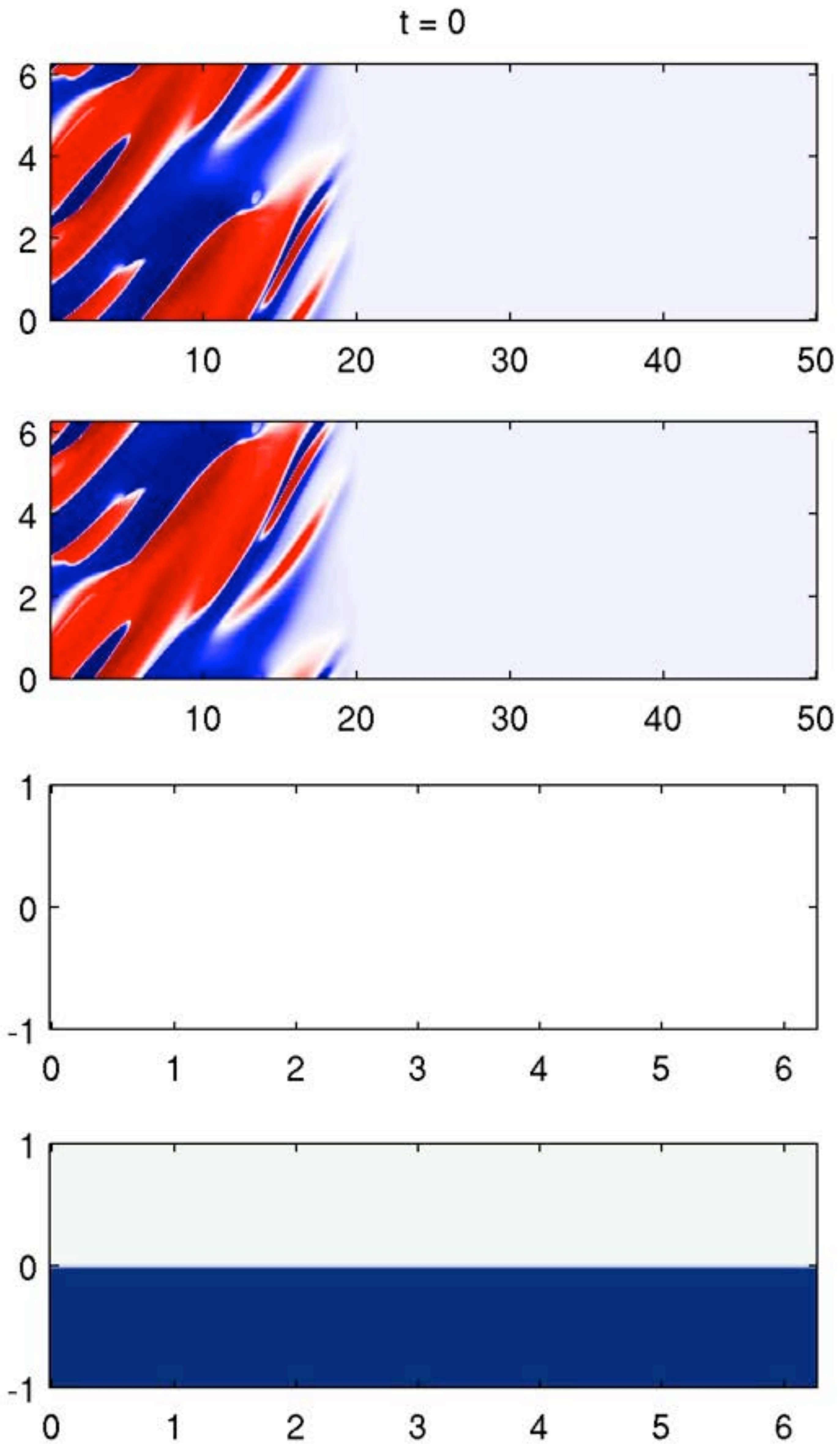
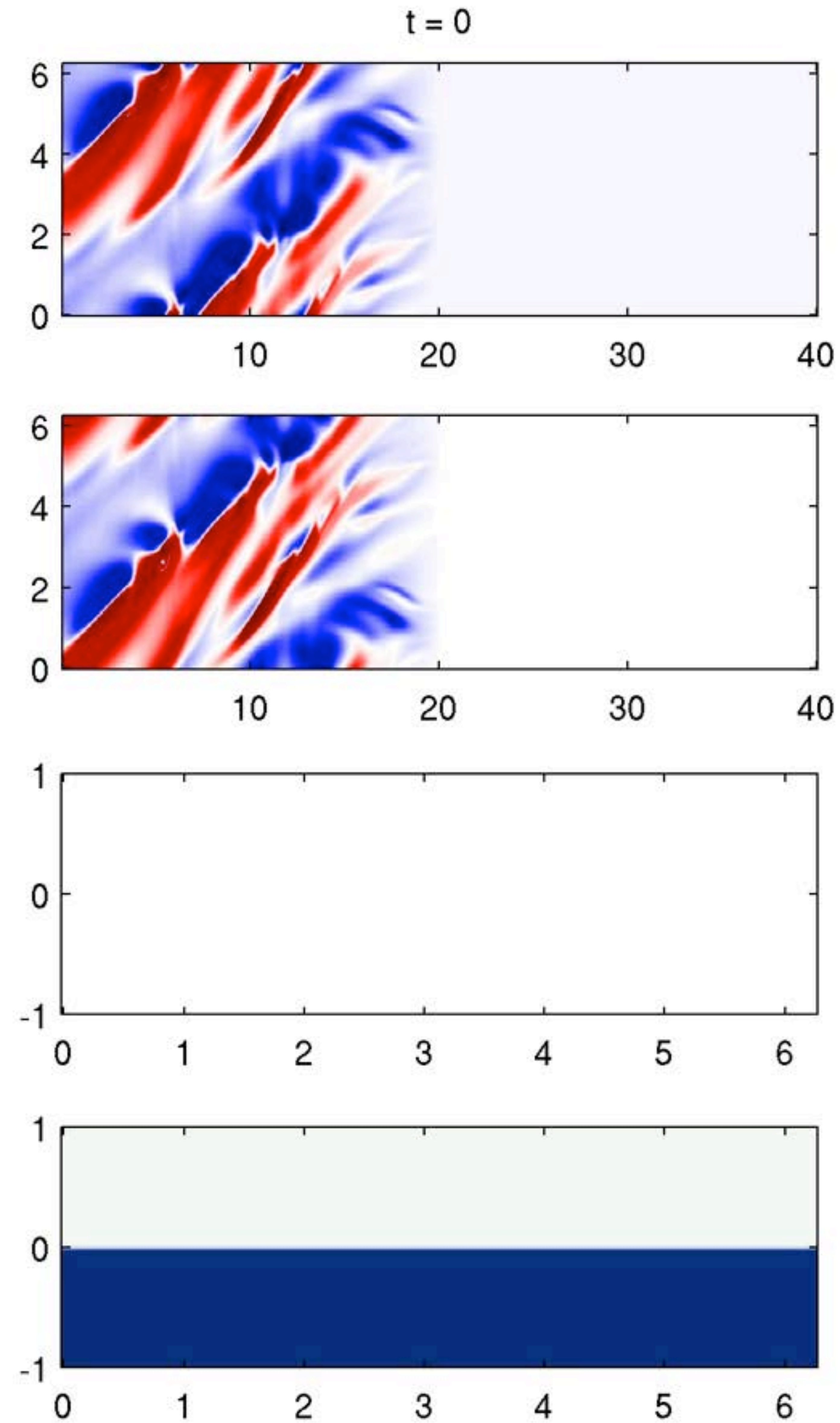
  - 3D/other flows straightforward but...
- What happens when flow is:
  - unstable?
