Bounds on scalar dissipation scale

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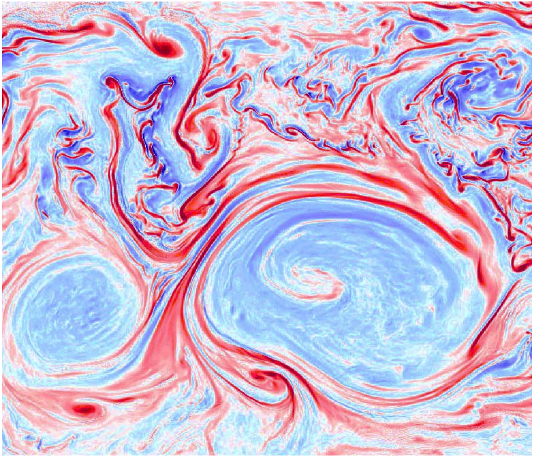
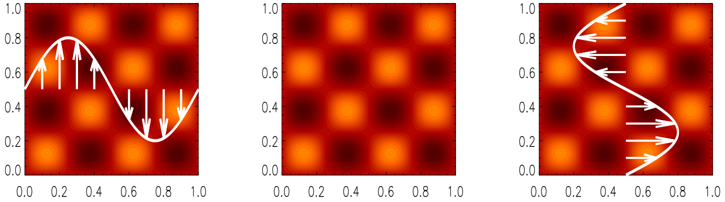
Turbulence transport and Mixing

How do we quantify mixing in the presence of sources and sinks?



Setup of the problem

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta + s,$$



$$\mathbf{u} = \mathbf{u}(\mathbf{x}/\ell_u, t/\tau_u) \qquad s = s(\mathbf{x}/\ell_s, t/\tau_s)$$

with $\langle s \rangle = 0$

& periodic boundary conditions

$$\langle f \rangle \equiv \lim_{T \to \infty} \frac{1}{V_\Omega T} \int_0^T dt \int_\Omega d\mathbf{x} \quad f(\mathbf{x}, t)$$

Parameters

$$\kappa, \quad U^2 = \langle \mathbf{u}^2 \rangle, \quad S^2 = \langle \mathbf{s}^2 \rangle,$$

$$\ell_u^{-2} = \langle (\nabla \mathbf{u})^2 \rangle / \langle \mathbf{u}^2 \rangle, \quad \ell_s^{-2} = \langle (\nabla s)^2 \rangle / \langle \mathbf{s}^2 \rangle$$

$$\tau_u, \quad \tau_s$$

Non-dimensional Numbers

$$\mathrm{Pe} \equiv U\ell_u/\kappa, \quad \rho \equiv \ell_u/\ell_s, \quad St_u \equiv \frac{\ell_u}{U\tau_u}, \qquad St_s \equiv \frac{\ell_u}{U\tau_s}, \qquad .$$

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$$\tau_u, \quad \tau_s$$

Non-dimensional Numbers

$$\mathrm{Pe} \equiv U\ell_u/\kappa, \quad \rho \equiv \ell_u/\ell_s, \quad St_u \equiv \frac{\ell_u}{U\tau_u} = 1, \quad St_s \equiv \frac{\ell_u}{U\tau_s} = 0, .$$

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Measures of Mixing

$$\langle (\nabla^n \theta)^m \rangle$$

with special cases

$$\varphi \equiv \langle |\nabla^{-1}\theta|^2 \rangle, \quad \sigma^2 \equiv \langle \theta^2 \rangle, \quad \chi \equiv \kappa \langle |\nabla \theta|^2 \rangle, \quad \eta \equiv \kappa \, \langle |\Delta \theta|^2 \rangle$$

Non-dimensional form using the scalar distribution in the absence of flow

$$\frac{\varphi}{\varphi_0} \simeq \frac{\varphi^2 \kappa^2}{S^2 \ell_s^6}, \qquad \frac{\sigma^2}{\sigma_0^2} \simeq \frac{\sigma^2 \kappa^2}{S^2 \ell_s^4}, \qquad \frac{\chi}{\chi_0} \simeq \frac{\chi \kappa}{S^2 \ell_s^2}, \qquad \frac{\eta}{\eta_0} \simeq \frac{\eta \kappa}{S^2}$$

