The 2D Magnetohydrodynamic Equations with Partial Dissipation

Jiahong Wu

Oklahoma State University

IPAM Workshop "Mathematical Analysis of Turbulence"
IPAM, UCLA, September 29-October 3, 2014

Outline I

Outline II

Introduction

The standard 2D incompressible MHD equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1}$$

where u denotes the velocity field, b the magnetic field, p the pressure, $\nu \geq 0$ the viscosity and $\eta \geq 0$ the magnetic diffusivity (resistivity).

The MHD equations model electrically conducting fluid in the presence of a magnetic field. The first equation is the Navier-Stokes equation with the Lorentz force generated by the magnetic field and the second equation is the induction equation for the magnetic field.

The MHD equations model many phenomena in physics, especially, in geophysics and astrophysics. The MHD equations have been studied analytically, numerically and experimentally. Many books and papers have been written on them.

Mathematically the 2D MHD equations may serve as a lower-dimensional model of the 3D hydrodynamics equations. They are naturally the next level of equations to study after the 2D Boussinesq equations. Due to the strong nonlinear coupling, it can be extremely challenging to deal with some of mathematical issues even in the 2D case.

We will focus on the 2D MHD equations with partial or fractional dissipation. For this purpose, we write the MHD equations in more general forms. The first one is the anisotropic MHD equations,

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu_1 u_{xx} + \nu_2 u_{yy} + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta_1 b_{xx} + \eta_2 b_{yy} + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \end{cases} \tag{2}$$

where $\nu_1 \geq 0, \nu_2 \geq 0, \eta_1 \geq 0$ and $\eta_2 \geq 0$. (2) will be called anisotropic MHD equations. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (2) becomes the standard MHD equations.

Another generalized form is the MHD equations with fractional dissipation, replacing the Laplacian by fractional Laplacians, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^\alpha u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = -\eta(-\Delta)^\beta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{3}$$

where $0 < \alpha, \beta \leq 1$, and the fractional Laplacian can be defined by the Fourier transform (or through Riesz potential),

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

This system will be called fractional MHD equations.

We consider the initial-value problems of the MHD equations with the initial data

$$u(x,0) = u_0(x), \qquad b(x,0) = b_0(x).$$

What we care about is the global regularity issue: Do these IVPs have a global solution for sufficiently smooth data (u_0, b_0)?

The global regularity problem on the 2D MHD equations has attracted considerable attention recently from the PDE community and progress has been made.

Consider the following seven cases:

- $\nu_1 = \nu_2 = \eta_1 = \eta_2 = 0$, ideal MHD
- $\nu_1 > 0$, $\nu_2 > 0$, $\eta_1 > 0$ and $\eta_2 > 0$,

  MHD with dissipation and magnetic diffusion
- $\nu_1 = \nu_2 > 0$, $\eta_1 = \eta_2 = 0$.

  dissipation but no magnetic diffusion
- $\eta_1 > 0$, $\eta_2 > 0$, $\nu_1 = \nu_2 = 0$.

  magnetic diffusion but no dissipation

- $\nu_1 > 0$ and $\eta_2 > 0$

  horizontal dissipation and vertical magnetic diffusion

  $\nu_2 > 0$ and $\eta_1 > 0$

  vertical dissipation and horizontal magnetic diffusion

- $\nu_1 > 0$ and $\eta_1 > 0$

  horizontal dissipation and horizontal magnetic diffusion

One general idea for proving the global (in time) existence and uniqueness. This is divided into two steps:

1) Local existence and uniqueness. This is in general done by the contraction mapping principle for $f(t) = G(f(t)) \equiv f(0) + \int_0^t N(f(\tau))\, d\tau$. This usually requires that the time interval is small.

2) Global bounds and the extension theorem. The nonlinear term is treated as bad part and the dissipation as good part.

• Ideal MHD equations

Ideal MHD equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \quad b(x,y,0) = b_0(x,y). \end{cases} \tag{4}$$

The global regularity problem remains open, although we do have local well-posedness and regularity criteria.

Theorem

Given $(u_0, b_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$. *Then there exists a unique local classical solution* $(u, b) \in C([0, T_0); H^s)$ *for some* $T_0 > 0$. *In addition, if*

$$\int_0^T (\|\omega\|_\infty + \|j\|_\infty)\, dt < \infty$$

for $T > T_0$, *then the solution remains in* H^s *for any* $t \leq T$.

The L^∞*-norm can replaced by BMO or* $B^0_{\infty,\infty}$.

Why is the global regularity problem hard? The global L^2-bound for (u, b) follows directly from the MHD equations

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

But global bounds for any Sobolev-norm appear to be impossible, for example, the H^1-norm. Consider the equations of $\omega = \nabla \times u$ and $j = \nabla \times b$,

$$\begin{cases} \omega_t + u \cdot \nabla \omega = b \cdot \nabla j, \\ j_t + u \cdot \nabla j = b \cdot \nabla \omega + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2) \end{cases}$$

Clearly,

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) = 2\int j\,\partial_x b_1 \partial_y u_1 + \cdots$$

Since there is no dissipation, we need

$$\int_0^T \|\omega\|_\infty \, dt < \infty \quad \text{or} \quad \int_0^T \|j\|_\infty \, dt < \infty$$

in order for this differential inequality to be closed. To bound Sobolev norms with more than one derivative, we need both.

In fact, for $Y = \|u(t)\|_{H^s}^2 + \|b\|_{H^s}^2$,

$$\frac{d}{dt} Y(t) \le C \left(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty} \right) Y(t).$$

Then one uses the logarithmic inequality,

$$\|\nabla u\|_{L^\infty} \le C \left(1 + \|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log^+ \|u\|_{W^{s+1,p}}\right), \quad p \in (1,\infty), \; s > d/p$$

Then

$$\frac{d}{dt} Y(t) \le C \left(\|\omega\|_{L^\infty} + \|j\|_{L^\infty} \right) Y(t) \log^+ Y(t).$$

To get a stronger regularity criterion, one uses BMO or $B^0_{\infty,\infty}$,

$$\|f\|_{L^\infty} \le C(1 + \|f\|_{BMO} \log^+ \|f\|_{W^{s,p}}), \quad p \in (1,\infty), \; s > d/p.$$

• Dissipative MHD equations

Fully dissipative MHD equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \end{cases}$$

The global regularity can be easily established.

Theorem

Let $(u_0, b_0) \in H^1(\mathbb{R}^2)$. Then there exists a unique global strong solution (u, b) satisfying, for any $T > 0$,

$$u,\ b \in L^\infty([0, T; H^1(\mathbb{R}^2)) \cap L^2([0, T]; H^2(\mathbb{R}^2))$$

The global regularity problem for the ideal MHD equations is extremely difficult while the problem for the fully dissipative MHD equations is very easy. Naturally we would like to consider the cases with intermediate dissipation.

• Dissipation only

The 2D MHD equations with no magnetic diffusion:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases}$$

where $\nu > 0$. The global regularity problem remains open.

The dissipation is NOT enough for global bounds in Sobolev spaces.

Again we have the global L^2-bound

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

To consider the H^1-norm, we use the equations for $\omega = \nabla \times u$ and $j = \nabla \times b$,

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \nu \Delta \omega + b \cdot \nabla j, \\ j_t + u \cdot \nabla j = b \cdot \nabla \omega + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2). \end{cases}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \nu\|\nabla\omega\|_{L^2}^2 = 2\int j\,\partial_x b_1 \partial_y u_1 + \cdots$$

To close the inequality, we need

$$\int_0^T \|\nabla u\|_{L^\infty}\, dt < \infty \quad \text{or} \quad \int_0^T \|\omega\|_\infty\, dt < \infty.$$

Even the global existence of weak solutions is unknown. The standard idea does not appear to work:

1) Mollify the equations and the data to obtain a global smooth solution (u^N, b^N);

2) Obtain uniform bounds, for any fixed $T > 0$, $s > 2$,

$$u^N \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \quad \partial_t u^N \in L^\infty(0,T;H^{-s}),$$
$$b^N \in L^\infty(0,T;L^2), \quad \partial_t b^N \in L^\infty(0,T;H^{-s});$$

3) Apply the local version of the Aubin-Lions Lemma

$$u^N \to u \quad \text{in} \quad L^2(0, T; L^2_{loc}),$$

$$b^N \to b \quad \text{in} \quad L^2(0, T; H^{-\delta}_{loc}) \ (\delta > 2)$$

The trouble is that this does not allow us to pass to the limit in

$$b^N \cdot \nabla b^N \ \to \ b \cdot \nabla b$$

in the distributional sense.

