Nonlocal turbulent cascades in nonlinear Schrödinger (Gross-Pitaevski) equation

N. Vladimirova\textsuperscript{1} and G. Falkovich\textsuperscript{2}

\textsuperscript{1} University of New Mexico
\textsuperscript{2} Weizmann Institute of Science

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"Non-local cascades" — oxymoron?  
Perhaps. But we can still talk about non-local turbulence.

The spectra (two-point correlation functions) can be non-local and non-universal (dependent on forcing and dissipation scales).

- There must be a high-order correlation function which is universal.

Kolmogorov turbulence:  
\[ \left\langle \left( \delta v_\parallel (r, \ell) \right)^3 \right\rangle = -\frac{4}{5} \epsilon \ell \]

NLS turbulence:  
???
Nonlinear Schrödinger (Gross-Pitaevski) equation as a model for wave turbulence

References cited in this talk are incomplete and subjective.
Formulation in terms of surface elevation $\eta(r, t)$ and velocity potential on the surface, $\Phi = \phi(r, \eta, t)$, where $v = \nabla \phi$.

Hamiltonian is expanded in powers of steepness, $\mu = \sqrt{|\nabla \eta|^2}$.

Complex canonical (normal) variables $a_k$ are introduced instead of real $\Phi(r, t)$ and $\eta(r, t)$.

$a_k$ is an elementary excitation (plane wave). Inverse cascade of $|a_k|^2$ is studied.

A. Korotkevich <alexkor@math.unm.edu>, private communications (2014)
Mid-range forcing and small-scale damping result in establishing of direct and inverse cascades and accumulation of wave action at small $k$. 

A. Korotkevich <alexkor@math.unm.edu>, private communications (2014)
Nonlinear Schrödinger (Gross-Pitaevskii) equation

\[ i\psi_t + \nabla^2 \psi \pm |\psi|^2 \psi = 0 \]

universal model
narrow wave packet
envelope of waves

Benney & Newell (1967) — general settings
Zakharov (1968) — deep water waves
Hasegawa & Tappert (1973) — optical fibers
Why universal?

Linear wave:

\[ \frac{\partial a}{\partial t} + \nu \frac{\partial a}{\partial x} = 0 \]
\[ \frac{\partial a_k}{\partial t} + i\omega a_k = 0 \]
\[ \frac{\partial a_k}{\partial t} = -i \frac{\partial H_2}{\partial a_k^*} \]
\[ H_2 = \int \omega_k |a_k|^2 dk \]

Nonlinearity:

\[ k_1 + k_2 = k_3 + k_4 \]
\[ k = k_0 + q_k, \quad q_k \ll k_0 \]
\[ H_4 = \ldots \]

\[ H = H_2 + H_4 = H_2 + \int T_{1234} a_1 a_2 a_3^* a_4^* \delta(k_1 + k_2 - k_3 - k_4) dk_1 dk_2 dk_3 dk_4 \]

Rewrite \[ \frac{\partial a_k}{\partial t} + i\omega a_k = -i \frac{\partial H_4}{\partial a_k^*} \] for the envelope, \[ a_k(t) = e^{-i\omega_0 t} \psi(q, t), \]

\[ \frac{\partial \psi_q}{\partial t} - i\omega_0 \psi_q + i\omega(q) \psi_q = -i T \int \psi_1^* \psi_2 \psi_3 \delta(q + q_1 - q_2 - q_3) dq_1 dq_2 dq_3 \]
Why universal?

\[ i \frac{\partial \psi_q}{\partial t} + \omega_0 \psi_q - \omega(q) \psi_q = T \int \psi_1^* \psi_2 \psi_3 \delta(q + q_1 - q_2 - q_3) dq_1 dq_2 dq_3 \]

Assume \( \omega = \omega(k) \) and expand for small \( q \)

\[ \omega(q) = \omega_0 + q_i \left( \frac{\partial \omega}{\partial k_i} \right)_0 + \frac{1}{2} q_i q_j \left( \frac{\partial^2 \omega}{\partial k_i \partial k_j} \right)_0 = \omega_0 + vq_\parallel + \frac{1}{2} \left( \omega'' q_\parallel^2 + \frac{v}{k_0} q_\perp^2 \right) \]

