Abstract framework for statistical solutions of evolution equations

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Aim

- Extend the theory of statistical solutions for the NSE to other eqs
  - Bénard problem (velocity field coupled with temperature)
  - MHD (velocity field coupled with magnetic field)
  - Quasi-geostrophic equations and other geophysical models
  - Other critical equations (wave eqs., dispersive eqs., reaction-diffusion eqs., etc.)
- Develop a sufficiently general abstract framework for some main results:
  - Initial-value problem: global existence
  - Initial-value problem: local uniqueness
  - Long-time behavior: Stationary statistical solutions
  - Long-time behavior: Ergodic properties
  - Etc.

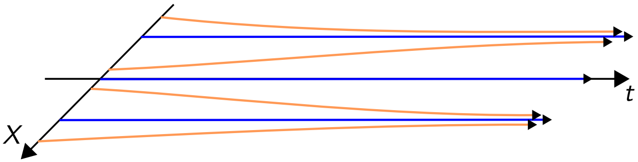
Statistical solutions

- Introduced for the study of turbulence in the Navier-Stokes equations
- Describe the evolution of probability distributions of initial conditions
- Related to the transport of measures by a semigroup... but without one

Evolution of solutions and probability distributions

Example: $x' = -x(x-1)(x+1)$.

- Individual solutions:



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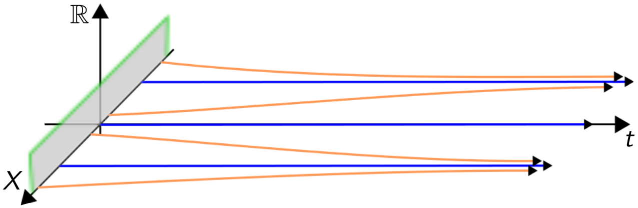
- Individual solutions with an initial probability measure:



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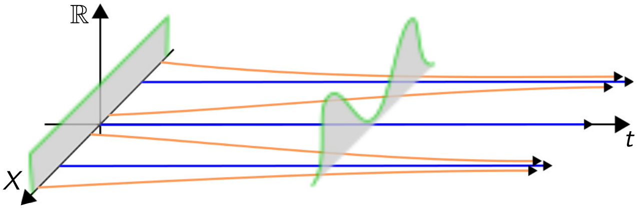
- Individual solutions and the evolution of the initial measure:



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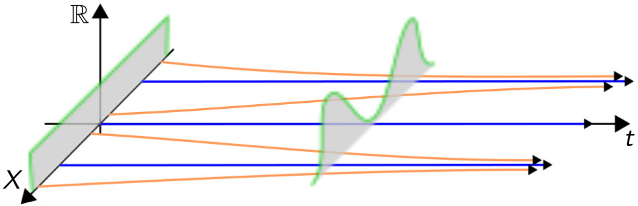

- Associated with the transport of the measure by the system

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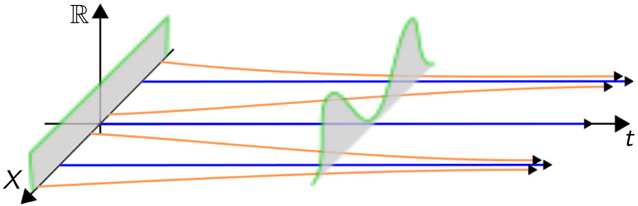

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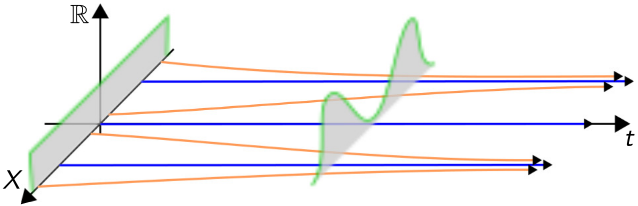

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- System is deterministic, not stochastic
- System might not be globally well-posed

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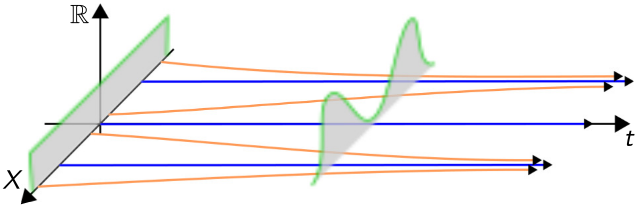


- Associated with the transport of the measure by the system
- System is deterministic, not stochastic
- System might not be globally well-posed
- In the conventional theory of turbulence:

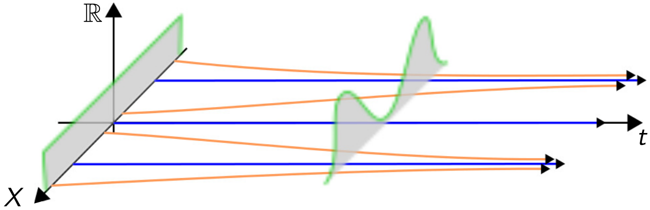
$$\int_X \Phi(\mathbf{u})\, \mathrm{d}\mu_t(\mathbf{u}) = \frac{1}{N}\sum_{j=1}^N \Phi(\mathbf{u}^{(j)}(t)).$$

Evolution of probability distributions in well-posed systems

- Diff. eq. with phase space X, and trajectory space $\mathcal{X}$:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(\mathbf{u}), \qquad \mathbf{u}(t) \in X, \ \mathbf{u} \in \mathcal{X}.$$

- If well-posed, $\mu_t = S(t)\mu_0$ (i.e. $\mu_t(E) = \mu_0(S(t)^{-1}E)$).



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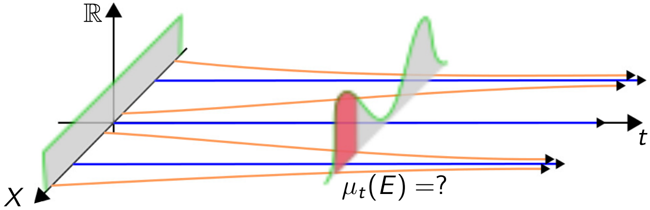

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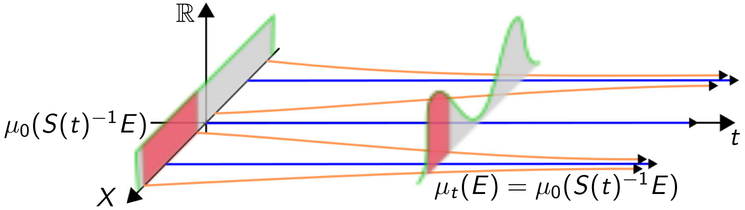


Motivation for statistical solutions

- Evolution of a probability distribution of initial conditions for equations without global well-posedness
- To make rigorous the notion of ensemble average in the conventional theory of turbulence.

"The theory of statistical solutions is to ensemble averages as the theory of weak solutions is to individual solutions."

Results

Pioneering formulations for NSE:

- Foias and Prodi (early 1970's)
- Vishik and Fursikov (mid 1970's)

Recent reformulation for NSE:

- Foias, Manley, Rosa, and Temam (2001)
- Foias, Rosa, and Temam (2011, 2013)

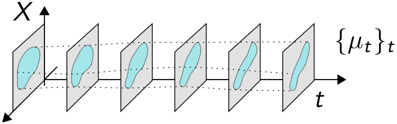
Abstract formulation and applications:

- Bronzi, Mondaini and Rosa (2014)
- Bronzi, Mondaini and Rosa (in preparation)

Statistical solutions in phase space and trajectory space

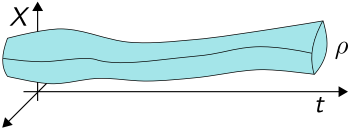
- Statistical solution in phase space (Foias-Prodi 1972): Family $\{\mu_t\}_t$ of Borel probability measures on X with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X \Phi(\mathbf{u})\,\mathrm{d}\mu_t(\mathbf{u}) = \int_X \langle \mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u})\rangle\,\mathrm{d}\mu_t,\ \forall\Phi.$$



- Statistical solution in trajectory space (Vishik-Fursikov 1977): Borel Probability measure ρ on $\mathcal{X}$ with

$$\rho(\mathcal{U}) = 1, \quad \text{where } \mathcal{U} = \{\text{ the set of all weak solutions }\}.$$



Incompressible 3D Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \nu\Delta\mathbf{u} + \mathbf{f}, \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \end{cases}$$

$\mathbf{u} = (u_1, u_2, u_3) =$ velocity field,

$\mathbf{x} = (x_1, x_2, x_3) =$ space variables,

$t =$ time variable,

$p =$ kinematic pressure,

$\mathbf{f} = (f_1, f_2, f_3) =$ density of volume forces,

$\nu =$ kinematic viscosity.

Functional setting

Basic function spaces

- $H = \{L^2 \text{ vector fields, } \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \text{ boundary cond}\}$;
- $V = \{H^1 \text{ vector fields, } \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \text{ boundary cond}\}$;
- $V' =$ dual of V (with $V \subset H = H' \subset V'$);
- $D(A^s) =$ domain of powers of the Stokes operator, $s \in \mathbb{R}$;
- $H_{\mathrm{w}} = H$ endowed with weak topology.