Doering, C. R. & Thiffeault, J.-L. 2006,
Shaw, T. A., Thiffeault, J.-L. & Doering C. R. 2007

Stirring estimates

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta + s,$$

Balance advection with injection

$$\frac{U\sigma}{\ell_u} \sim S \to \sigma \sim \frac{S\ell_u}{U}$$

Balance advection with dissipation

$$\frac{U\sigma}{\ell_u} \sim \frac{\kappa\sigma}{\ell^2} \to \ell \sim \sqrt{\frac{\kappa}{U\ell_u}}$$

Definition: Batchelor scale

$$\ell_B \equiv \sqrt{\frac{\kappa\ell_u}{U}} = \ell_u \sqrt{\mathrm{Pe}}$$

Upper and lower bounds

Can we derive bounds on mixing?

Do we learn something new from them?

A lower bound

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta + s,$$

Multiply by a smooth field $\psi(\mathbf{x})$ (eg $\psi = s/S$) & space-time average:

$$\langle \theta \mathbf{u} \cdot \nabla \psi \rangle + \kappa \langle \theta \Delta \psi \rangle = -\langle s \psi \rangle.$$

$$\sigma \quad \geqslant \frac{S}{U \sup_{\mathbf{x}} |\nabla s| + \kappa \langle |\Delta s|^2 \rangle^{\frac{1}{2}}} \quad = \frac{S \ell_s}{U} \frac{1}{c_1 + \mathrm{Pe}^{-1} \rho c_2}$$

Lower Bound I

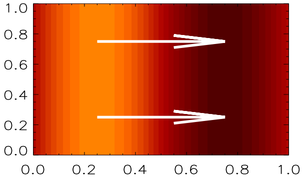
$$\frac{\sigma \kappa}{S \ell_s^2} \geqslant \rho \mathrm{Pe}^{-1} \frac{1}{c_1 + \mathrm{Pe}^{-1} \rho c_2}$$

THIFFEAULT, J.-L., DOERING C. R. & GIBBON, J. D. 2004

A lower Bound

$$\sigma \geqslant \frac{S\ell_s}{U}\frac{1}{c_1 + \mathrm{Pe}^{-1}\rho c_2}$$

- The bound captures the right scaling with κ in the $\kappa \to 0$ limit.
- It is sharp for certain classes of flows & source-sink distributions
- There is no effect of the velocity shear $\nabla \mathbf{u}$
- There is an ℓ_s where one expects to find a ℓ_u



$$s = \sqrt{2} S \cos(x/\ell_s), \qquad \mathbf{u} = e_x U$$

$$\theta = \sqrt{2} S \frac{(\kappa/\ell_s^2)\cos(x/\ell_s) + (U/\ell_s)\sin(x/\ell_s)}{(\kappa/\ell_s^2)^2 + (U/\ell_s)^2}$$

PLASTING, S. C. & YOUNG, W. R. 2006

Introducing the dissipation scale ℓ_d

Definition: Dissipation scale

$$k_d^2 \equiv \ell_d^{-2} \equiv \frac{\langle |\nabla\theta|^2 \rangle}{\langle \theta^2 \rangle} = \frac{\chi}{\kappa\sigma^2}$$

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Using

$$\chi = \kappa\langle|\nabla\theta|^2\rangle = \langle\theta s\rangle$$

$$\sigma = \frac{\langle\theta s\rangle}{\sigma}\cdot\frac{\sigma^2}{\chi}$$

.

$$\frac{\sigma\kappa}{S\ell_s^2} = \left(\frac{\langle\theta s\rangle}{S\sigma}\right)\times\left(\frac{\ell_d^2}{\ell_s^2}\right)$$