Back to \( r \)-space \((k_0 || \hat{z})\):

\[ i \left( \frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial z} \right) + \frac{\omega''}{2} \frac{\partial^2 \psi}{\partial z^2} + \frac{v}{2k_0} \nabla_\perp^2 \psi = T |\psi|^2 \psi \]

- \( \frac{\partial \psi}{\partial t} \) in moving frame
- dispersion
- diffraction
- nonlinearity

Rescale \( \psi \) and spatial coordinates:

\[ i \psi_t + \nabla^2 \psi \pm |\psi|^2 \psi = 0 \]
Connection to nonlinear optics

\[
\frac{1}{c^2} \left( \epsilon E \right)_{tt} - \nabla^2 E = 0
\]

Stationary envelope: \( E = \frac{1}{2} \psi(x, y, z)e^{ikz-i\omega t} \), with \( \omega = \frac{kc}{\sqrt{\epsilon_0}} \).

Kerr nonlinearity: \( \epsilon = \epsilon_0 + \epsilon_2 |E|^2 = \epsilon_0 + \epsilon_2 |\psi|^2 \).

\[
\frac{1}{c^2} (i\omega)^2 (\epsilon_0 + \epsilon_2 |\psi|^2) \psi - \left[ \nabla^2 \psi + 2ik\psi_z - k^2 \psi \right] = 0
\]

Neglecting \( \frac{\partial^2 \psi}{\partial z^2} \) and using \( kx \rightarrow x, \frac{1}{2} kz \rightarrow z \), and \( \psi|_{\frac{\epsilon_2}{k\epsilon_0}^{\frac{1}{2}}} \rightarrow \psi \),

\[
i\psi_z + \nabla_{\perp}^2 \psi - T|\psi|^2 \psi = 0, \quad \text{with} \quad T = \pm 1
\]
Connection to hydrodynamics

\[ i\psi_t + \nabla^2 \psi - T|\psi|^2\psi = 0 \]

Change of variables: \( \psi = Ae^{i\phi} , \quad \rho = A^2 , \quad \mathbf{v} = 2\nabla \phi . \)

\[ \mathbf{v}_t + \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) = -\frac{1}{\rho} \nabla \rho \]

\[ \rho_t + \nabla (\rho \mathbf{v}) = 0 \]

“Equation of state”: \[ \frac{1}{\rho} \nabla \rho = \nabla \left[ 2T\rho - \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right] \]
Collapses in focusing NSE

\[ i\psi_t + \nabla^2 \psi + |\psi|^2 \psi = 0 \]

Integrals of motion

\[ N = \int |\psi|^2 \, d^D r \]
\[ \mathcal{H} = \int \left( |\nabla \psi|^2 - \frac{1}{2} |\psi|^4 \right) \, d^D r \]

Within the packet

\[ |\psi|^2 \sim \frac{N}{L^D} \]
\[ \mathcal{H} \sim NL^{-2} - N^2 L^{-D} \]

\( Zakharov & Kuznetsov (1986) \)
Cascades of turbulence

\[ \mathcal{H} = \int \omega_k |a_k|^2 \, dk \]

\[ N = \int |a_k|^2 \, dk \]

\[ N_1 + N_3 = N_2 \]

\[ \omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2 \]

\[ N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1} \approx N_2 \]

\[ N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ll N_2 \]

\[ \omega_1 N_1 \ll \omega_2 N_2 \]

\[ \omega_3 N_3 \approx \omega_2 N_2 \]

Dyachenko, Newell, Pushkarev, & Zakharov (1992)
Modulational instability

\[ i\psi_t = -\frac{1}{2}\omega'' \nabla^2 \psi + T|\psi|^2\psi \]

Exact solution (condensate):

\[ \psi = \sqrt{N_0} e^{-iT_0 t} \]

For small perturbation \( \psi := \Psi + \psi \),

\[ i\psi_t = -\frac{1}{2}\omega'' \nabla^2 \psi + 2TN_0 \psi + T\psi^2 \psi^* + O(|\psi|^2). \]