Consider no-slip boundary conditions on a smooth, bounded domain or space-periodic conditions with zero average.

$$\mathbf{f} \in L^\infty(0, \infty; H), \text{ or even } \mathbf{f} \in L^2_{\mathrm{loc}}(0, \infty; V')$$

Foias (1972)

Definition (of time-dependent Foias-Prodi statistical solution)

It is a family $\{\mu_t\}_{t\geq 0}$ of Borel prob. measures on H satisfying

- $t \mapsto \int_H \varphi(\mathbf{u})\,\mathrm{d}\mu_t(\mathbf{u})$ measurable on $t \geq 0$, $\forall \varphi \in \mathcal{C}_{\text{bdd}}(H)$
- $t \mapsto \int_H \|\mathbf{u}\|_{L^2}^2\,\mathrm{d}\mu_t(\mathbf{u})$ in $L^\infty(0,\infty)$
- $t \mapsto \int_H \|\boldsymbol{\nabla}\otimes\mathbf{u}\|_{L^2}^2\,\mathrm{d}\mu_t(\mathbf{u})$ in $L^1_{\mathrm{loc}}(0,\infty)$
- $\int_H \varphi(\|\mathbf{u}\|_{L^2}^2)\,\mathrm{d}\mu_t(\mathbf{u})$ is continuous at $t=0$
- Mean energy inequality for all smooth $\psi \geq 0$, $0 \leq \psi'(r) \leq c$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_H \psi(\|\mathbf{u}\|_{L^2}^2)\,\mathrm{d}\mu_t(\mathbf{u}) + \nu\int_H \psi'(\|\mathbf{u}\|_{L^2}^2)\|\boldsymbol{\nabla}\otimes\mathbf{u}\|_{L^2}^2\,\mathrm{d}\mu_t(\mathbf{u}) \leq \int_H \psi'(\|\mathbf{u}\|_{L^2}^2)(\mathbf{f},\mathbf{u})_{L^2}\,\mathrm{d}\mu_t(\mathbf{u}),$$

- Satisfies Statistical NSE for "cylindrical" test functions:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_H \Phi(\mathbf{u})\,\mathrm{d}\mu_t(\mathbf{u}) = \int_H \big(\mathbf{F}(\mathbf{u}),\Phi'(\mathbf{u})\big)_{L^2}\,\mathrm{d}\mu_t(\mathbf{u})$$

Cylindrical test functions

Definition (of cylindrical test functions)

A cylindrical test function is a continuous function $\Phi : H \to \mathbb{R}$ of the form

$$\Phi(\mathbf{u}) = \psi((\mathbf{u}, \mathbf{v}_1)_H, \ldots, (\mathbf{u}, \mathbf{v}_n)_H),$$

where $\psi \in \mathcal{C}_{\mathrm{c}}^1(\mathbb{R}^n)$, $n \in \mathbb{N}$, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Such a function is Fréchet diferentiable $\Phi'(\mathbf{u}) \in V$, for all $\mathbf{u} \in H$, given by

$$\Phi'(\mathbf{u}) = \sum_{k=1}^{n} \partial_k \psi((\mathbf{u}, \mathbf{v}_1)_H, \ldots, (\mathbf{u}, \mathbf{v}_n)_H)\mathbf{v}_k.$$

- Each $\Phi \in \mathcal{C}_{\mathrm{b}}(H_{\mathrm{w}})$;
- Stone-Weierstrass theorem implies $\{\Phi|_{B_H(R)}\}$ is dense in $\mathcal{C}(B_H(R)_{\mathrm{w}})$.

Vishik and Fursikov (1979)

Work in a "trajectory space"

$$\mathcal{X} = L^2_{\mathrm{loc}}([0,\infty), H) \cap \mathcal{C}_{\mathrm{loc}}([0,\infty), D(A^{-s/2})), s \geq 2.$$

Definition (of trajectory statistical solution of Vishik-Fursikov)

Borel probability measure ρ on $\mathcal{X}$ such that

- $\exists W \subset \{$weak solutions not necessarily of Leray-Hopf type$\}$;
- $\rho(W) = 1$,
- W is closed in the space

$$\tilde{\mathcal{L}} = \Big\{\mathbf{u} \in L^\infty(0,\infty; H) \cap L^2_{\mathrm{loc}}(0,\infty; V); \\ \mathbf{u}' \in L^{4/3}_{\mathrm{loc}}((0,\infty), D(A^{-s/2}))\Big\};$$

- $\displaystyle\int_{\mathcal{X}} \left(|\mathbf{u}(t)|^2_{L^2} + \|\mathbf{u}\|^2_{\tilde{\mathcal{L}}} \right) \,\mathrm{d}\rho(\mathbf{u}) \leq C \left(\int_{\mathcal{X}} |\mathbf{u}(0)|^2_{L^2} \,\mathrm{d}\rho(\mathbf{u}) + 1 \right),$

Functional approach

Spaces:

- FP use (in the proof) $X = H$ and $\mathcal{X} = L^1(0, T; H)$.
- VF use $\mathcal{X} = L^2_{\mathrm{loc}}([0, \infty), H) \cap \mathcal{C}_{\mathrm{loc}}([0, \infty), D(A^{-s/2})), s \geq 2$.
- FRT use $X = H_{\mathrm{w}}$ and $\mathcal{X} = \mathcal{C}_{\mathrm{loc}}([0, \infty), H_{\mathrm{w}})$.