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Two ways to reduce σ: (1) Minimise $\langle\theta s\rangle/S\sigma \Rightarrow$ **transport**
(2) Minimise $\ell_d^2/\ell_s^2 \Rightarrow$ **stirring**

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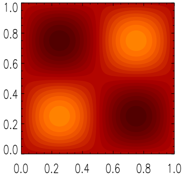
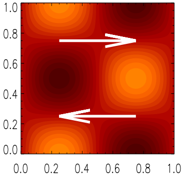
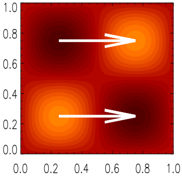
Diffusion, Stirring, Transport, and expectations







Diffusion	Stirring	Transport
$\sigma \propto \frac{S\ell_s^2}{\kappa}$	$\sigma \propto \frac{S\ell_u}{U}$	$\sigma \propto \frac{S\ell_s}{U}$
$\langle \theta s \rangle \propto S\sigma$	$\langle \theta s \rangle \propto S\sigma$	$\langle \theta s \rangle \propto S\sigma \mathrm{Pe}^{-1}\rho$
$\ell_d^2 \propto \ell_s^2$	$\ell_d^2 \propto \mathrm{Pe}^{-1}\ell_u^2$	$\ell_d^2 \propto \ell_s^2$
$k_d^2\ell_B^2 \propto \rho^2\mathrm{Pe}$	$k_d^2\ell_B^2 \propto 1$	$k_d^2\ell_B^2 \propto \rho^2\mathrm{Pe}$

Bounding the dissipation scale (1)

What are the bounds for $k_d^2 \ell_B^2$?

Bounding the dissipation scale (1)

Using the Thiffeault, Doering & Gibbon 2004 bound:

$$
\begin{aligned}
k_d^2 \ell_B^2 &= \frac{\langle |\nabla \theta|^2 \rangle}{\sigma^2} \; \frac{\kappa \ell_u}{U} \\
&= \frac{\langle \theta s \rangle}{\kappa \sigma^2} \; \frac{\kappa \ell_u}{U} \\
&\leqslant \frac{S}{\kappa \sigma} \; \frac{\kappa \ell_u}{U} \\
&\leqslant \frac{\ell_u}{\ell_s} (c_1 + \mathrm{Pe}^{-1} \rho c_2)
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\end{aligned}
$$

Upper bound I

$$k_d^2 \ell_B^2 \leqslant \rho (c_1 + \mathrm{Pe}^{-1} \rho c_2)$$

Bounding the dissipation scale (2)

New constraint obtained from $\langle|\nabla\theta|^2\rangle$ evolution equation:

$$\partial_t \nabla\theta + \mathbf{u}\cdot\nabla(\nabla\theta) = \kappa\Delta\nabla\theta - (\nabla\mathbf{u})^{\dagger}\cdot\nabla\theta + \nabla s.$$

$$\eta = \kappa\,\langle|\Delta\theta|^2\rangle = -\langle\nabla\theta(\nabla\mathbf{u})^{\mathrm{sym}}\nabla\theta\rangle + \langle\nabla\theta\cdot\nabla s\rangle.$$

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Using Hölder, Cauchy-Schwartz:

$$\eta \leqslant sup|(\nabla\mathbf{u})^{\mathrm{sym}}|\,\chi/\kappa + \frac{S}{\ell_s}\sqrt{\chi/\kappa}$$

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Using $\chi^2 \leqslant \kappa^2\langle\theta^2\rangle\langle|\Delta\theta|^2\rangle = \kappa\sigma^2\eta$:

Upper bound II

$$k_d^2\ell_B^2 \leqslant \frac{1}{2}c_3\left\{1 + \sqrt{1 + 4\rho^3\mathrm{Pe}^{-1}(c_1c_2 + \rho c_2^2\mathrm{Pe}^{-1})c_3^{-2}}\right\}$$

Bounding the dissipation scale (3)