In \( k \)-space, using \((\psi^*)_k = \psi^*_{-k} \),

\[
\begin{align*}
i \frac{d}{dt} \psi_k &= \left( \frac{1}{2} \omega'' k^2 + 2TN_0 \right) \psi_k + T\psi^2 \psi^*_{-k}, \\
-i \frac{d}{dt} \psi^*_{-k} &= \left( \frac{1}{2} \omega'' k^2 + 2TN_0 \right) \psi^*_{-k} + T\psi^2 \psi_k.
\end{align*}
\]
Modulational instability

Looking for the solution in the form

$$\psi_k = \alpha e^{-i(TN_0 + \Omega_k)t} \quad \text{and} \quad \psi^*_k = \beta e^{i(TN_0 - \Omega_k)t},$$

rewrite the system as

$$\begin{pmatrix}
\frac{1}{2} \omega''' k^2 + TN_0 - \Omega_k \\
T \psi^*^2
\end{pmatrix}
\begin{pmatrix}
T \psi^2 \\
\frac{1}{2} \omega'' k^2 + TN_0 + \Omega_k
\end{pmatrix}
\begin{pmatrix}
\alpha e^{-iTN_0 t} \\
\beta e^{iTN_0 t}
\end{pmatrix} = 0$$

Bogoliubov dispersion relation:

$$\Omega^2_k = \omega'' TN_0 k^2 + \frac{1}{4} \omega''^2 k^4$$

Instability: $\omega'' T < 0$ (focusing nonlinearity).

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*Bogoliubov (1947)*
Why turbulence?

- Wide energy spectra; cascades
- Statistical description
- High probability of extreme events (intermittency)
- Coherent structures — condensate or collapses
- Steady (with damping/forcing) or decaying
Direct and inverse cascades in 2D NLS equation with defocusing nonlinearity before onset of the condensate
“Non-local cascades” — oxymoron?
Perhaps. But we can still talk about non-local turbulence.

- The spectra (two-point correlation functions) can be non-local and non-universal (dependent on forcing and dissipation scales).
- There must be a high-order correlation function which is universal.

Kolmogorov turbulence:
\[
\langle (\delta \nu_{\parallel}(r, \ell))^3 \rangle = -\frac{4}{5} \epsilon \ell
\]

NLS turbulence:

???
Defocusing nonlinearity, forcing in $k$-space:

$$i\psi_t + \nabla^2 \psi - |\psi|^2 \psi = i\hat{f}_k \psi + i\hat{g}_k.$$  

Pumping: $g_k = |g_k|e^{i\phi_k}$, $|g_k| \propto \sqrt{(k^2 - k_i^2)(k_f^2 - k^2)}$, random $\phi_k$, $k_i < k < k_f$. Deposition rate $\alpha = \dot{N} \equiv |\psi|^2$.

Small-scale damping: $f_k = -\beta(k/k_d)^4(k/k_d - 1)^2$, $k > k_d$.

Large-scale friction: $f_k = -(1, 1, \frac{1}{\sqrt{2}}) \gamma$ for $k = (0, 1, \sqrt{2})k_{\text{min}}$. 

\begin{align*}
\text{Numerical setup} & \\

k_{\text{min}} &= 2\pi/L \quad \text{friction} \\
 k_f &= 1.5k_{\text{min}} \quad \text{pumping} \\
 k_i &< k < k_r \quad \text{damping} \quad k_{\text{d}} = 256 \quad k_{\text{max}} = 512
\end{align*}
Inverse Cascade

Evolution of spectra in simulations without friction
Spectra stabilized by friction
Comparison to nonlinear theory
Inverse cascade: time evolution of non-stabilized spectra

Early stage: Equipartitioned distribution of wave action.
Intermediate stage: Thermal quasi-equilibrium with chemical potential.
Late stage: Nonlinearity effects, moving pile-up at low $k$. 
Thermal equilibrium with chemical potential $\mu = k^2_{\mu}$

Assumption of $T(t) \to \text{const}$ leads to $k_{\mu} = Ae^{-\alpha t}$.

From balance of wave action, $T(k_{\mu}) \propto (t - a)/(t - b)$.