• Global solutions near equilibrium

Some very recent efforts are devoted to global solutions near an equilibrium. Progress has been made:

- F. Lin, L. Xu, and P. Zhang, Global small solutions to 2-D incompressible MHD system, arXiv:1302.5877v2 [math.AP] 4 Jun 2013.
- X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, *J. Functional Anal.* **267** (2014), 503-541.

- J. Wu, Y. Wu and X. Xu, Global small solution to the 2D MHD system with a velocity damping term, arXiv:1311.6185v1 [math.AP] 24 Nov 2013.
- T. Zhang, An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, arXiv:1404.5681v1 [math.AP] 23 Apr 2014.
- X. Hu and F. Lin, Global Existence for Two Dimensional Incompressible Magnetohydrodynamic Flows with Zero Magnetic Diffusivity, arXiv: 1405.0082v1 [math.AP] 1 May 2014.

Why near an equilibrium?

Mathematically, the lack of magnetic diffusion makes it extremely difficult to obtain global solutions, even small global solutions.

Rewriting the equations near equilibrium generates favorable terms.

Since $\nabla \cdot b = 0$, write $b = \nabla^\perp \phi = (-\partial_y, \partial_x)\phi$ and

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + \nabla^\perp \phi \cdot \nabla \nabla^\perp \phi, \\ \phi_t + u \cdot \nabla \phi = 0, \\ \nabla \cdot u = 0. \end{cases}$$

Clearly, $(u, \phi) = (0, y)$ is a steady solution.

Setting $\phi = y + \psi$ yields

$$\begin{cases} \partial_t \psi + u \cdot \nabla \psi + u_2 = 0, \\ \partial_t u_1 + u \cdot \nabla u_1 - \nu \Delta u_1 + \partial_1 \partial_2 \psi = -\partial_1 p - \nabla \cdot (\partial_1 \psi \nabla \psi), \\ \partial_t u_2 + u \cdot \nabla u_2 - \nu \Delta u_2 + \partial_1^2 \psi = -\partial_2 p - \nabla \cdot (\partial_2 \psi \nabla \psi). \end{cases} \tag{5}$$

The aim is to look for global small solutions of this system.

The work of Lin, Xu and Zhang reformulated the system in Lagrangian coordinates. More precisely, they define

$$Y(x,y,t) = X(x,y,t) - (x,y),$$

where $X = X(x,y,t)$ be the particle trajectory determined by u. Y satisfies

$$Y_{tt} - \Delta Y_t - \partial_x^2 Y = f(Y, q(y)), \quad q = p + |\nabla\psi|^2.$$

They then estimate the Lagrangian velocity Y_t in $L^1_t Lip_x$, using anisotropic Littlewood-Paley theory and anisotropic Besov space techniques.

Due to their use of the Lagrangian coordinates, they need to impose a compatibility condition on the initial data ψ_0, more precisely, $\partial_y \psi_0$ and $(1 + \partial_y \psi_0, -\partial_x \psi_0)$ are admissible on $0 \times \mathbb{R}$ and $\mathrm{supp} \partial_y \psi_0(\cdot, y) \subset [-K, K]$ for some K.

$\partial_y \psi_0$ and $(1 + \partial_y \psi_0, -\partial_x \psi_0)$ are admissible on $0 \times \mathbb{R}$ if

$$\int_{\mathbb{R}} \partial_y \psi_0(X(a,t))dt = 0 \quad \text{for all } a \in 0 \times \mathbb{R}.$$

where X is the particle trajectory defined by $(1 + \partial_y \psi_0, -\partial_x \psi_0)$.

Theorem

Given u_0 and ψ_0 satisfying $(u_0, \nabla\psi_0) \in H^s \cap \dot{H}^{s_2}$ with $s_1 > 1$, $s_2 \in (-1, -\frac{1}{2})$ and $s > s_1 + 2$, and

$$\|\nabla\psi_0\|_{H^{s_1+2}} \leq 1, \quad \|(\nabla\psi_0, u_0)\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} + \|\partial_y\psi_0\|_{H^{s_1+2}} \leq \epsilon_0$$

for some ϵ_0 small. Assume that $\partial_y\psi_0$ and $(1 + \partial_y\psi_0, -\partial_x\psi_0)$ are admissible on $0 \times \mathbb{R}$ and $\partial_y\psi_0(\cdot, y) \subset [-K, K]$ for some K. Then the 2D MHD equations with no magnetic diffusion has a unique global solution (ψ, u, p).

Work of X. Ren, J. Wu, Z. Xiang and Z. Zhang

X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, *J. Functional Anal.* **267** (2014), 503-541.

The aim here is twofold: 1) to do direct energy estimates without Lagrangian coordinates and remove the compatibility assumption; 2) to confirm the numerical observation that the energy of the MHD equations is dissipated at a rate as that for the linearized equations.

Definition

Let $\sigma, s \in \mathbb{R}$. The anisotropic Sobolev space $\mathring{H}^{\sigma,s}(\mathbb{R}^2)$ is defined by

$$\mathring{H}^{\sigma,s}(\mathbb{R}^2) = \Big\{ f \in S'(\mathbb{R}^2) : \|f\|_{\mathring{H}^{\sigma,s}} < +\infty \Big\},$$

where

$$\|f\|_{\mathring{H}^{\sigma,s}} = \Big\| \big\{ 2^{js} 2^{\sigma k} \|\Delta_j \Delta_k^h f\|_{L^2} \big\}_{j,k} \Big\|_{\ell^2}.$$

or

$$\|f\|_{\mathring{H}^{\sigma,s}} = \left[\int_{\mathbb{R}^2} |\xi|^{2s} \, \xi_1^{2\sigma} |\widehat{f}(\xi)|^2 \, d\xi \right]^{\frac{1}{2}}.$$

Theorem

Assume $(\nabla\psi_0, u_0) \in H^8(\mathbb{R}^2)$. *Let* $s \in (0, \frac{1}{2})$. *There exists a small positive constant* ε *such that, if,* $(\nabla\psi_0, u_0) \in \mathring{H}^{-s,-s} \cap \mathring{H}^{-s,8}(\mathbb{R}^2)$, *and*

$$\|(\nabla\psi_0, u_0)\|_{H^8} + \|(\nabla\psi_0, u_0)\|_{\mathring{H}^{-s,-s}} + \|(\nabla\psi_0, u_0)\|_{\mathring{H}^{-s,8}} \le \varepsilon,$$

then (5) has a unique global solution (ψ, u) *satisfying*

$$(\nabla\psi, u) \in C([0, +\infty); H^8(\mathbb{R}^2)).$$

Theorem

Moreover, the solution decays at the same rate as that for the linearized solutions,

$$\|\partial_x^k \nabla\psi\|_{L^2} + \|\partial_x^k u\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{s+k}{2}},$$

for any $t \in [0, +\infty)$ *and* $k = 0, 1, 2.$

Ideas in the proof: First, we consider the linearized equation

Proposition

Consider the linearized equation

$$\begin{cases} \partial_t u_1 - \Delta u_1 - \partial_{x_1 x_2}\psi = 0, \\ \partial_t u_2 - \Delta u_2 + \partial_{x_1 x_1}\psi = 0, \\ \partial_t \psi + u_2 = 0, \\ u(x,0) = u_0(x), \quad \psi(x,0) = \psi_0(x). \end{cases}$$