  - turbulent...a **minimal mixing seed?**

Forcing to T=20



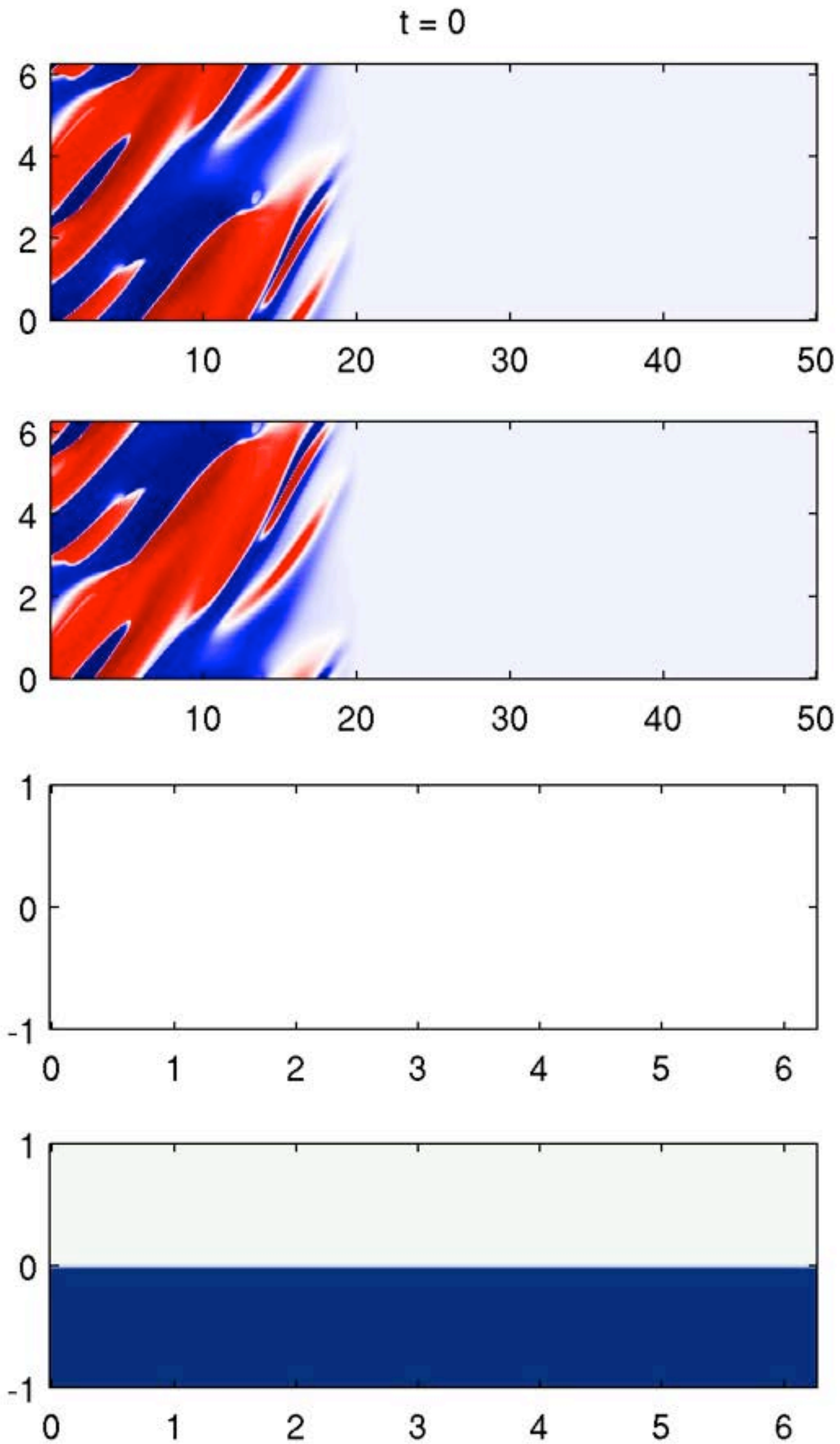
$$\int_0^T \int_0^L |\mathbf{u}_\pm|^{20}\, dx\, dt = 1$$

Horizontal forcing at the boundaries
Symmetry between walls
Use p-norm to constrain maximum value
Uses much more energy
Still exploit Taylor dispersion
Three stage mixing still evident...