Existence of statistical solutions in trajectory space $\mathcal{X}$

- FP and VF use convergence of Galerkin approximations
- FRT use convergence of convex combination of Dirac measures

Projections:

- FP use a complicate representation of the dual of $L^1(0, T; H)$ to project the operator at each time.
- VF do not project
- FRT project using the continuity of the operator

$$\Pi_t : \mathcal{X} = \mathcal{C}_{\mathrm{loc}}([0, \infty), X) \to X, \quad \Pi_t \mathbf{u} = \mathbf{u}(t), \quad \forall t \geq 0$$

FRT (2010)

Work in the "trajectory space" $\mathcal{X} = \mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})$ and consider

$$\mathcal{U} = \{\mathbf{u} \in \mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}});\ \mathbf{u} = \text{weak solutions on } [0,\infty)\}$$

$$\mathcal{U}^\sharp = \{\mathbf{u} \in \mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}});\ \mathbf{u} = \text{ weak solution on } (0,\infty)\}$$

Have characterization $\mathcal{U} = \{\mathbf{u} \in \mathcal{U}^\sharp; \mathbf{u} = \text{ strongly continuous at } t = 0\}$.

Definition (of Vishik-Fursikov measure)

A measure ρ on $\mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})$ satisfying

(i) ρ is carried by the closure $\mathcal{U}^\sharp$ of $\mathcal{U}$ in $\mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})$;

(ii) $t \mapsto \displaystyle\int_{\mathcal{U}^\sharp} |\mathbf{u}(t)|_{L^2}^2 \,\mathrm{d}\rho(\mathbf{u}) \in L^\infty_{\mathrm{loc}}(0,\infty)$;

(iii) $t \mapsto \displaystyle\int_{\mathcal{U}^\sharp} \psi\left(|\mathbf{u}(t)|_{L^2}^2\right) \,\mathrm{d}\rho(\mathbf{u})$ is continuous at $t = 0$ for every smooth $\psi \geq 0$, $\psi' \geq 0$, ψ' bounded.

FRT (2010) - Equivalent definition

Work in the "trajectory space" $\mathcal{X} = \mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})$ and consider

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Definition (of Vishik-Fursikov measure)

A measure ρ on $\mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})$ satisfying

(i) $\rho(\mathcal{U}) = 1$;

(ii) $t \mapsto \int_{\mathcal{U}} |\mathbf{u}(t)|_{L^2}^2 \, \mathrm{d}\rho(\mathbf{u}) \in L^\infty_{\mathrm{loc}}(0,\infty)$;

Statistical solutions of Vishik-Fursikov type - FRT 2010

Projecting to phase space we obtain a family of measures which is a statistical solution in the Foias-Prodi sense.

Definition (of time-dependent Vishik-Fursikov statistical solution)

A family $\{\mu_t\}_{t\geq 0}$ obtained as the projection

$$\mu_t = \Pi_t \rho$$

of a Vishik-Fursikov measure ρ, where

$$\Pi_t : \mathcal{C}_{\mathrm{loc}}([0,\infty), H_{\mathrm{w}})) \mapsto H_{\mathrm{w}},\ \Pi_t \mathbf{u} = \mathbf{u}(t),$$

$\mu_t = \Pi_t \rho$:

$$\int_H \varphi(\mathbf{v}) \,\mathrm{d}\mu_t(\mathbf{v}) = \int_{\mathcal{U}^\sharp} \varphi(\mathbf{u}(t)) \,\mathrm{d}\mu(\mathbf{u}), \qquad \forall \varphi \in \mathcal{C}_{\mathrm{b}}(H_{\mathrm{w}}).$$

Abstract framework

We consider, in general:

- Phase space: $X =$ Hausdorff topological space;
- Time interval: $I = [t_0, T)$ or $[t_0, \infty)$;
- The space of continuous paths with the compact-open topology:

$$\mathcal{X} = \mathcal{C}_{\mathrm{loc}}(I, X);$$

- Initial probability distribution:

$$\mu_0 = \text{ tight Borel probability measure on } X;$$

- Set of "solutions/trajectories" $\mathcal{U} \subset \mathcal{X}$.