Assume $(u_0, \nabla\psi_0) \in H^4$ *and* $|D_1|^{-s} u_0 \in H^{1+s}$ *and* $|D_1|^{-s}\nabla\psi_0 \in H^{1+s}$ *for* $s > 0$, *then, for* $k = 0, 1, 2$,

$$\|\partial_{x_1}^k u\|_{L^2} + \|\partial_{x_1}^k \nabla\psi\|_{L^2} \le C(1+t)^{-\frac{k+s}{2}}.$$

Proof. For $\varepsilon_1 > 0$, define

$$
\begin{aligned}
D_0(t) =& \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla^2\psi\|_{L^2}^2 + 2\varepsilon_1\langle u_2, \nabla\psi\rangle, \\
H_0(t) =& \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \varepsilon_1\|\nabla\partial_1\psi\|_{L^2}^2 - \varepsilon_1\|\nabla u_2\|_{L^2}^2 - \varepsilon_1\langle\Delta u_2, \Delta\psi\rangle. \\
E_s(t) =& \||D_1|^{-s}u\|_{L^2}^2 + \||D_1|^{-s}\nabla\psi\|_{L^2}^2 \\
& + \||D|^{1+s}|D_1|^{-s}u\|_{L^2}^2 + \||D|^{1+s}|D_1|^{-s}\nabla\psi\|_{L^2}^2.
\end{aligned}
$$

We can show

$$
\frac{d}{dt}D_0(t) + C\,H_0(t) \le 0, \qquad \frac{d}{dt}E_s(t) \le 0.
$$

By interpolation inequalities,

$$D_0(t) \leq E_s(t)^{\frac{1}{1+s}} H_0(t)^{\frac{s}{1+s}}, \quad H_0(t) \geq E_s(0)^{-\frac{1}{s}} D_0(t)^{1+\frac{1}{s}}.$$

Thus,

$$\frac{d}{dt} D_0(t) + C\, E_s(0)^{-\frac{1}{s}} D_0(t)^{1+\frac{1}{s}} \leq 0.$$

$$E(t) \leq (E(0)^{-\frac{1}{s}} + C(s)t)^{-s} = E_0 \left(E_0^{\frac{1}{s}} C(s) t + 1 \right)^{-s}.$$

We return to the full nonlinear system

$$\begin{cases} \partial_t \psi + u \cdot \nabla \psi + u_2 = 0, \\ \partial_t u_1 + u \cdot \nabla u_1 - \nu \Delta u_1 + \partial_1 \partial_2 \psi = -\partial_1 p - \nabla \cdot (\partial_1 \psi \nabla \psi), \\ \partial_t u_2 + u \cdot \nabla u_2 - \nu \Delta u_2 + \partial_1^2 \psi = -\partial_2 p - \nabla \cdot (\partial_2 \psi \nabla \psi). \end{cases} \tag{6}$$

The frame work to prove the global existence of small solutions is the Bootstrap Principle.

T. Tao, Local and global analysis of dispersive and wave equations, p.21.

Lemma (Abstract Bootstrap Principle)

Let I be an interval. Let $C(t)$and $H(t)$ be two statements related to $t \in I$. If $C(t)$ and $H(t)$satisfy

(a) *If $H(t)$ is true, then $C(t)$ is true for the same t,*

(b) *If $C(t_1)$ is true, then $H(t)$ is true for t in a neighborhood of t_1,*

(c) *If $C(t_k)$ is true for a sequence $t_k \to t$, then $C(t)$is true,*

(d) *$C(t)$ is true for at least one $t_0 \in I$,*

then, $C(t)$ is true for all $t \in I$.

What we do here is to:

1) obtain decay rates under the assumption that the solution is small;

2) show that the solution is even smaller if the initial data is small.

Then the Bootstrap principle would imply that the solution remain small for all time.

We use anisotropic Sobolev and Besov spaces due to the anisotropicity,

$$u_{tt} - \Delta u_t - \partial_x^2 u = 0$$

The characteristic equation satisfies

$$\lambda^2 + |\xi|^2 \lambda + \xi_1^2 = 0,$$

which has two roots

$$\lambda_\pm = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_1^2}}{2}.$$

As $|\xi| \to \infty$,

$$\lambda_-(\xi) \to -\frac{\xi_1^2}{|\xi|^2} \sim \begin{cases} -1, & |\xi| \sim |\xi_1|, \\ 0, & |\xi| \gg |\xi_1| \end{cases}$$

The dissipation is weak in the case of $|\xi| \gg |\xi_1|$.

Proposition

If $(u, \nabla\psi)$ *satisfies*

$$\|(u(t), \nabla\psi(t))\|_{H^4} \leq \delta$$

for some $\delta > 0$ *and for* $t \in [0, T]$, *then we can show*

$$\frac{d}{dt} D_0 + C H_0 \leq 0 \quad \text{for } t \in [0, T].$$

Define For $l = 1, 2$, we define

$$
\begin{aligned}
D_l(t) &= \sum_{j,k} 2^{2lk}\big(\|\Delta_j\Delta_k^h u\|_{L^2}^2 + \|\Delta_j\Delta_k^h \nabla u\|_{L^2}^2 + \|\Delta_j\Delta_k^h \nabla\psi\|_{L^2}^2 \\
&\quad + \|\Delta_j\Delta_k^h \nabla^2\psi\|_{L^2}^2 + 2\varepsilon_1\langle \Delta_j\Delta_k^h u_2, \Delta_j\Delta_k^h \Delta\psi\rangle\big),
\end{aligned}
$$

$$
\begin{aligned}
H_l(t) &= \sum_{j,k} 2^{2lk}\big(\|\Delta_j\Delta_k^h \nabla u\|_{L^2}^2 + \|\Delta_j\Delta_k^h \nabla^2 u\|_{L^2}^2 + \varepsilon_1\|\Delta_j\Delta_k^h \nabla\partial_1\psi\|_{L^2}^2 \\
&\quad - \|\Delta_j\Delta_k^h \nabla u_2\|_{L^2}^2 - \varepsilon_1\langle \Delta_j\Delta_k^h \Delta u_2, \Delta_j\Delta_k^h \Delta\psi\rangle\big).
\end{aligned}
$$

Proposition

Let $e(t) = \|(u, \nabla\psi)\|_{H^8 \cap H^{-s,-s} \cap H^{-s,8}}$. *If*

$$\sup_{t\in[0,T]} e(t) \le \delta$$

for some sufficiently small δ, *then*

$$\frac{d}{dt} D_l(t) + C H_l(t) \le 0, \qquad t \in [0, T].$$

We further define

$$E_{s,s_1} = \|(u, \nabla\psi)\|^2_{\dot{H}^{-s,s_1}} + \|(u, \nabla\psi)\|^2_{\dot{H}^{-s,s_1+1}},$$

$$\varepsilon_{s,k}(t) = E_{s,0}(t) + E_{s,s+k}(t).$$

Proposition

Assume, for $k = 0, 1, 2,$

$$\sup_{t\in[0,T]} e(t) \leq \delta, \qquad \sup_{t\in[0,T]} \varepsilon_{s,k}(t) \leq C\epsilon^2,$$

then

$$\|\partial^l_{x_1}(u, \nabla\psi)\|_{L^2} + \|\partial^l_{x_1}(\nabla u, \nabla^2\psi)\|_{L^2} \leq C(1+t)^{-\frac{l+s}{2}}.$$

Proposition

If

$$e(0) = \|(u_0, \nabla\psi_0)\|_{H^8 \cap \dot{H}^{-s,8} \cap \dot{H}^{-s,-s}} \leq r_0,$$

then $(u, \nabla\psi)$ *satisfies*

$$e(t) = \|(u, \nabla\psi)\|_{H^8 \cap \dot{H}^{-s,8} \cap \dot{H}^{-s,-s}} \leq 2r_0$$

We can choose r_0 to be sufficiently small so that $2r_0 < \delta$.