$$\int_0^T \int_0^L |\mathbf{u}_\pm|^{20}\, dx\, dt = 2$$

Forcing to T=20



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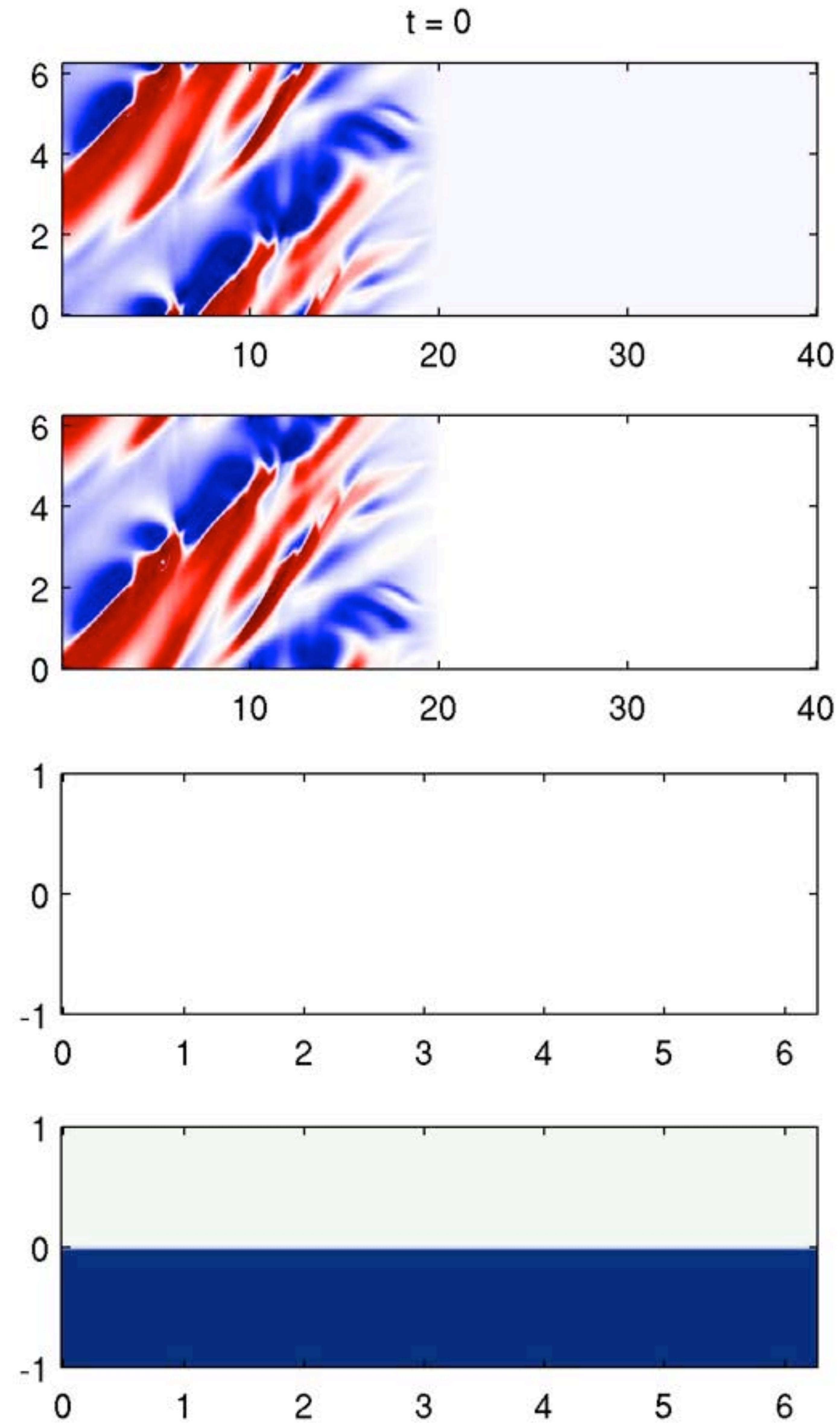