Deviation is due to non-linear effects, not due to limited domain size.
Non-linear effects in large boxes

\[ n_k = \frac{T(1 + c_1 c_2 k^2 \ln k)}{k_{\mu}^2 + k^2 + c_2 k^4}, \]

Three intervals: equipartitioned, DNPZ-1992, \( n_k \propto k^{-2} \ln k \).
Hump location moves as \( t^{-1} \), amplitude grows as \( t^{3/2} \) (bottleneck?)
Pumping at lower rate \( \alpha \) reduces piling-up and extends the spectrum.
Stabilized spectra: effect of forcing and friction

- Deviation from $n_k \sim k^{-2}$ is small.
- Weak turbulence, four-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/3}$ scaling.
- Too high or too low $\gamma$ leads to the distortion of spectrum at small $k$. 
Stabilized spectra with high nonlinearities

- At large $k$, deviation from $n_k \sim k^{-2}$ is small; unlike at weak nonlinearity, compensated spectra have negative slopes.

- Strong turbulence, three-wave interactions are dominant, resulting in $n_k \sim \alpha^{1/2}$ scaling.

- Nonlinearity makes equipartitioned part of the spectrum wider.
Stabilized spectra: effect of domain size

Can we extend the universal part of the spectrum by reducing $k_{\text{min}}$?

- For given $\alpha$, domain size does not affect $k^{-2}$ part of the spectrum.
- Pushing $k_{\text{min}} \to 0$ widens equipartitioned part, with $k_\mu = \text{const}$.
- Adjustment of friction does not extend universal part.
- Longer spectrum is expected for lower pumping rate $\alpha$. 
Stabilized Spectra of Direct Cascade
Comparison to Weakly-Nonlinear Theory
Three-wave interactions are dominant, $n_k \sim \alpha^{1/2}$.

Spectra at larger scales are distorted due to nonlinearity and sensitive to friction, $\gamma$.

Spectra at small scales are universal and well-described by Malkin’s theory (1996).
Comparison to weakly-nonlinear theory (Malkin, 1996)

Implicit description in terms of the fraction of wave action contained within a sphere of radius $k$, $N_k/N$, and energy flux $P$,

$$\frac{n_k k^2}{k_{\min}^2} = \frac{C}{2\pi} \left[ \ln \frac{N}{N_k} \right]^\frac{1}{3}, \quad \frac{C}{N} \ln \frac{k_d}{k} = p\left(\frac{N_k}{N}\right).$$

Here, $p(m) = \int_m^1 \left[ \ln y^{-1} \right]^{-\frac{1}{3}} dy$ and $C \propto P^{\frac{1}{3}}$. We show that $C \propto \alpha^{\frac{1}{2}}$. 
Comparison to weakly-nonlinear theory (Malkin, 1996)

The parametric representation does not provide explicit expression for $n_k(k)$. Using approximation $p_{\text{approx}}(m) = \frac{3}{2}(1 - m)^{\frac{2}{3}}$, we obtain

$$\frac{n_k k^2}{k_{\text{min}}^2} = \frac{C}{2\pi} \ln^{\frac{1}{3}} \left[ 1 - \left( \frac{2C}{3N} \ln \frac{k_d}{k} \right)^{\frac{3}{2}} \right].$$

Low pumping rates (smaller nonlinearity) might extent the range of applicability.
Fluxes of Wave Action and Energy
Flux of wave action

\[ N = \langle |\psi|^2 \rangle \] grows in time and long modes appear, but

\[ \langle |\psi_1 - \psi_2|^2 \rangle = \text{const} \]

\[ \langle |\psi_1 - \psi_2|^2 \rangle = \int |\psi_k|^2 (1 - \cos kr) dk \]

\[ \langle |\psi_1 - \psi_2|^2 \rangle \sim \int_{1/r}^{\infty} |\psi_k|^2 dk = \text{const} \]

Take time derivative of \[ \langle |\psi_1 - \psi_2|^2 \rangle = 2N - \langle \psi_1 \psi_2^* + \psi_1^* \psi_2 \rangle \] to obtain,

\[ Q(r) \equiv 2 \text{Im} \langle \psi_1^* |\psi_2|^2 \psi_2 \rangle = -\dot{N} \]

\[ Q(r) \] does not depend on distance between two points, \( r \).