• Global small solutions for a damped system

J. Wu, Yifei Wu and Xiaojing Xu, Global small solution to the 2D MHD system with a velocity damping term, arXiv: 1311.6185 [math.AP] 24 Nov 2013.

Consider the following 2D MHD equation

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \vec{u} + \nabla P = -\mathrm{div}(\nabla\phi\otimes\nabla\phi), \quad (t,\, x,\, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t\phi + \vec{u}\cdot\nabla\phi = 0, \\ \nabla\cdot\vec{u} = 0, \\ \vec{u}|_{t=1} = \vec{u}_0(x,\, y), \quad \phi|_{t=1} = \phi_0(x,\, y), \end{cases} \tag{7}$$

where $\vec{u} = (u,\, v)$.

Letting $\phi = y + \psi$ in (6) yields

$$\begin{cases} \partial_t u + u\,\partial_x u + v\partial_y u + u + \partial_x \tilde{P} = -\Delta\psi\partial_x\psi, \\ \partial_t v + u\,\partial_x v + v\partial_y v + v + \partial_y \tilde{P} = -\Delta\psi - \Delta\psi\partial_y\psi, \\ \partial_t \psi + u\partial_x\psi + v\partial_y\psi + v = 0, \\ \partial_x u + \partial_y v = 0, \end{cases} \tag{8}$$

where $\tilde{P} = P + \frac{1}{2}|\nabla\phi|^2$. By $\nabla \cdot \vec{u} = 0$,

$$\Delta\widetilde{P} = -\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) - \nabla\cdot(\Delta\psi\nabla\psi) - \Delta\partial_y\psi.$$

Therefore, (32) can be written as

$$\partial_t u + u - \partial_{xy}\psi = N_1, \tag{9}$$

$$\partial_t v + v + \partial_{xx}\psi = N_2, \tag{10}$$

$$\partial_t \psi + v = -u\partial_x\psi - v\partial_y\psi, \tag{11}$$

where

$$N_1 = -\vec{u}\cdot\nabla u + \partial_x\Delta^{-1}\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) - \Delta\psi\,\partial_x\psi + \partial_x\Delta^{-1}\nabla\cdot(\Delta\psi\,\nabla\psi),$$

$$N_2 = -\vec{u}\cdot\nabla v + \partial_y\Delta^{-1}\nabla\cdot(\vec{u}\cdot\nabla\vec{u}) - \Delta\psi\,\partial_y\psi + \partial_y\Delta^{-1}\nabla\cdot(\Delta\psi\,\nabla\psi).$$

Taking the time derivative leads to

$$\begin{cases} \partial_{tt} u + \partial_t u - \partial_{xx} u = F_1, \\ \partial_{tt} v + \partial_t v - \partial_{xx} v = F_2, \\ \partial_{tt}\psi + \partial_t\psi - \partial_{xx}\psi = F_0, \\ \vec{u}|_{t=1} = \vec{u}_0(x,\, y), \quad \vec{u}_t|_{t=1} = \vec{u}_1(x,\, y) \\ \psi|_{t=1} = \psi_0(x,\, y), \quad \psi_t|_{t=1} = \psi_1(x,\, y), \end{cases} \tag{12}$$

where $\vec{u}_1 = (u_1(x,\, y),\, v_1(x,\, y)),\ \psi_0 = \phi_0 - y$, and

$$u_1 = (-u + \partial_{xy}\psi + N_1)|_{t=1},$$

$$v_1 = (-v - \partial_{xx}\psi + N_2)|_{t=1},$$

$$\psi_1 = (-u\partial_x\psi - v\partial_y\psi - v)|_{t=1},$$

and

$$F_0 = -\vec{u}\cdot\nabla\psi - \partial_t(\vec{u}\cdot\nabla\psi) - N_2,$$

$$F_1 = \partial_t N_1 - \partial_{xy}(\vec{u}\cdot\nabla\psi),$$

$$F_2 = \partial_t N_2 + \partial_{xx}(\vec{u}\cdot\nabla\psi).$$

We consider the linear equation

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = 0, \tag{13}$$

with the initial data

$$\Phi(0,x,y) = \Phi_0(x,y),\ \ \Phi_t(0,x,y) = \Phi_1(x,y).$$

Taking the Fourier transform on the equation (13), we have

$$\partial_{tt}\hat{\Phi} + \partial_t\hat{\Phi} + \xi^2\hat{\Phi} = 0, \tag{14}$$

where the Fourier transform $\hat{\Phi}$ is defined as

$$\hat{\Phi}(t,\xi,\eta) = \int_{\mathbb{R}^2} e^{ix\xi+iy\eta}\Phi(t,x,y)\,dxdy.$$

Solving (14) by a simple ODE theory, we have

$$\begin{aligned}\hat{\Phi}(t,\xi,\eta) =& \frac{1}{2}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\right)\widehat{\Phi_0}(\xi,\eta) \\ &+ \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\left(e^{(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2})t} - e^{(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2})t}\right) \\ &\qquad \left(\frac{1}{2}\widehat{\Phi_0}(\xi,\eta) + \widehat{\Phi_1}(\xi,\eta)\right).\end{aligned}$$

Definition

Let the operators $K_0(t,\partial_x), K_1(t,\partial_x)$ be defined as

$$\widehat{K_0(t,\partial_x)f}(t,\xi,\eta) = \frac{1}{2}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\right)\hat{f}(t,\xi,\eta);$$

and

$$\widehat{K_1(t,\partial_x)f}(t,\xi,\eta) = \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} - e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\right)\hat{f}(t,\xi,\eta).$$

where $\sqrt{-1} = i$.

Therefore, the solution Φ of the equation (13) is written as

$$\Phi(t,x,y) = K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\big(\frac{1}{2}\Phi_0 + \Phi_1\big).$$

Moreover, consider the inhomogeneous equation,

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = F, \tag{15}$$

with initial data $\Phi(1,x) = \Phi_0, \partial_t\Phi(1,x) = \Phi_1$.

Then we have the following standard Duhamel formula,

$$\Phi(t,x,y) = K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\Big(\frac{1}{2}\Phi_0 + \Phi_1\Big) \tag{16}$$

$$+ \int_1^t K_1(t-s,\partial_x)F(s,x,y)\,ds. \tag{17}$$

The rest of the proof is to apply this formula to rewrite (12) and then verify the continuity principle. We will need the following estimates on K_0 and K_1.

Lemma

Let K_0, K_1 *be defined in Definition* (4.10), *then*

1) $\left\| |\xi|^\alpha \widehat{K}_i(t,\cdot) \right\|_{L^q_\xi(|\xi|\le\frac{1}{2})} \lesssim t^{-\frac{1}{2}(\frac{1}{q}+\alpha)}$, *for any* $\alpha \ge 0$, $1 \le q \le \infty$, $i = 0, 1$.
2) $\left\| \partial_t \widehat{K}_i(t,\cdot) \right\|_{L^q_\xi(|\xi|\le\frac{1}{2})} \lesssim t^{-1-\frac{1}{2q}}$, $i = 0, 1$.
3) $\left| \widehat{K}_i(t,\xi) \right| \lesssim e^{-\frac{1}{2}t}$, *for any* $|\xi| \ge \frac{1}{2}$, $i = 0, 1$.
4) $\left| \langle\xi\rangle^{-1} \partial_t \widehat{K_0}(t,\xi) \right|, \left| \partial_t \widehat{K_1}(t,\xi) \right| \lesssim e^{-\frac{1}{2}t}$, *for any* $|\xi| \ge \frac{1}{2}$.