Abstract trajectory statistical solution

Definition

A $\mathcal{U}$-**trajectory statistical solution** is a Borel probability measure ρ on $\mathcal{X}$ such that

- ρ is tight in $\mathcal{X}$; and
- $\rho(\mathcal{X} \setminus \mathcal{V}) = 0$, for some $\mathcal{V} \subset \mathcal{U}$ Borel.

Problem

Given μ_0 tight Borel probability measure on X, find ρ such that

- ρ *is a* $\mathcal{U}$-**trajectory statistical solution**
- $\Pi_0 \rho = \mu_0$.

Existence of solutions of the initial value problem

Theorem (Bronzi-Mondaini-Rosa (2015?))

Suppose $\mathcal{U}$ satisfies

1. $\Pi_{t_0}\mathcal{U} = X$;
2. *There exists a family of sets $\mathfrak{K}'(X) \subset \mathfrak{B}_X$ such that*
    1. *Every $K \in \mathfrak{K}'(X)$ is compact in X;*
    2. *Every tight Borel probability measure μ_0 on X is inner regular with respect to the family $\mathfrak{K}'(X)$;*
    3. *For every $K \in \mathfrak{K}'(X)$, $\Pi_{t_0}^{-1}K \cap \mathcal{U}$ is compact in $\mathcal{X}$.*

Then, for any μ_0 tight Borel probability measure on X, there exists $\mathcal{U}$-trajectory statistical solution with

$$\Pi_0\rho = \mu_0.$$

Sketch of proof - part 1

- Choose (increasing) compact sets $K_n \in \mathfrak{K}'(X)$ such that

$$\mu_0(X \setminus K_n) \to 0;$$

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- Construct ρ_n carried by the compact set $\mathcal{U} \cap \Pi_0^{-1} K_n$ with
$$\Pi_0 \rho_n = \mu_0|_{K_n \setminus K_{n-1}}.$$

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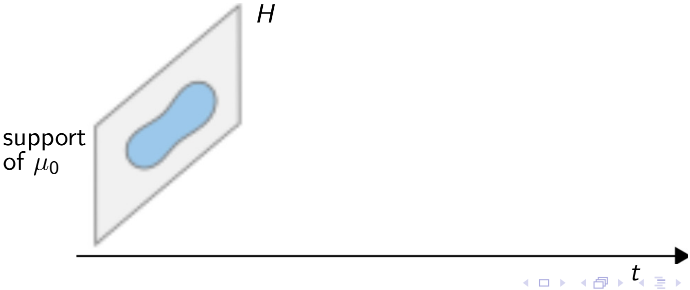
$$\Pi_0 \rho_n = \mu_0|_{K_n \setminus K_{n-1}}.$$
- Sum up to find measure
$$\rho = \sum_n \rho_n$$
with
$$\Pi_0 \rho = \mu_0 \quad \text{and} \quad \rho(\mathcal{V}) = 0,$$
where
$$\mathcal{V} = \bigcup_n (\mathcal{U} \cap \Pi_0^{-1} K_n) = \text{ Borel subset } \subset \mathcal{U}.$$

Sketch of proof - part 2

- When μ_0 is carried by some $K \subset \mathfrak{K}'(X)$



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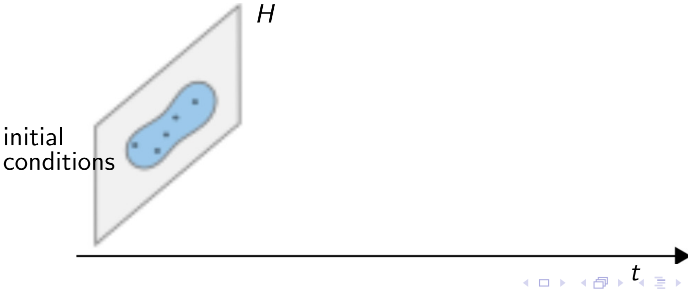
- When μ_0 is carried by some $K \subset \mathfrak{K}'(X)$
- Via Krein-Milman, approximate initial measure $\sum_{j=1}^{J} \theta_j^n \delta_{\mathbf{u}_{0j}^n} \stackrel{wsc}{\rightharpoonup} \mu_0$



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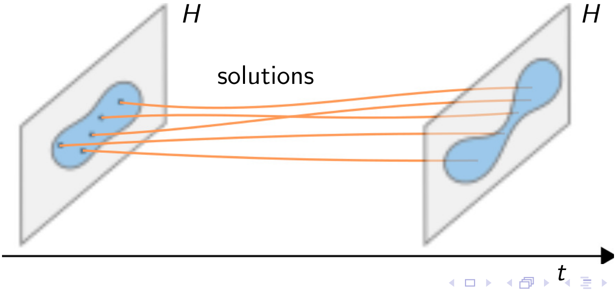
- Construct time-dependent measures

$$\rho_n = \sum_{j=1}^{J} \theta_j^n \delta_{\mathbf{u}_j^n(\cdot)}$$

with $\mathbf{u}_j^n$ weak solution with $\mathbf{u}_j^n(t_0) = \mathbf{u}_{0j}^n$.