Analog of Kolmogorov’s 4/5-law!
Flux of wave action in inverse cascade, $r \gg r_p$

![Graph](image)

$$Q(r) \equiv 2 \text{Im} \langle \psi_1^* | \psi_2 |^2 \psi_2 \rangle = -\dot{N}$$

Simulations confirm:

- $-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha$ for all scales.
- $Q(r) = \dot{N}$ for $r_p \lesssim r \lesssim L/16$.

$Q(r)$ is constant across the scales in inverse cascade.
Flux of energy in direct cascade, $r \ll r_p$

Simulations show:

$-Q(r) \propto \dot{N} = \tilde{\alpha} \approx 0.9\alpha$ for all scales.

$-Q''(r) = \text{const}$, therefore $P \sim Q r^{-2} = \text{const}$ for $r \ll r_p$.

$P(r)$ is constant across the scales in direct cascade.
NLS turbulence after onset of the condensate
Defocusing nonlinear Schrödinger equation

\[ i\psi_t + \nabla^2 \psi - |\psi|^2 \psi = i\hat{f}\psi \]

Condensate

\[ \psi = \sqrt{N_0} \exp(-iN_0 t) \]

Notation:

\[ N = |\psi|^2 \]
\[ N_0 = |\bar{\psi}|^2 \]
\[ n = N - N_0 = \int |\psi_k|^2 d^2 k \]

We consider large condensate

\[ N_0 \gg n \]

Statistically quasi-steady

\[ t \sim 10^4 \gg \frac{1}{\omega} \sim 10^{-3} \]
Onset of condensate

The graph shows the onset of condensate over time, with three curves labeled $N$, $N_0$, and $n$. The curve $N$ represents the total condensate, $N_0$ represents the condensate, and $n$ represents the over-condensate. The horizontal axis represents time ($t$), and the vertical axis represents the number of condensate ($N$, $N_0$, and $n$).
Onset of condensate

$t = 100 : \quad N_0 = 58, \quad n = 160$

$t = 1500 : \quad N_0 = 751, \quad n = 20$
Effect of forcing

\[ i\psi_t + \nabla^2 \psi - |\psi|^2 \psi = i\hat{f}\psi \]

Instability-driven force

\[ i\psi_t + \nabla^2 \psi - |\psi|^2 \psi = i\hat{F} \]

Random force
Phase transitions: breakdown of symmetries

$N = 219$

$N = 771$

$N = 1166$

$N = 4202$

amplitude deviation

$10^{-3} - 10^{-2} - 1 0 1 2 3$

spectrum $n_k$

$10^{-16} - 10^{-10} - 10^{-4} - 10^{2}$

$10^{2}$

$10^{6}$

$10^{10}$

$10^{16}$

$10^{-2}$

$10^{-6}$

$10^{-10}$

$10^{-16}$

$10^{2}$

$10^{6}$

$10^{10}$

$10^{16}$
Phase transitions: breakdown of symmetries

Higher condensate ⇒ more ordered system
Long-range orientational, short-range positional order
What happens at even larger $N$?

Vladimirova, Derevyanko, & Falkovich (2012)
Small perturbations

Compare quadratic and cubic terms in Hamiltonian

\[ \langle \mathcal{H}_2 \rangle = \Omega_k n = N_0^{1/2} kn \]
\[ \langle \mathcal{H}_3 \rangle = \sum_{k_1, k_2, k_3} V_{123} \langle \psi_{k_1} \psi_{k_2} \psi_{k_3}^* \rangle \delta(k_1 + k_2 - k_3) \]
\[ \approx \sum_{k_1, k_2, k_3} |V_{123}|^2 n_1 n_2 \delta(k_1 + k_2 - k_3) \delta(\Omega_1 + \Omega_2 - \Omega_3) \]
\[ \approx \frac{|V|^2 n^2 c^2}{k^3} \frac{k}{c} \approx \frac{n^2 k}{N_0^{1/2}} \]

Effective nonlinearity parameter is small,

\[ \frac{\mathcal{H}_3}{\mathcal{H}_2} \approx \frac{n}{N_0}. \]

But: weak turbulence assumes random phases.