Let X_0 be the Banach space defined by the following norm

$$\|(\vec{u}_0,\, \psi_0)\|_{X_0} = \|\langle\nabla\rangle^N(\vec{u}_0,\, \nabla\psi_0)\|_{L^2_{xy}}$$
$$+ + \|\langle\nabla\rangle^{6+}(\vec{u}_0,\, \psi_0)\|_{L^1_{xy}} + \|\langle\nabla\rangle^{6+}(\vec{u}_1,\, \psi_1)\|_{L^1_{xy}},$$

where $\langle\nabla\rangle = (I-\Delta)^{\frac{1}{2}}$, $N \gg 1$and $a+$ denotes $a+\epsilon$ for small $\epsilon > 0$.

The solution spaces X is defined by

$$\|(\vec{u},\, \psi)\|_X = \sup_{t\geq 1}\Big\{ t^{-\varepsilon}\|\langle\nabla\rangle^N(\vec{u}(t),\, \nabla\psi(t))\|_2 + t^{\frac{1}{4}}\|\langle\nabla\rangle^3\psi\|_2$$
$$+ t^{\frac{1}{4}}\|\langle\nabla\rangle^3\psi\|_2 + t^{\frac{3}{2}}\|\partial_{xx}\psi\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle^2\partial_{xx}\psi\|_2 + t^{\frac{3}{2}}\|\partial_{xxx}\psi\|_2$$
$$+ t^{\frac{3}{2}}\|\partial_t\vec{u}\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle\partial_t\vec{u}\|_2 + t\|\langle\nabla\rangle\partial_x\vec{u}\|_\infty + t^{\frac{3}{2}}\|\partial_x\partial_t v\|_2\Big\}.$$

Our main result can be stated as follows:

Theorem

There exists a small constant $\varepsilon > 0$ *such that, if the initial data satisfies* $\|(\vec{u}_0, \psi_0)\|_{X_0} \leq \varepsilon$*, then* (6) *possesses a unique global solution* $(u, v, \psi) \in X$*. Moreover, the following decay estimates hold*

$$\|u(t)\|_{L_x^\infty} \lesssim \varepsilon t^{-1}; \quad \|v(t)\|_{L_x^\infty} \lesssim \varepsilon t^{-\frac{3}{2}}; \quad \|\psi(t)\|_{L_x^\infty} \lesssim \varepsilon t^{-\frac{1}{2}}.$$

The proof of this theorem relies on the continuity argument.

Lemma (Continuity Argument)

Suppose that $(\vec{u}, \psi)$ *with the initial data* $(\vec{u}_0, \psi_0)$, *satisfies*

$$\|(\vec{u}, \psi)\|_X \lesssim \|(\vec{u}_0, \psi_0)\|_{X_0} + C \, \|\vec{u}, \psi)\|_X^\beta \tag{18}$$

with $\beta > 1$. *Then, there exists* r_0 *such that, if*

$$\|(\vec{u}_0, \psi_0)\|_{X_0} \lesssim r_0,$$

then $\|(\vec{u}, \psi)\|_X \lesssim 2r_0$.

Proposition

Let $K(t,\partial_x)$ be a Fourier multiplier operator satisfying

$$\left\|\widehat{\partial_x^\alpha K}(t,\xi)\right\|_{L^1_\xi(|\xi|\le\frac12)}<\infty,\quad \left\|\widehat{K}(t,\xi)\right\|_{L^\infty_\xi(|\xi|\ge\frac12)}<\infty,\quad \alpha\ge 0.$$

Then, for any space-time Schwartz function f,

$$\begin{aligned}\left\|\partial_x^\alpha K(t,\partial_x)f\right\|_{L^\infty_{xy}} \lesssim &\left(\left\|\widehat{\partial_x^\alpha K}(t,\xi)\right\|_{L^1_\xi(|\xi|\le\frac12)}+\left\|\widehat{K}(t,\xi)\right\|_{L^\infty_\xi(|\xi|\ge\frac12)}\right)\\ &\times\left\|\langle\nabla\rangle^{\alpha+1+\epsilon}\partial_y f\right\|_{L^1_{xy}}. \end{aligned}\tag{19}$$

Lemma

For any $s \geq 1$,

$$\left\| \langle \nabla \rangle^5 |\nabla|^{\frac{1}{2}-\epsilon} F_1(s,\cdot) \right\|_{L^1_{xy}} \lesssim s^{-\frac{3}{2}-\varepsilon} \, \|(\vec{u}, \psi)\|_Y^2. \qquad (20)$$

Using the Duhamel formula, namely (16),

$$\psi(t,x,y) = K_0(t,\partial_x)\psi_0 + K_1(t,\partial_x)(\frac{1}{2}\psi_0 + \psi_1) + \int_1^t K_1(t-s,\partial_x)F_0(s$$

Therefore,

$$\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty \lesssim \|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty + \|\langle\nabla\rangle\partial_{xx}K_1(t)(\frac{1}{2}\psi_0+\psi_1)\|_\infty + \|\int_1^t \langle\nabla\rangle\partial_{xx}K_1(t-s)F_0(s)ds\|_\infty.$$

By Corollary 4.13 and Lemma 2,

$$\|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty$$
$$\lesssim \left(\|\widehat{\partial_{xx}K_0}(t,\xi)\|_{L^1_\xi(|\xi|\le\frac{1}{2})} + \|\widehat{K_0}(t,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})}\right)\|\langle\nabla\rangle^{2+\varepsilon}\partial_{xx}\partial_y\psi_0\|_{L^1_{xy}}$$
$$\lesssim \left(t^{-\frac{3}{2}} + e^{-t}\right)\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{L^1_{xy}} \lesssim t^{-\frac{3}{2}}\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{X_0}.$$

Since the estimates for K_0 and K_1 are the same, we also have

$$\|\langle\nabla\rangle\partial_{xx}K_1(t)(\frac{1}{2}\psi_0+\psi_1)\|_\infty \lesssim t^{-\frac{3}{2}}\Big\|\langle\nabla\rangle^{5+\varepsilon}(\frac{1}{2}\psi_0+\psi_1)\Big\|_{X_0}.$$

Moreover,

$$\begin{aligned}&\left\|\int_1^t \langle\nabla\rangle\partial_{xx}K_1(t-s)\,F_0(s)\,ds\right\|_\infty\\ &\lesssim \int_1^t \|\partial_{xx}K_1(t-s)\,\langle\nabla\rangle F_0(s)\|_\infty\,ds\\ &\lesssim \int_1^{\frac{t}{2}} \|\partial_{xx}K_1(t-s)\,\langle\nabla\rangle F_0(s)\|_\infty\,ds + \int_{\frac{t}{2}}^t \|\partial_x K_1(t-s)\,\langle\nabla\rangle\partial_x F_0(s)\|_\infty\,ds\end{aligned}$$

• Magnetic Diffusion only, $\nu_1 = \nu_2 = 0$, $\eta_1 = \eta_2 > 0$

The 2D MHD equations with no dissipation:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{21}$$

The global regularity problem remains open.

Global weak solutions have been established

Theorem

Let $\kappa > 0$. *Let* $(u_0, b_0) \in H^1$. *Then (21) has a global weak solution* (u, b) *with*

$$(u, b) \in L^\infty([0, \infty); H^1).$$

In this case, we can show that (u, b) admits global H^1-bound.