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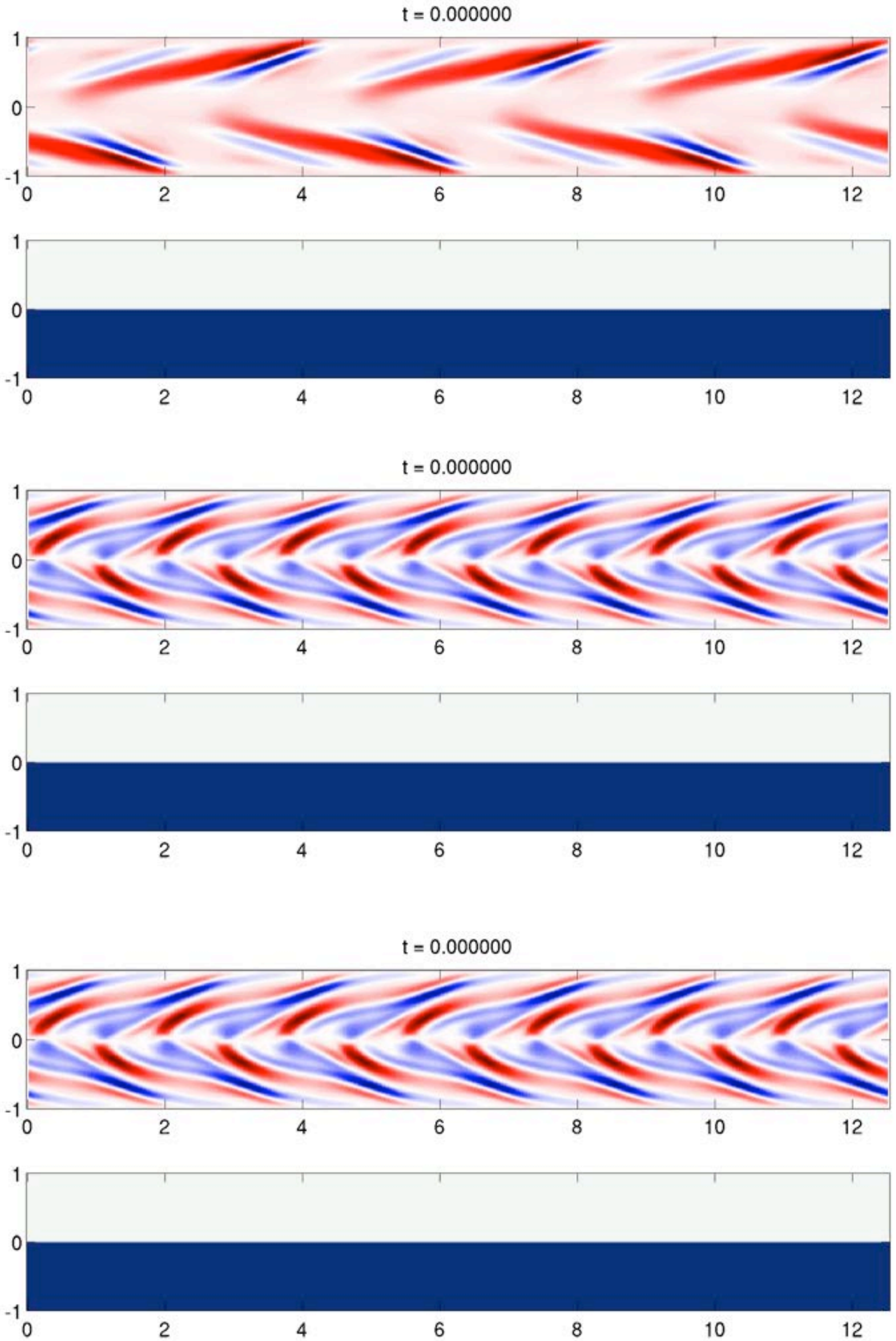

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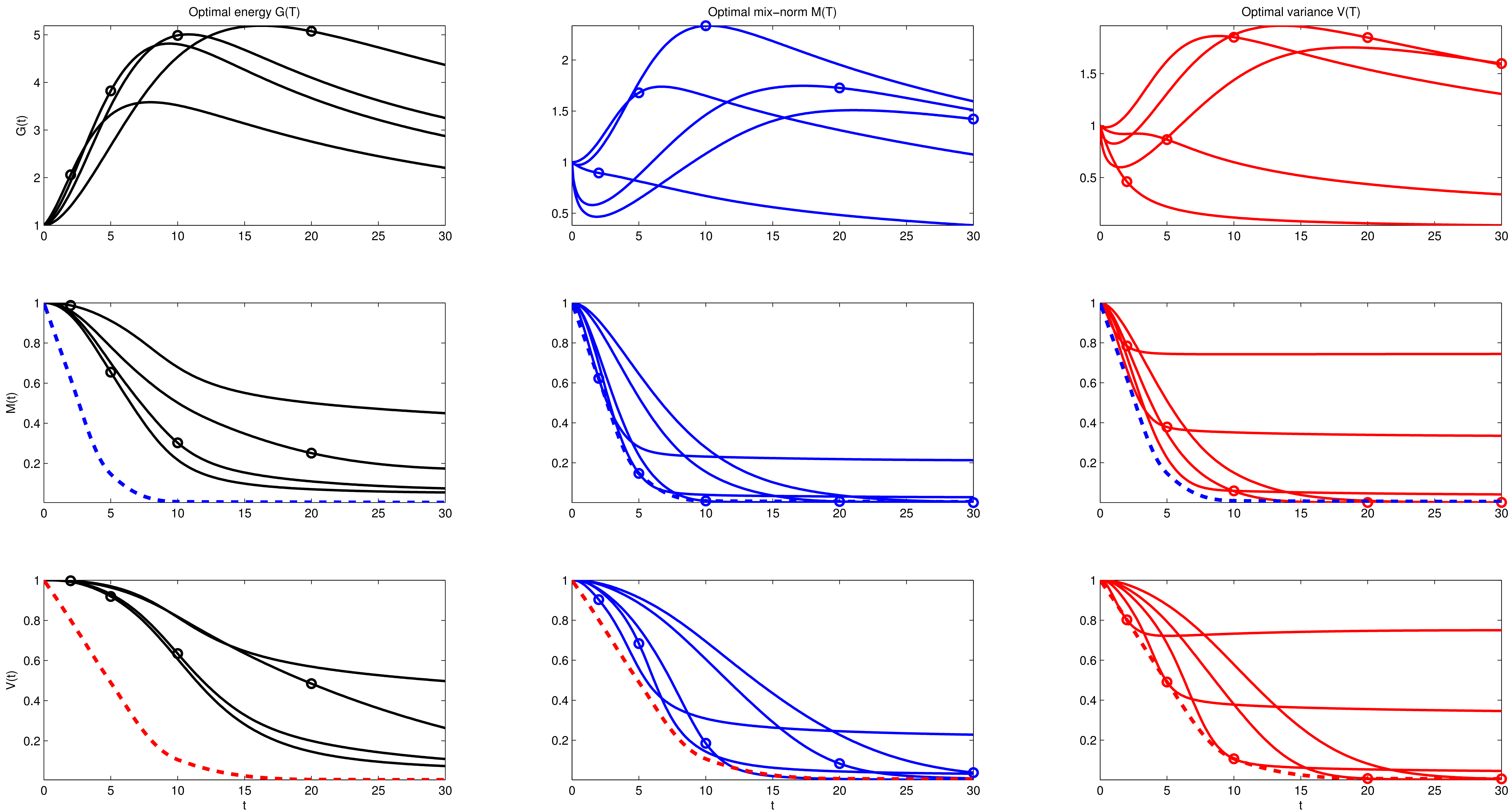