For monochromatic sources $\Delta s = -\ell_s^{-2} s$ it follows that $|\langle \nabla\theta \cdot \nabla s \rangle| = c_2 k_s^2 \langle \theta s \rangle = k_s^2 \chi$:

$$\kappa \langle |\Delta\theta|^2 \rangle = -\langle \nabla\theta (\nabla \mathbf{u})^{\mathrm{sym}} \nabla\theta \rangle + k_s^2 \chi .$$

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$$\eta \leqslant \left(c_3 \frac{U}{\ell_u \kappa} + c_2 \frac{1}{\ell_s^2} \right) \chi.$$

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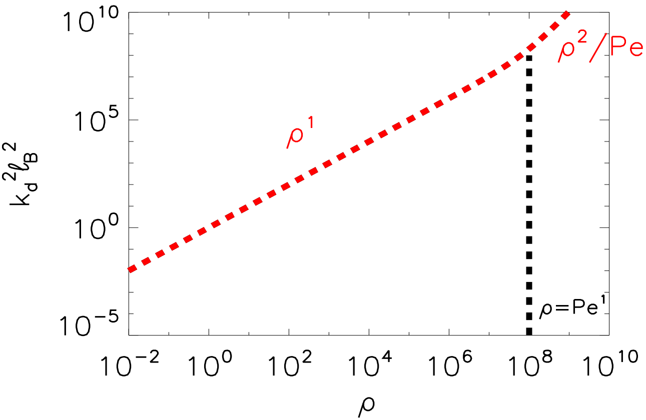
and using $\chi^2 \leqslant \kappa \sigma^2 \eta$

Upper Bound III

$$k_d^2 \ell_B^2 \leqslant c_3 + c_2 \rho^2 \mathrm{Pe}^{-1}.$$

Upper bounds on $k_d^2\ell_B^2$



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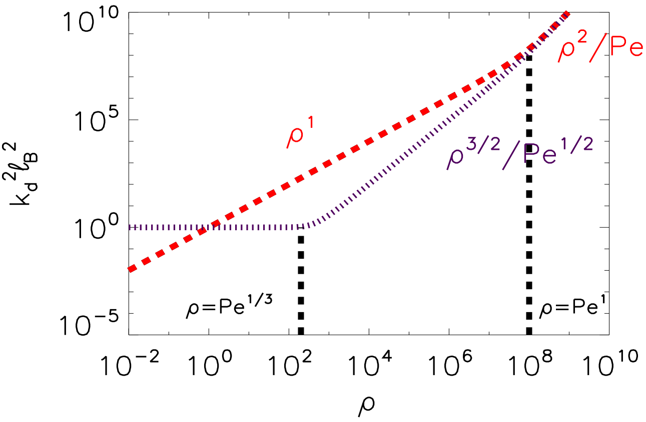

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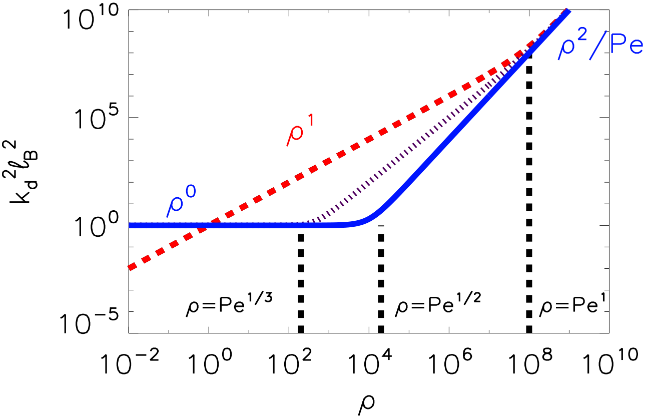