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \nu \Delta \omega + b \cdot \nabla j, \\ j_t + u \cdot \nabla j = b \cdot \nabla \omega + 2\partial_x b_1 (\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1 (\partial_y b_1 + \partial_x b_2). \end{cases}$$

$$\frac{1}{2}\,\frac{d\,\|\omega\|_2^2}{dt} = \int b\cdot\nabla j\,\omega\,dxdy,$$
$$\frac{1}{2}\,\frac{d\,\|j\|_2^2}{dt} + \eta\|\nabla j\|_2^2 = \int b\cdot\nabla\omega\,j\,dxdy + \ 2\int j\,\partial_x b_1\,\partial_x u_2\,dxdy + \cdots$$

Since

$$\int b\cdot\nabla j\,\omega\,dxdy + \int b\cdot\nabla\omega\,j\,dxdy = 0,$$

we have, for $X(t) = \|\omega(t)\|_2^2 + \|j(t)\|_2^2$,

$$\frac{d\,X(t)}{dt} + 2\eta\,\|\nabla j\|_2^2 \le C\,\|\nabla u\|_2\,\|\nabla b\|_4\,\|j\|_4,$$

$$\|\nabla u\|_2 = \|\omega\|_2, \quad \|\nabla b\|_4 \le \|j\|_4, \quad \|j\|_4^2 \le \|j\|_2 \, \|\nabla j\|_2$$

and Young's inequality, we find

$$\frac{d\,X(t)}{dt} + 2\eta\,\|\nabla j\|_2^2 \le \frac{C}{\eta}\|\omega\|_2^2\,\|j\|_2^2 + \eta\,\|\nabla j\|_2^2.$$

In particular,

$$\frac{d\,X(t)}{dt} + \eta\,\|\nabla j\|_2^2 \le \frac{C}{\eta}\|j\|_2^2\,X(t).$$

By Gronwall's inequality,

$$X(t) + \eta \int_0^t \|\nabla j(\tau)\|_2^2\,d\tau \le X(0)\,\exp\left(\frac{C}{\eta}\int_0^t \|j\|_2^2\,d\tau\right).$$

Remark. It remains open whether or not two H^1-weak solutions must coincide.

Remark. It remains open whether or not the H^1-weak solution becomes regular when (u_0, b_0) is more regular, say $(u_0, b_0) \in H^2$.

The global regularity problem for the 2D MHD equations with only Laplacian magnetic diffusion remains open.

The main difficulty is the lack of the global bound for $\|\omega\|_{L^\infty}$, although we do have global L^p-bound.

Proposition

For any $p \in (2, \infty)$ *and* $q \in (2, \infty)$, *the solution* (u, b) *obeys, for any* $T > 0$,

$$\|\omega\|_{L^\infty(0,T;L^p)} \le C, \qquad \|b\|_{L^q(0,T;W^{2,p})} \le C,$$

where C *is a constant depending on* p, q, T *and the initial data only.*

It is not clear how $\|\omega\|_{L^p}$ depends on p.

• Global regularity for MHD equation with $(-\Delta)^\beta b$

Consider

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b + u \cdot \nabla b + (-\Delta)^\beta b = b \cdot \nabla u, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{22}$$

C. Cao, J. Wu, B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, SIAM J Math Anal., 46 (2014), No. 1, 588-602.

Q. Jiu and J. Zhao, A Remark On Global Regularity of 2D Generalized Magnetohydrodynamic Equations, arXiv:1306.2823 [math.AP] 12 Jun 2013.

Theorem (C. Cao, J. Wu and B. Yuan)

Consider (22) with $\beta > 1$. Assume that $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$ and $j_0 = \nabla \times b_0$ satisfying

$$\|\nabla j_0\|_{L^\infty} < \infty.$$

Then (22) has a unique global solution (u, b) satisfying, for any $T > 0$,

$$(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)), \quad \nabla j \in L^1([0, T]; L^\infty(\mathbb{R}^2))$$

where $j = \nabla \times b$.

• 2D MHD with $\nu_2 > 0$ and $\eta_1 > 0$

The 2D MHD equations with vertical dissipation and horizontal magnetic diffusion

$$u_t + u \cdot \nabla u = -\nabla p + \nu\, u_{yy} + b \cdot \nabla b,$$

$$b_t + u \cdot \nabla b = \eta\, b_{xx} + b \cdot \nabla u,$$

$$\nabla \cdot u = 0, \quad \nabla \cdot b = 0.$$

C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, *Advances in Mathematics* **226** (2011), 1803-1822.

Theorem

Assume $u_0 \in H^2(\mathbb{R}^2)$ *and* $b_0 \in H^2(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then the aforementioned MHD equations have a unique global classical solution* (u, b). *In addition,* (u, b) *satisfies*

$$(u, b) \in L^\infty([0,\infty); H^2),$$
$$\omega_y \in L^2([0,\infty); H^1), \quad j_x \in L^2([0,\infty); H^1).$$

The main efforts are devoted to global *a priori* bounds in H^1 and H^2. We need a lemma.

Lemma

Assume that f, g, g_y, h and h_x are all in $L^2(\mathbb{R}^2)$. Then,

$$\iint |f\, g\, h|\, dxdy \le C\, \|f\|_{L^2}\, \|g\|_{L^2}^{1/2}\, \|g_y\|_{L^2}^{1/2}\, \|h\|_{L^2}^{1/2}\, \|h_x\|_{L^2}^{1/2}.$$

H^1-bounds

Proposition

If (u, b) is a solution of the aforementioned MHD equations, then

$$\|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \nu \int_0^t \|\omega_y(\tau)\|_2^2 \, d\tau + \eta \int_0^t \|j_x(\tau)\|_2^2 \, d\tau$$

$$\leq C(\nu, \eta) \, \left(\|\omega_0\|_2^2 + \|j_0\|_2^2\right)$$

where $C(\nu, \eta)$ denotes a constant depending on ν and η only, $\omega_0 = \nabla \times u_0$ and $j_0 = \nabla \times b_0$.

H^2-bounds

Proposition

If (u, b) *is a solution of the aforementioned MHD equations, then*

$$\|\nabla\omega(t)\|_2^2 + \|\nabla j(t)\|_2^2 + \nu \int_0^t \|\nabla\omega_y(\tau)\|_2^2 \, d\tau + \eta \int_0^t \|\nabla j_x(\tau)\|_2^2 \, d\tau$$

$$\leq C(\nu, \eta, t) \, \left(\|\nabla\omega_0\|_2^2 + \|\nabla j_0\|_2^2\right)$$

where $C(\nu, \eta, t)$ *depends on* ν, η *and* t *only.*

• 2D MHD with $\nu_1 > 0$ and $\eta_2 > 0$

The 2D MHD equations with horizontal dissipation and vertical magnetic diffusion

$$u_t + u \cdot \nabla u = -\nabla p + \nu\, u_{xx} + b \cdot \nabla b,$$

$$b_t + u \cdot \nabla b = \eta\, b_{yy} + b \cdot \nabla u,$$

$$\nabla \cdot u = 0, \quad \nabla \cdot b = 0.$$

For any initial data $(u_0, b_0) \in H^2$, this system of equations also possess a unique global solution.

In fact, the case $\nu\, u_{xx}$ and ηb_{yy} can be converted into the case νu_{yy} and ηb_{xx}. Set

$$U_1(x,y,t) = u_2(y,x,t), \quad U_2(x,y,t) = u_1(y,x,t),$$
$$B_2(x,y,t) = b_1(y,x,t), \quad B_1(x,y,t) = b_2(y,x,t),$$
$$P(x,y,t) = p(y,x,t).$$

Then $U = (U_1, U_2)$, P and $B = (B_1, B_2)$ satisfy

$$U_t + U \cdot \nabla U = -\nabla P + \nu\, U_{yy} + B \cdot \nabla B,$$
$$B_t + U \cdot \nabla B = \eta\, B_{xx} + B \cdot \nabla U,$$
$$\nabla \cdot U = 0, \quad \nabla \cdot B = 0.$$

• 2D MHD with $\nu_1 > 0$ and $\eta_1 > 0$

The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \partial_{xx} u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = \partial_{xx} b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{23}$$

where we have set $\nu_1 = \eta_1 = 1$.

The global regularity for this case is almost obtained, but this case appears to be very difficult.

Theorem (Expected Theorem)

Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then, (23) has a unique global solution* (u, b) *satisfying, for any* $T > 0$ *and* $t \le T$,

$$u,\ b \in L^\infty([0,T]; H^2(\mathbb{R}^2)), \quad \partial_x u,\ \partial_x b \in L^2([0,T]; H^2(\mathbb{R}^2)).$$

References

C. Cao, D. Regmi and J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, *J. Differential Equations* **254** (2013), No.7, 2661-2681.