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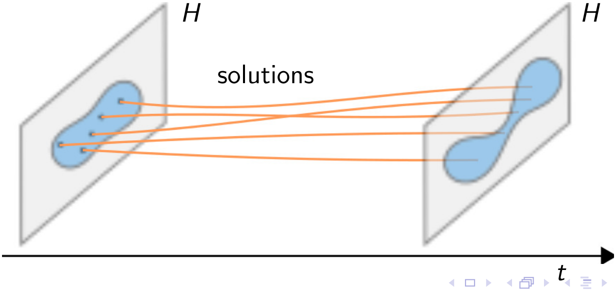
with $\mathbf{u}_j^n$ weak solution with $\mathbf{u}_j^n(t_0) = \mathbf{u}_{0j}^n$.

- Use compactness of $\mathcal{U} \cap \Pi^{-1}K$ to pass to the limit $\rho_n \overset{wsc}{\rightharpoonup} \rho$



Framework for phase-space statistical solution

- X = Hausdorff topological space
- A Banach space $Y \subset X \subset Y'_{w^*}$;
- The space

$$\mathcal{Z} = \{u \in \mathcal{C}_{\mathrm{loc}}(I, X) \cap W^{1,1}_{\mathrm{loc}}(I, Y');\ u(t) \in Y \text{ for almost all } t \in I\}.$$

- Assume $\mathcal{U} \subset \mathcal{Z} \subset \mathcal{X}$.
- $F : I \times Y \to Y'$ such that

$$u_t = F(t, u), \qquad \text{a.e. in } I,\ \forall u \in \mathcal{U}.$$

Abstract phase-space statistical solution

Definition

Family of Borel measures $\{\rho_t\}_t$ on X is a **(phase-space) statistical solution** of $u_t = F(t,u)$ if

- $t \mapsto \int_X \varphi(u)\mathrm{d}\rho_t(u)$ is continuous, $\forall \varphi \in \mathcal{C}_b(X)$;
- $\rho_t(Y) = 1$ for a.e. $t \in I$;
- $u \mapsto F(t,u)$ is ρ_t-integrable and $t \mapsto \int_X \|F(t,u)\|_{Y'}\mathrm{d}\rho_t(u)$ is $L^1_{\mathrm{loc}}(I)$;
- For any cylindrical test function Φ and all $t', t \in I$:

$$\int_X \Phi(u)\mathrm{d}\rho_t(u) = \int_X \Phi(u)\mathrm{d}\rho_{t'}(u) + \int_{t'}^t \int_X \langle F(s,u), \Phi'(u)\rangle_{Y',Y}\mathrm{d}\rho_s(u)\mathrm{d}s$$

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- $\rho_t(Y) = 1$ for a.e. $t \in I$;
- $u \mapsto F(t, u)$ is ρ_t-integrable and $t \mapsto \int_X \|F(t, u)\|_{Y'}\mathrm{d}\rho_t(u)$ is $L^1_{\mathrm{loc}}(I)$;
- For any cylindrical test function Φ and all $t', t \in I$:

$$\int_X \Phi(u)\mathrm{d}\rho_t(u) = \int_X \Phi(u)\mathrm{d}\rho_{t'}(u) + \int_{t'}^{t} \int_X \langle F(s, u), \Phi'(u)\rangle_{Y',Y}\mathrm{d}\rho_s(u)\mathrm{d}s$$

Definition

A **projected statistical solution** is a phase-space statistical solution $\{\rho_t\}_t$ such that $\rho_t = \Pi_t\rho$, for a trajectory statistical solution ρ.

Projection of trajectory statistical solutions

Theorem (Bronzi-Mondaini-Rosa (2015?))

Suppose

1. $\mathfrak{B}(Y) \subset \mathfrak{B}(X)$;
2. ρ *is a* $\mathcal{U}$*-trajectory statistical solution;* $\rho(\mathcal{V}) = 1$, $\mathcal{V} \subset \mathcal{U}$ *Borel in* $\mathcal{X}$;
3. $F : I \times Y \to Y'$ *is an* $(\mathfrak{L}(I) \otimes \mathfrak{B}(Y), \mathfrak{B}(Y'))$*-measurable function such that*
$$t \mapsto \int_{\mathcal{V}} \|F(t, u(t))\|_{Y'} \mathrm{d}\rho(u) \in L^1_{loc}(I),$$

Then, $\{\rho_t\}_t$ *is a projected statistical solution, where* $\rho_t = \Pi_t \rho$.