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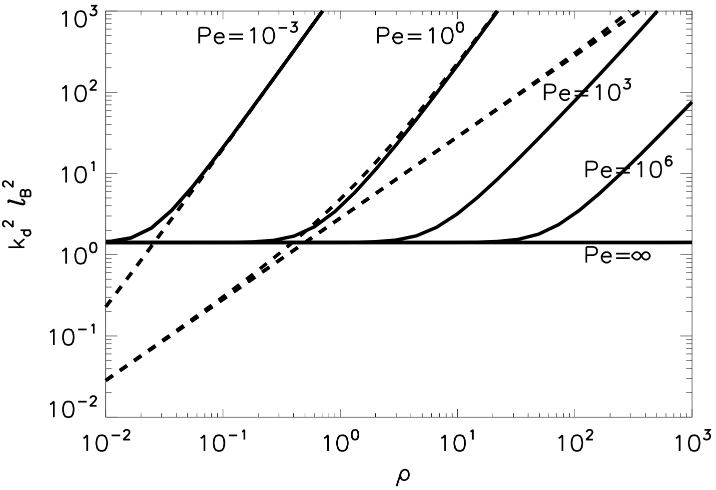

The $\rho \ll 1$ limit: homogenization theory

$$\partial_t \bar{\theta} = \nabla \cdot \mathbf{K} \cdot \nabla \bar{\theta} + s, \quad \rho \ll 1$$

where $\mathbf{K} = \kappa(\mathbf{I} + \mathbf{K}_T)$.

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$\times \bar{\theta}$ and take $\langle \cdot \rangle$

$$k_d^2 = \frac{\chi}{\kappa \sigma^2} \approx \frac{\langle \nabla \bar{\theta} (\mathbf{I} + \mathbf{K}_T) \nabla \bar{\theta} \rangle}{\langle \bar{\theta}^2 \rangle}.$$

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Now $||\mathbf{K}_T|| \sim \mathrm{Pe}^{\alpha}$ where $0 < \alpha \leqslant 2$.

cellular $\alpha = 1/2$, chaotic $\alpha = 1$, shear $\alpha = 2$.

MAJDA & KRAMER 1999, KRAMER & KEATING 2009

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Majda & Kramer 1999, Kramer & Keating 2009

Homogenization result

$$k_d^2 \ell_B^2 \sim \rho^2 \mathrm{Pe}^{\alpha - 1}, \quad \rho \ll \min\{1, \mathrm{Pe}^{1-\alpha}\}.$$

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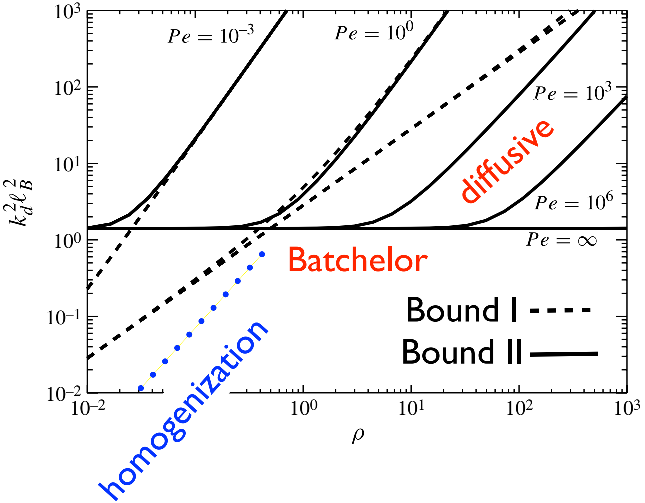
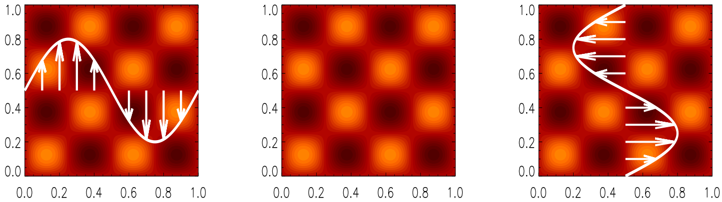


An example flow and source



$$\mathbf{u}(\mathbf{x}/\ell_u, t) = \begin{bmatrix} \Theta(\tau/2 - t \bmod \tau)\,\sqrt{2}U\sin(y/\ell_u + \phi_1) \\ \Theta(t \bmod \tau - \tau/2)\,\sqrt{2}U\sin(x/\ell_u + \phi_2) \end{bmatrix},$$

PIERREHUMBERT 1994.