C. Cao, D. Regmi, J. Wu and X. Zheng, Global regularity for the 2D magnetohydrodynamics equations with horizontal dissipation and horizontal magnetic diffusion, preprint.

The major effort is devoted to obtaining global bounds. We have global bound for the L^2-norm:

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\partial_x u(\tau)\|_2^2 d\tau + 2\int_0^t \|\partial_x b(\tau)\|_2^2 d\tau \\ = \|u_0\|_2^2 + \|b_0\|_2^2,$$

The trouble arises when we try to obtain the global H^1-bound. If we resort to the equations of ω and $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_x^2 \omega + b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = \partial_x^2 j + b \cdot \nabla \omega \\ \qquad\qquad +2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \end{cases}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_2^2 + \|j\|_2^2\right) + \|\partial_x \omega\|_2^2 + \|\partial_x j\|_2^2$$
$$= 2\int j\left(\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)\right)\,dxdy.$$

If we use the anisotropic Sobolev inequalities stated in the previous lemma,

$$\iint |f\,g\,h|\,dxdy \le C\,\|f\|_2\,\|g\|_2^{\frac{1}{2}}\,\|g_y\|_2^{\frac{1}{2}}\,\|h\|_2^{\frac{1}{2}}\,\|h_x\|_2^{\frac{1}{2}},$$

two terms can be bounded suitably. But $\int j\partial_x b_1 \partial_y u_1$ and $\int j\partial_x u_1\,\partial_y b_1$ can not be controlled.

If we do know that

$$\int_0^T \|(u_1, b_1)\|_\infty^2 \, dt < \infty, \tag{24}$$

then

$$\left| \int j\partial_x b_1 \partial_y u_1 + \int j\partial_x u_1 \, \partial_y b_1 \right| \le \frac{1}{2} \left(\|\partial_x \omega\|_2^2 + \|\partial_x j\|_2^2 \right) + C \, \|(u_1, b_1)\|_\infty^2 \left(\|\omega\|_2^2 + \|j\|_2^2 \right).$$

Then we can close the differential inequality and get a global bound for $\|\omega\|_2^2 + \|j\|_2^2$. (24) also allows us to get a global bound for $\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2$.

It appears to be extremely hard to prove (24) directly. Motivated by our recent work on the 2D Boussinesq equations with partial dissipation,

C. Cao and J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, *Arch. Rational Mech. Anal.* **208** (2013), 985-1004,

we bound the L^r-norm of (u_1, b_1) suitably.

Theorem

Let (u, b) be a solution of (23). Let $2 < r < \infty$. Then,

$$\|(u_1, b_1)(t)\|_{L^r} \le B_0 \sqrt{r \log r} + B_1(t), \qquad (25)$$

The proof of this bound uses the symmetric structure of (23), namely

$$w^{\pm} = u \pm b$$

satisfies

$$\begin{cases} \partial_t w^+ + (w^- \cdot \nabla) w^+ = -\nabla p + \partial_x^2 w^+, \\ \partial_t w^- + (w^+ \cdot \nabla) w^- = -\nabla p + \partial_x^2 w^-, \\ \nabla \cdot w^+ = 0, \quad \nabla \cdot w^- = 0. \end{cases} \tag{26}$$

We then bound $\|w_1^{\pm}\|_{L^r}$.

Multiplying the first component of the first equation of (26) by $w_1^+|w_1^+|^{2r-2}$ and integrating with respect to space variable, we obtain, after integration by parts,

$$\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + (2r-1)\int |\partial_x w_1^+|^2|w_1^+|^{2r-2}$$
$$= (2r-1)\int p\,\partial_x w_1^+|w_1^+|^{2r-2}. \qquad (27)$$

The main effort is devoted to bounding the pressure. If we knew that

$$\int_0^T \|p\|_{L^\infty}\,dt < \infty,$$

then we can easily show that

$$\|w_1^+\|_{2r} \le C\sqrt{r}$$

In order to get the bounds for the pressure, we showed that

$$\|(u_1, b_1)\|_{2r} \leq C_1 e^{C_2 r^3} \quad \text{for any } r$$

$$\|(u_2, b_2)(t)\|_{L^{2r}} \leq C, \quad r = 2, 3, \tag{28}$$

Since

$$-\Delta p = \nabla \cdot (w^- \cdot \nabla w^+)$$

and

$$\|p\|_q \leq C \|w^-\|_{2q} \|w^+\|_{2q},$$

we can show that, for any $1 < q \leq 3$,

$$\|p(t)\|_q \leq C, \tag{29}$$

where C is a constant depending on T and the initial data.

We can also show that, for any $s \in (0,1)$,

$$\int_0^T \|p(\tau)\|_{H^s}^2 \, d\tau < C.$$

Since

$$\begin{aligned}
\|\Lambda^s p\|_2 &\leq \|\Lambda^s (-\Delta)^{-1} \partial_x (w_1^- \partial_x w_1^+ + w_1^+ \partial_x w_1^-)\|_2 \\
&\quad + \|\Lambda^s (-\Delta)^{-1} \partial_y (w_1^+ \partial_x w_2^- + w_1^- \partial_x w_2^+)\|_2 \\
&\leq C \left(\|\partial_x w^+\|_2 + \|\partial_x w^-\|_2\right) \left(\|w_1^+\|_{\frac{2}{1-s}} + \|w_1^-\|_{\frac{2}{1-s}}\right),
\end{aligned}$$

To start, we fix $R > 0$ (to be specified later) and write

$$(2r-1)\int p\,\partial_x w_1^+ |w_1^+|^{2r-2} = J_1 + J_2,$$

where

$$J_1 = (2r-1)\int \overline{p}\,\partial_x w_1^+\,|w_1^+|^{2r-2}, \qquad J_2 = (2r-1)\int \widetilde{p}\,\partial_x w_1^+\,|w_1^+|^{2r-2}$$

with $\overline{p}$ and $\widetilde{p}$ as defined as follows.

As for the 2D Boussinesq equations, we decompose the pressure into low and high frequency parts and bound each part accordingly.

Lemma

Let $f \in H^s(\mathbb{R}^2)$ with $s \in (0,1)$. Let $R \in (0,\infty)$. Denote by $B(0,R)$ the box centered at zero with each side R and by $\chi_{B(0,R)}$ the characteristic function on $B(0,R)$. Write

$$f = \overline{f} + \widetilde{f} \quad \text{with} \quad \overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)}\mathcal{F}f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1-\chi_{B(0,R)})\mathcal{F}f)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Then we have the following estimates for $\overline{f}$ and $\widetilde{f}$.

Lemma

(1) *For a pure constant C_0 (independent of s),*

$$\|\overline{f}\|_\infty \le \frac{C_0}{\sqrt{1-s}}\, R^{1-s}\, \|f\|_{H^s(\mathbb{R}^2)},$$

(2) *For any $2 \le q < \infty$ satisfying $1 - s - \frac{2}{q} < 0$, there is a constant C_1 independent of s, q, R and f such that*

$$\|\widetilde{f}\|_q \le C_1\, q\, R^{1-s-\frac{2}{q}}\, \|f\|_{H^s(\mathbb{R}^2)}.$$

By Hölder's and Young's inequalities, we find

$$\begin{aligned} |J_1| &\leq (2r-1)\|\overline{p}\|_\infty \||w_1^+|^{r-1}\|_2 \|\partial_x w_1^+ (w_1^+)^{r-1}\|_2 \\ &\leq (2r-1)\|\overline{p}\|_\infty^2 \||w_1^+|^{r-1}\|_2^2 + \frac{2r-1}{4}\|\partial_x w_1^+ (w_1^+)^{r-1}\|_2^2. \end{aligned}$$

Applying Lemma 5, we have

$$\|\overline{p}\|_\infty \leq \frac{C_0}{\sqrt{1-s}} R^{1-s} \|p\|_{H^s}, \tag{30}$$

We will skip more details.