Angle of interaction: \( k/c \sim k/\sqrt{N_0}, \) where \( c = \sqrt{2N_0}. \)
Angle of interaction

Arch grows in $k$-space from the condensate to a preset mode, $k_0$. Arch equation:

\[
\omega(k_0) = \omega(k) + \omega(|k_0 - k|)
\]

\[
\omega^2(k) = 2N_0 k^2 + k^4
\]

Angle of interaction:

\[
\phi_{\text{max}} \approx \frac{k}{\sqrt{3N_0/2}} \sim \frac{k}{c}
\]
Condensate-turbulence oscillations

- The system periodically oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- Predator-prey model?
Phase coherence

\[ n_k \quad \theta_k = 2\phi_0 - \phi_k - \phi_{-k} \]

\[ 2\phi_0 - \phi_k - \phi_{-k} = \pi \]
Three-wave model

Consider condensate interacting with two waves

\[ \psi_{\pm k} = \sqrt{n} \exp(\pm ikx + iN_0 t + i\phi_{\pm k}) \]

with \( \theta = 2\phi_0 - \phi_k - \phi_{-k} \).

Hamiltonian:

\[ H = 2k^2 n + \frac{1}{2} N^2 + 2n(N - 2n)(1 + \cos \theta) + n^2 \]

Equations of motion:

\[ \dot{n} = 2n(N - 2n) \sin \theta \]
\[ \dot{\theta} = 2k^2 + 2(N - 3n) + 2(N - 4n) \cos \theta \]

Stability points:

\[ \theta = \pi, \quad n = -\frac{1}{2} k^2 \quad \Rightarrow \quad \text{unphysical} \]
\[ \theta = 0, \quad n = \frac{(4N + k^2)}{14} \quad \Rightarrow \quad \text{too high} \ n \]

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Falkovich (2011), Miller, Vladimirova & Falkovich (2013)
Predictions of three-wave model

\[ \dot{n} = 2n(N - 2n) \sin \theta \]
\[ \dot{\theta} = 2k^2 + 2(N - 3n) + 2(N - 4n) \cos \theta \]

For \( n \ll N \):

- the system spends most of its time around \( \theta = \pi \) state
- the frequency of oscillations \( 2\Omega \approx 2\sqrt{2Nk^2 + k^4} \)
- the amplitude \( a \equiv \sqrt{n(t)} \) exhibits complicated cusped shape
Individual modes in turbulence

In turbulence, $n \ll N$ condition is well satisfied.

As predicted:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations approaches $2\Omega = 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a = \sqrt{n(t)}$ exhibits complicated cusped shape

However:

The 3-wave model cannot grasp closed trajectories with $\theta \approx \pi$. 
Collective oscillations

- The system periodically oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- The condensate imposes the phase coherence between the pairs of counter-propagating waves (anomalous correlation).
- Collective oscillations are not of a predator-prey type; they are due to phase coherence and anomalous correlations.
Conclusions - I

- When the driving term corresponds to an instability (but not a random force) high levels of condensate lead to a phase transitions — spontaneous breakdown of symmetries of small-scale over-condensate fluctuations: from the 2-fold to 3-fold to 4-fold.

- Collective oscillations are not of a predator-prey type; they are due to phase coherence, imposed by condensate, and anomalous correlations.

Wave spectra (second-order moments) are close to slightly (logarithmically) distorted thermal equilibrium in both cascades.

Correction by Dyachenko, Newell, Pushkarev, Zakharov (1992) for inverse cascade spectra works for intermediate $k$.

Correction by Malkin (1996) for direct cascade spectra works.

Analog of Kolmogorov’s 4/5 law:

$$Q(r) \equiv 2 \text{Im} \langle \psi_1^* | \psi_2 | ^2 \psi_2 \rangle = - \dot{N} \quad \text{for} \quad r > r_p;$$

the flux of wave action is independent of scale in inverse cascade, while the flux of energy is independent of scale in direct cascade.