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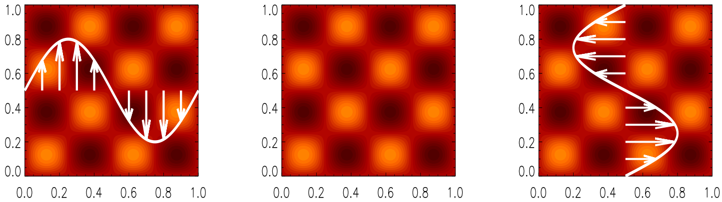

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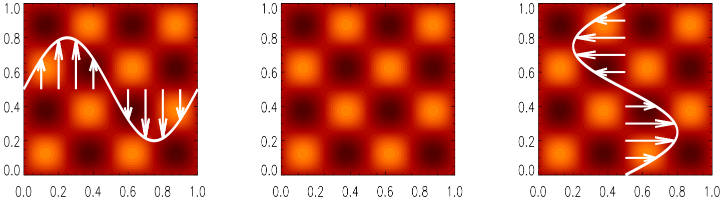
PIERREHUMBERT 1994.

$$s(\mathbf{x}/\ell_s) = 2S\sin(x/\ell_s)\sin(y/\ell_s).$$



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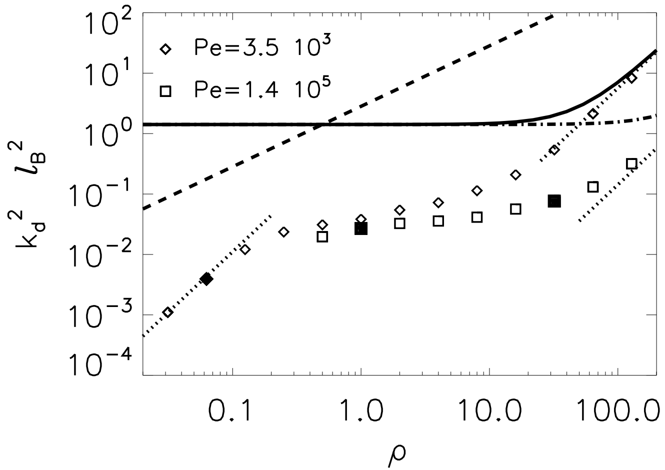
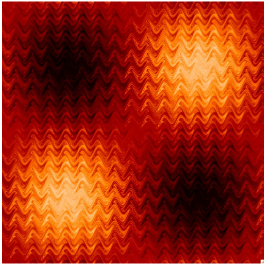
$$\mathbf{K}_T = \frac{U^2\tau}{8\kappa}\mathbf{I} = \frac{\pi \mathrm{Pe}}{4\sqrt{2}\,St}\mathbf{I}, \quad c_1 = 2\sqrt{2}, \quad c_2 = 2, \quad c_3 = \sqrt{2}.$$

Numerical Results

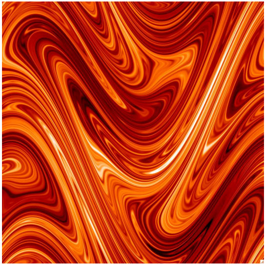


Spatial structures



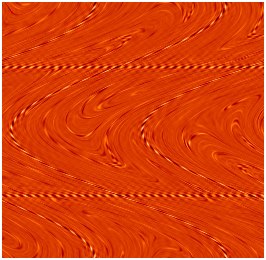
$\rho = 1/16$

$\mathrm{Pe} = 3.5 \times 10^3$



$\rho = 1$

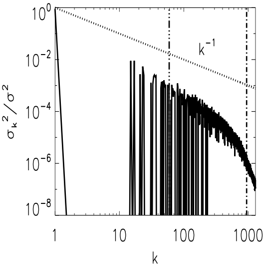
$\mathrm{Pe} = 1.4 \times 10^5$



$\rho = 32$

$\mathrm{Pe} = 1.4 \times 10^5$

Spectra



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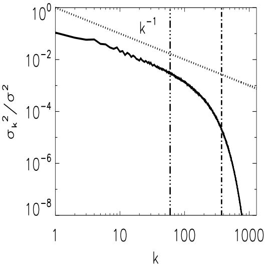