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Remark: I second countable $\Rightarrow \mathfrak{B}(I \times \mathcal{V}) = \mathfrak{B}(I) \otimes \mathfrak{B}(\mathcal{V}) \Rightarrow$ Nemytskii operator $(t, u) \mapsto \tilde{F}(t, u) = F(t, u(t))$ is $(\mathfrak{L}(I) \otimes \mathcal{V}, \mathfrak{B}(Y'))$-measurable

Existence of a projected trajectory statistical solutions

Theorem (Bronzi-Mondaini-Rosa (2015?))

Suppose

1. $\mathfrak{B}(Y) \subset \mathfrak{B}(X)$;
2. $F : I \times Y \to Y'$ *is an* $(\mathfrak{L}(I) \otimes \mathfrak{B}(Y), \mathfrak{B}(Y'))$*-measurable function such that*
$$\int_{t_0}^{t} \|F(s, u(s))\|_{Y'} \mathrm{d}s \leq \gamma(t, u(t_0)), \quad \forall t \in I, \ \forall u \in \mathcal{U},$$
for some $(\mathfrak{L}(I) \times \mathfrak{B}(X))$*-measurable function* $\gamma : I \times X \to \mathbb{R}$;
3. μ_0 *is a tight Borel probability measure with*
$$\int_X \gamma(t, u_0) \, \mathrm{d}\mu_0(u_0) < \infty.$$

Then, $\exists$ *projected statistical solution* $\{\rho_t\}_t$ *of* $u_t = F(t, u)$ *with* $\rho_{t_0} = \mu_0$.

Energy inequalities

Theorem (Bronzi-Mondaini-Rosa (2015?))

Suppose $\alpha : I \times X \to \mathbb{R}$ *and* $\beta : I \times Y \to \mathbb{R}$ *satisfy*

1. $(t, u) \mapsto \alpha(t, u(t)) \in L^1(J \times \mathcal{V}, \lambda \times \rho)$, $\forall J \subset I$ *compact;*
2. $(t, u) \mapsto \beta(t, u(t)) \in L^1(J \times \mathcal{V}, \lambda \times \rho)$, $\forall J \subset I$ *compact;*
3. *For* ρ*-almost every* $u \in \mathcal{V}$ *and for all* $\varphi \in \mathcal{C}_c^\infty(I, \mathbb{R})$, $\varphi \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t, u(t)) + \beta(t, u(t)) \leq 0,$$

in the sense of distributions on I*.*

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \alpha(t, u(t))\,\mathrm{d}\rho(u) + \int_{\mathcal{V}} \beta(t, u(t))\,\mathrm{d}\rho(u) \leq 0,$$

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Incompressible 3D Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \nu \Delta \mathbf{u} + \mathbf{f}, \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \end{cases}$$

No-slip boundary conditions on a smooth, bounded domain or space-periodic conditions with zero average.

- $H = \{L^2$ vector fields, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, boundary cond$\}$;
- $V = \{H^1$ vector fields, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, boundary cond$\}$;
- $V' =$ dual of V (with $V \subset H = H' \subset V'$);
- $D(A^s) =$ domain of powers of the Stokes operator, $s \in \mathbb{R}$;
- $H_{\mathrm{w}} = H$ endowed with weak topology.
- $\mathbf{f} \in L^2_{\mathrm{loc}}(0, \infty; V')$

Framework for NSE

- $I = [0, \infty)$, $X = H_{\mathrm{w}}$, $Y = V$, $Y' = V'$
- $\mathcal{U} = \{$ weak solutions of the NSE on $[0, \infty)\}$
- $F(t, u) = \mathbf{f}(t) - \nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u}) : Y \to Y'$
- $\mathfrak{K}'(X) = \{K;\ K$ strongly compact in $H,\ n \in \mathbb{N}\}$

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Conditions for existence of trajectory statistical solution

- $\exists$ global weak solutions for initial conditions in $H \Rightarrow \Pi_0\mathcal{U} = X$.
- Galerkin projection $\Rightarrow$ any tight μ_0 in H is regular w.r.t. $\mathfrak{K}'(X)$
- A priori estimates $\Rightarrow \mathcal{U} \cap \Pi_0^{-1}K$ is compact in $\mathcal{U}$, $\forall K \in \mathfrak{K}(X)$

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Conditions for existence of phase-space statistical solution

- $F : I \times Y \to Y'$ is continuous, hence measurable
- $\mathfrak{B}(V) = \mathfrak{B}(H) \cap V \subset \mathfrak{B}(H)$
- $\forall \mathbf{u} \in \mathcal{U}$, $\mathbf{u} \in \mathcal{C}_{\mathrm{loc}}(I, X) \cap L^2_{\mathrm{loc}}(I; Y)$, $\mathbf{u}_t \in L^{4/3}_{\mathrm{loc}}(I; Y')$, thus

  $$\mathcal{U} \subset \mathcal{Z} = \{u \in \mathcal{C}_{\mathrm{loc}}(I, X) \cap W^{1,1}_{\mathrm{loc}}(I, Y');\ u(t) \in Y \text{ for almost all } t \in I\}$$