3D Results for T=2,5,10,20,30



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- Late times ~5% better than 2D...quite different initial structure, and finite scale in 3D as time increases...

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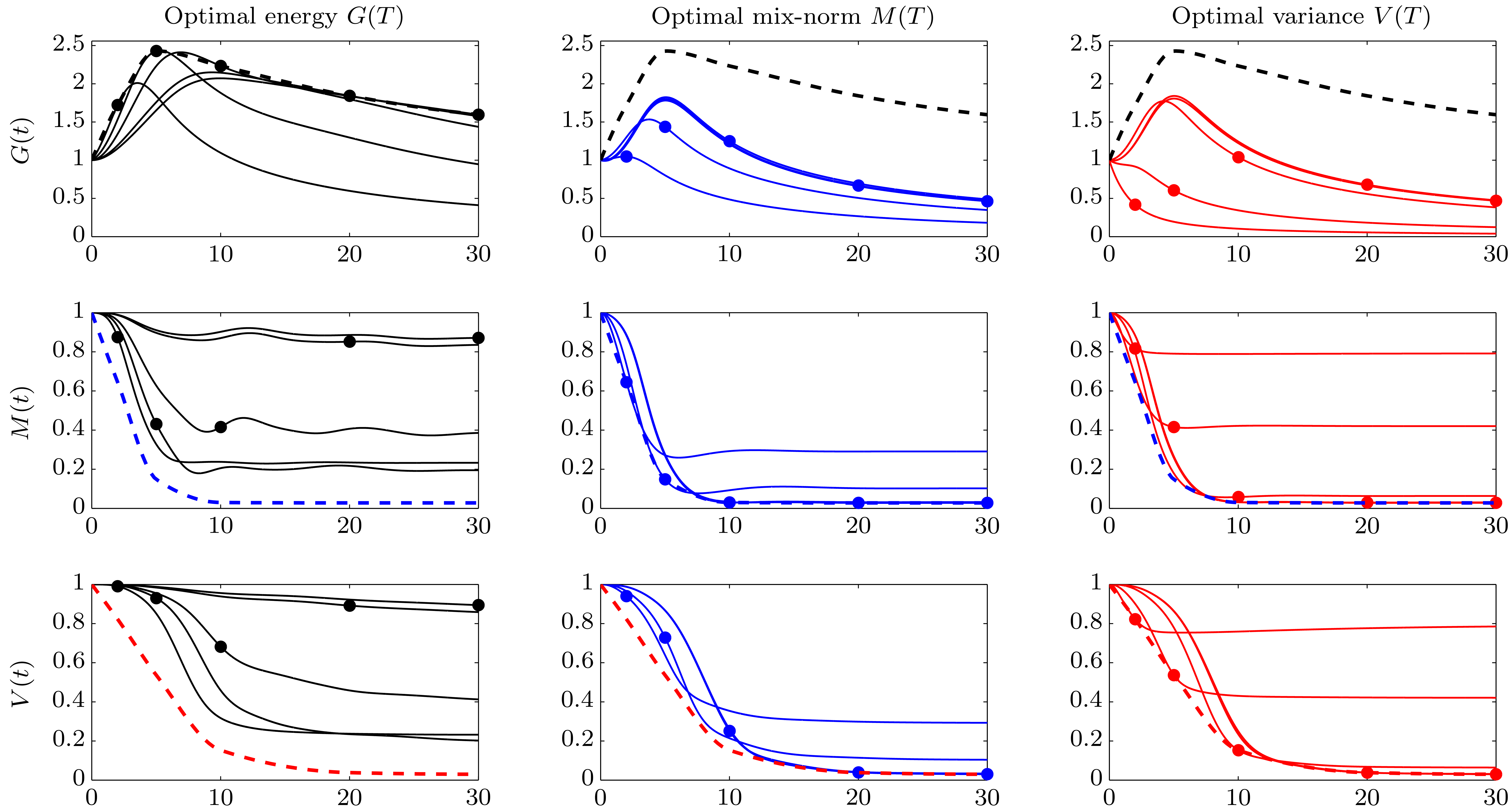

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