We need the bound for the L^∞-norm and we have a suitable bound for L^r-norm. The bridge is the following interpolation inequality.

Proposition

Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$. Then there exists a constant C depending on s only such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C \sup_{r\ge 2} \frac{\|f\|_r}{\sqrt{r\log r}} \left[\log(e+\|f\|_{H^s(\mathbb{R}^2)})\log\log(e+\|f\|_{H^s(\mathbb{R}^2)})\right]^{\frac{1}{2}}.$$

Proof of the proposition on interpolation inequality: By the Littlewood-Paley decomposition, we can write

$$f = S_{N+1} f + \sum_{j=N+1}^{\infty} \Delta_j f,$$

where Δ_j denotes the Fourier localization operator and

$$S_{N+1} = \sum_{j=-1}^{N} \Delta_j.$$

The definitions of Δ_j and S_N are now standard. Therefore,

$$\|f\|_\infty \leq \|S_{N+1} f\|_\infty + \sum_{j=N+1}^{\infty} \|\Delta_j f\|_\infty.$$

We denote the terms on the right by I and II. By Bernstein's inequality, for any $q \geq 2$,

$$|I| \leq 2^{\frac{2N}{q}} \|S_{N+1} f\|_q \leq 2^{\frac{2N}{q}} \|f\|_q.$$

Taking $q = N$, we have

$$|I| \leq 4\|f\|_N \leq 4\sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}}.$$

By Bernstein's inequality again, for any $s > 1$,

$$\begin{aligned} |II| &\leq \sum_{j=N+1}^{\infty} 2^j \|\Delta_j f\|_2 = \sum_{j=N+1}^{\infty} 2^{-j(s-1)} 2^{sj} \|\Delta_j f\|_2 \\ &= C\, 2^{-(N+1)(s-1)} \|f\|_{B^s_{2,2}}. \end{aligned}$$

where C is a constant depending on s only. By identifying $B^s_{2,2}$ with H^s, we obtain

$$\|f\|_\infty \le 4\sqrt{N\log N}\,\sup_{r\ge 2}\frac{\|f\|_r}{\sqrt{r\log r}} + C\,2^{-(N+1)(s-1)}\,\|f\|_{H^s}.$$

We obtain the desired inequality (31) by taking

$$N = \left[\frac{1}{s-1}\log_2(e+\|f\|_{H^s})\right],$$

where $[a]$ denotes the largest integer less than or equal to a.

2D MHD with fractional dissipation

Consider the 2D fractional MHD equations

$$\begin{cases} u_t + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b + \eta(-\Delta)^\beta b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{31}$$

where

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

The aim is at the smallest α and β for which (31) has a global regular solution.

The results we have indicate three cases:

The subcritical case: $\alpha + \beta > 1$;

The critical case: $\alpha + \beta = 1$;

The supercritical case: $\alpha + \beta < 1$.

The global regularity results we currently have are for subcritical cases.

1. $\nu = 0$ and $\beta > 1$
   C. Cao, J. Wu, B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, SIAM J Math Anal., 46 (2014), No. 1, 588-602.
   Q. Jiu and J. Zhao, Global Regularity of 2D Generalized MHD Equations with Magnetic Diffusion, arXiv:1309.5819 [math.AP] 23 Sep 2013.

1. $\alpha > 0$ and $\beta = 1$

   J. Fan, G. Nakamura, Y. Zhou, Global Cauchy problem of 2D generalized MHD equations, preprint.

2. $\alpha \geq 2$ (or with logarithmic improvement):

   K. Yamazaki, Remarks on the global regularity of two-dimensional magnetohydrodynamics system with zero dissipation, arXiv:1306.2762v1 [math.AP] 13 Jun 2013.

Open problems:

1) $\nu = 0$ and $\beta = 1$

2) $\eta = 0$ and $1 \leq \alpha < 2$

3) $1 < \alpha + \beta < 2,\ 0 < \beta < 1$

J. Wu, Generalized MHD equations, *J. Differential Equations* **195** (2003), 284-312.

C. Trann, X. Yu and Z. Zhai, On global regularity of 2D generalized magnetohydrodynamic equations, *J. Differential Equations* **254** (2013), 4194-4216.

B. Yuan and L. Bai, Remarks on global regularity of 2D generalized MHD equations, arXiv:1306.2190v1 [math.AP] 11 Jun 2013.

Q. Jiu and J. Zhao, A Remark On Global Regularity of 2D Generalized Magnetohydrodynamic Equations, arXiv:1306.2823 [math.AP] 12 Jun 2013.

2D Compressible MHD

Two recent papers are devoted to the compressible MHD with only velocity dissipation:

Jiahong Wu and Yifei Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, preprint, April, 2014.

Xianpeng Hu, Global Existence for Two Dimensional Compressible Magnetohydrodynamic Flows with Zero Magnetic Diffusivity, arXiv:1405.0274v1 [math.AP] 1 May 2014.

The 2D compressible MHD system can be written as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \\ \partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) - \Delta \vec{u} - \lambda \nabla (\nabla \cdot \vec{u}) + \nabla P = -\frac{1}{2} \nabla \left(|\vec{b}|^2 \right) + \vec{b} \cdot \nabla \vec{b}, \\ \partial_t \vec{b} + \vec{u} \cdot \nabla \vec{b} = \vec{b} \cdot \nabla \vec{u}, \\ \nabla \cdot \vec{b} = 0. \end{cases}$$

with the initial data

$$\rho|_{t=0} = \rho_0(x, y), \quad \vec{u}|_{t=0} = \vec{u}_0(x, y), \quad \vec{b}|_{t=0} = \vec{b}_0(x, y).$$

This paper of Wu and Wu achieves three goals:

- It establishes the global well-posedness of smooth solutions of when the initial data $(\rho_0, \vec{u}_0, \vec{b}_0)$ is smooth and close to the equilibrium state $(1, \vec{0}, \vec{e}_1)$, where we denote $\vec{0} = (0, 0)$ and $\vec{e}_1 = (1, 0)$;
- It offers a new way of diagonalizing a complex system of linearized equations;
- It obtains explicit and sharp large-time decay rates for the solutions in various Sobolev spaces.

In the 2D case, $\nabla \cdot \vec{b} = 0$ implies that for a scalar function ϕ,

$$\vec{b} = \nabla^{\perp}\phi \equiv \left(\partial_y\phi, -\partial_x\phi\right).$$

With this substitution, (6) becomes

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\vec{u}) = 0, \quad (t,\, x,\, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \\ \partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) - \Delta\vec{u} - \lambda\nabla(\nabla\cdot\vec{u}) + \rho^2\nabla\rho = -\nabla\phi\Delta\phi, \\ \partial_t\phi + \vec{u}\cdot\nabla\phi = 0. \end{cases} \tag{32}$$

Theorem

Assume $|\lambda| \leq c_0$ *for some absolute constant* $c_0 > 0$, *and let* $n = \rho - 1, \psi = \phi - y$, $n_0 = \rho_0 - 1, \psi_0 = \phi_0 - y$. *Then there exists a small constant* $\delta > 0$ *such that, if the initial data* $(n_0, \vec{u}_0, \phi_0)$ *satisfies* $\|(n_0, \vec{u}_0, \psi_0)\|_{X_0} \leq \delta$, *then there exists a unique global solution* $(\rho, u, v, \phi) \in X$ *to the MHD system. Moreover,*

$$\|(n, u, v, \phi)\|_X \lesssim \delta.$$

Especially, the following decay estimates hold

$$\|n(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-\frac{1}{2}}; \quad \|\vec{u}(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-1}; \quad \|\nabla\psi(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-\frac{1}{2}}.$$

Thank You Very Much!