- $\int_{t_0}^t \|\mathbf{u}_t\|_{V'}\,\mathrm{d}t \leq \frac{c}{\nu^{3/4}}\|\mathbf{u}_0\|^2_{L^2} + C(\nu, \lambda_1, \mathbf{f})$
- $\int_H \|\mathbf{u}_0\|^2_{L^2(\Omega)}\,\mathrm{d}\mu_0(\mathbf{u}_0) < \infty$

Reaction-Diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - f(u,t) + g(t), \\ u|_{\partial\Omega} = 0, \end{cases}$$

- Domain $\Omega \subset \mathbb{R}^n$ bounded, with smooth boundary $\partial\Omega$
- $g \in L^2_{\mathrm{loc}}(I; V')$, $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$
- $\eta|v|^p - C_1 \leq f(v,s)v$, some $p \geq 2$
- $|f(v,s)|^{\frac{p}{p-1}} \leq C_2(|v|^p + 1)$, some $\eta > 0$

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Global weak solutions for $u_0 \in L^2(\Omega)$ (Chepyzhov & Vishik (2000)):

- $\mathbf{u} \in L^p_{\mathrm{loc}}(I, L^p(\Omega)) \cap L^2_{\mathrm{loc}}(I, H^1_0(\Omega)) \cap \mathcal{C}_{\mathrm{loc}}(I, H)$
- $\mathbf{u}_t \in L^q_{\mathrm{loc}}(I, H^{-r}(\Omega))$, $1/p + 1/q = 1$, $r \geq \max\{1, n(1/2 - 1/p)\}$

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Use

- $X = L^2(\Omega)$, $Y = H^1_0(\Omega) \cap L^p(\Omega)$
- $\mathcal{U} = \{$weak solutions on $[0,\infty)\}$
- $\Pi_{t_0}\mathcal{U} = X$; $\mathcal{U} \cap \Pi_{t_0}^{-1}K$ compact for every $K \subset X$ compact
- $\int_X \|u_0\|^2_{L^2(\Omega)} \,\mathrm{d}\mu(u_0) < \infty$

Nonlinear wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^r u = f, \\ u|_{\partial\Omega} = 0. \end{cases}$$

- Domain $\Omega \subset \mathbb{R}^n$ bounded, with smooth boundary $\partial\Omega$
- $f \in L^2_{\mathrm{loc}}(I; L^2(\Omega))$ and $r > 0$ (critical value is $2/(n-2)$)

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- $u_0 \in H^1_0(\Omega) \cap L^p(\Omega)$, $p = r + 2$, $u_{0t} \in L^2(\Omega)$
- $\mathbf{u} \in L^\infty_{\mathrm{loc}}(I, H^1_0(\Omega) \cap L^p(\Omega)) \cap \mathcal{C}_{\mathrm{loc}}(I, (H^1_0(\Omega) \cap L^p(\Omega))_{\mathrm{w}})$
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Write as a first order system for $U = (u, u_t)$ and use

- $X = (H^1_0(\Omega) \cap L^p(\Omega))_{\mathrm{w}} \times L^2(\Omega)_{\mathrm{w}}$
- $Y = H^1_0(\Omega) \cap L^p(\Omega) \times L^2(\Omega)$
- $\mathcal{U} = \{\text{weak solutions on } [0, \infty)\}$
- $\mathcal{U} \cap \Pi_{t_0}^{-1} K$ compact for every $K \subset Y$ (strongly) compact
- $\int_X \|u_0\|^2_{L^2(\Omega)} \, \mathrm{d}\mu(u_0) < \infty$

Approximate statistical solutions

- **Alternate proof of existence:** By Galerkin approximation, as the weak-star limit of

$$\Lambda_m(\varphi) = \int_{B_H(R_0)} \varphi(S_m(\cdot)\mathbf{u}_0)\,\mathrm{d}\mu_0(\mathbf{u}_0), \quad \forall \varphi \in \mathcal{C}(\mathcal{Y}(R_0)),$$

  for μ_0 restricted to $B_H(R_0)$, $R_0 \to \infty$, where $\mathcal{Y}(R_0)$ is a compact subset of $\mathcal{C}([0,T], B_H(R)_{\mathrm{w}}) \cap L^2(0,T;H)$ based on the a priori estimates.

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- **General idea of approximation:** For any approximation with

$$\mathcal{U}_\varepsilon \to \mathcal{U}$$

in a "reasonable" sense, get

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho,$$

with

$$\rho(\mathcal{U}) = 1.$$

Thank you for your attention!