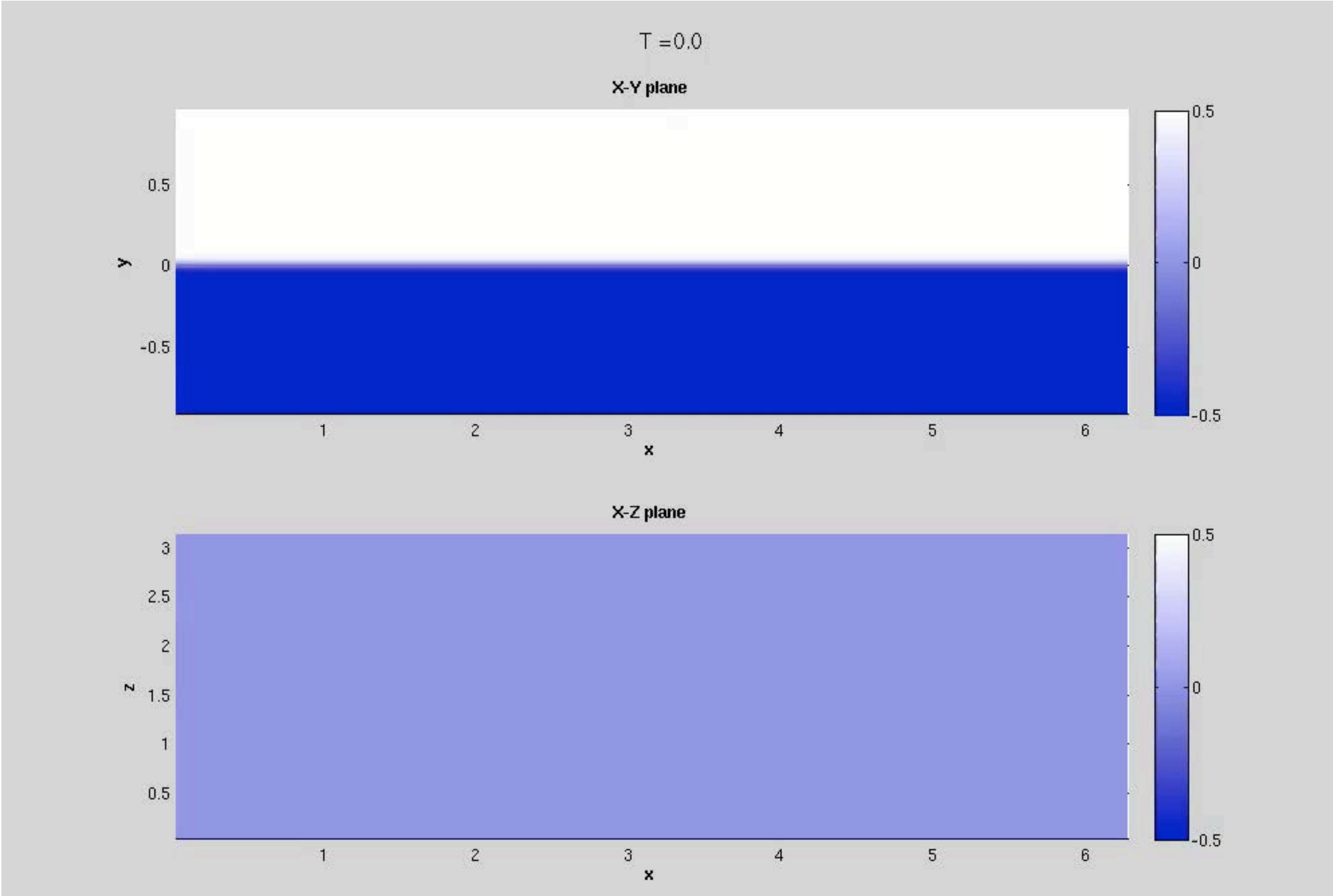
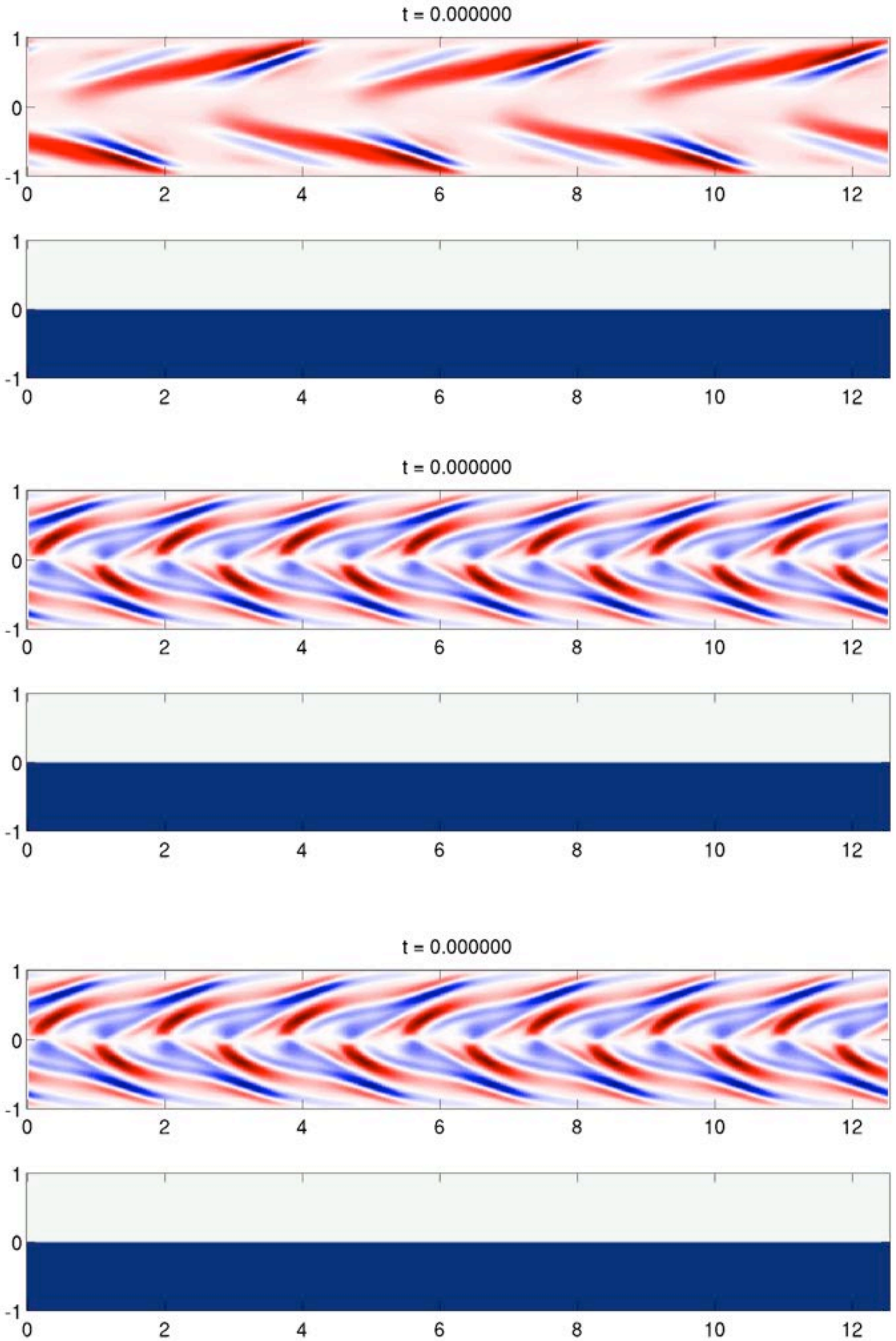
3D: Variance T=10





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