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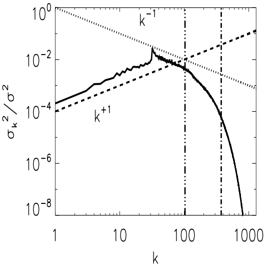

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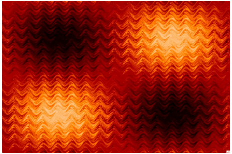
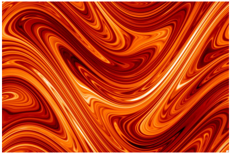
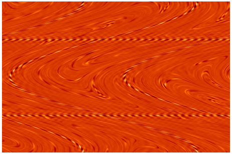
Movies



$\rho \ll 1$

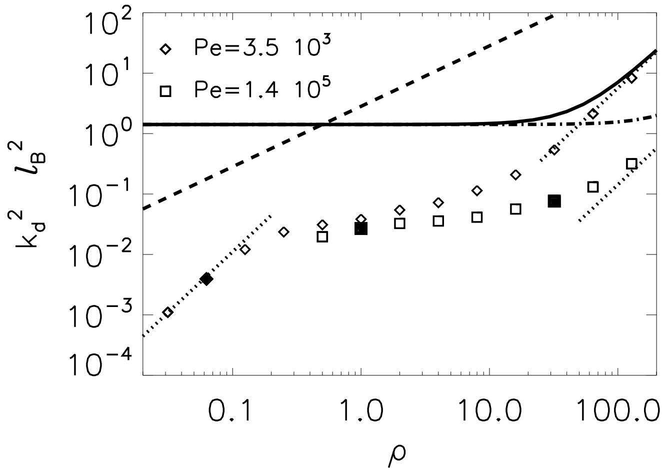


$\rho \sim 1$

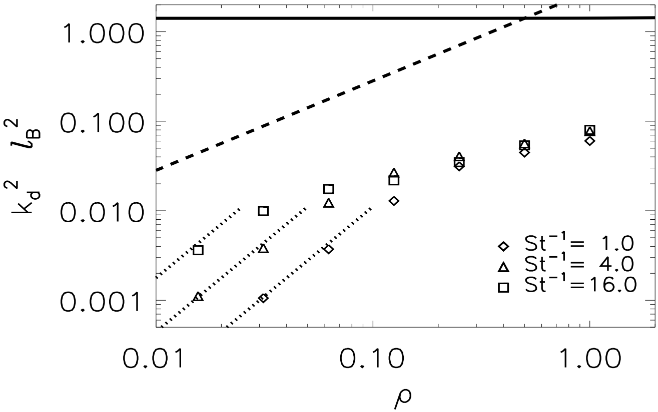


$\rho \gg 1$

Is there something more?



Is there something more?



$$St_u \equiv \frac{\ell_u}{U\tau_u}$$

$$St = 1 \quad k_d^2\ell_B^2 \sim \rho^2 \quad \rho \ll 1$$

$$St = 0 \quad k_d^2\ell_B^2 \sim \rho^2 \mathrm{Pe}^1 \quad \rho \ll \mathrm{Pe}^{-1}$$

Summary

Variance is controlled by both transport and stirring

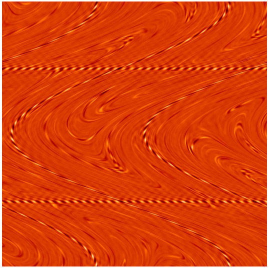
Different mechanisms imply different scalings

Bounds & homogenization describe well the dependence of k_d with system parameters

Some thoughts...

In multiscale (turbulent) flows different scales control mixing & different scales control transport

The pattern in $\rho \gg 1$ suggest that higher moments would be interesting to look at



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Thank you