On the convergence of 2D second-grade fluid equations to Euler equations in bounded domains

Helena J. Nussenzveig Lopes

Instituto de Matemática, Universidade Federal do Rio de Janeiro

Workshop I: Mathematical Analysis of Turbulence
IPAM, Los Angeles, Sept. 29 to Oct. 3, 2014.

Collaborators:

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Dragoş Iftimie (Univ. Lyon)

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Dragoş Iftimie (Univ. Lyon)

Milton Lopes Filho (IM-UFRJ)

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Dragoş Iftimie (Univ. Lyon)

Milton Lopes Filho (IM-UFRJ)

Edriss Titi (Weizmann Institute and Texas A & M Univ.)

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Dragoş Iftimie (Univ. Lyon)

Milton Lopes Filho (IM-UFRJ)

Edriss Titi (Weizmann Institute and Texas A & M Univ.)

Aibin Zang (Yichun University, China)

Collaborators:

A. Valentina Busuioc (Univ. Lyon)

Dragoş Iftimie (Univ. Lyon)

Milton Lopes Filho (IM-UFRJ)

Edriss Titi (Weizmann Institute and Texas A & M Univ.)

Aibin Zang (Yichun University, China)

Introduction

The Euler-α equations are given by

Introduction

The Euler-α equations are given by

$$\left\{ \right.$$

Introduction

The Euler-α equations are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \\ \\ \end{cases}$$

Introduction

The Euler-α equations are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \\ \text{div } u = 0, \\ \\ \end{cases}$$

Introduction

The Euler-α equations are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \text{div } u = 0, \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Introduction

The Euler-α equations are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Above, p is the (scalar) pressure and v is the (unfiltered) velocity

Introduction

The Euler-α equations are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \\ \operatorname{div} u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Above, p is the (scalar) pressure and v is the (unfiltered) velocity (and $u = (I - \alpha^2 \Delta)^{-1} v$ is the filtered velocity).

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

1. Euler-α is a *regularization* of the Euler equations;

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

1. Euler-α is a *regularization* of the Euler equations;
2. Inviscid sub-grid scale model in turbulence;

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

1. Euler-α is a *regularization* of the Euler equations;
2. Inviscid sub-grid scale model in turbulence;
3. Solutions are geodesics in space of volume-preserving diffeomorphisms with H^1_α-metric ($\sqrt{\|u\|^2 + \alpha^2\|\nabla u\|^2}$);

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

1. Euler-α is a *regularization* of the Euler equations;

2. Inviscid sub-grid scale model in turbulence;

3. Solutions are geodesics in space of volume-preserving diffeomorphisms with H^1_α-metric ($\sqrt{\|u\|^2 + \alpha^2\|\nabla u\|^2}$);

4. Zero-viscosity second-grade fluid model;

Euler-α aka *LAE-α – Lagrangian Averaged Euler*:

1. Euler-α is a *regularization* of the Euler equations;
2. Inviscid sub-grid scale model in turbulence;
3. Solutions are geodesics in space of volume-preserving diffeomorphisms with H^1_α-metric ($\sqrt{\|u\|^2 + \alpha^2\|\nabla u\|^2}$);
4. Zero-viscosity second-grade fluid model;
5. In $2D$ it gives rise to a *vortex blob* method.

The second-grade fluid model is given by

The second-grade fluid model is given by

The second-grade fluid model is given by

$$\left\{ \begin{array}{l} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \\ \\ \end{array} \right.$$

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ \end{cases}$$

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α:

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side,

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side, instead of $\nu \Delta v$.

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side, instead of $\nu \Delta v$.

Second grade fluids:

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side, instead of $\nu \Delta v$.

Second grade fluids: special case of *differential type* of model for viscoelastic fluids.

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \operatorname{div} u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side, instead of $\nu \Delta v$.

Second grade fluids: special case of *differential type* of model for viscoelastic fluids.

Model subject to controversy in rheology.

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu \Delta u$ on right-hand-side, instead of $\nu \Delta v$.

Second grade fluids: special case of *differential type* of model for viscoelastic fluids.

Model subject to controversy in rheology. Nevertheless it is simple model,

The second-grade fluid model is given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \\ \\ \text{div } u = 0, \\ \\ v = u - \alpha^2 \Delta u. \end{cases}$$

Compare with LANS-α: $\nu\Delta u$ on right-hand-side, instead of $\nu\Delta v$.

Second grade fluids: special case of *differential type* of model for viscoelastic fluids.

Model subject to controversy in rheology. Nevertheless it is simple model, mathematically interesting.

Second grade equations can be written as

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect)

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect) *only* infinitesimal past.

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect) *only* infinitesimal past.

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha^2 A^2 + \alpha^2(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A \nabla u), \tag{2}$$

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \rightarrow$ memory (elastic effect) *only* infinitesimal past.

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha^2 A^2 + \alpha^2(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A \nabla u), \tag{2}$$

$$A = A(u) = \nabla u + (\nabla u)^t.$$

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect) *only* infinitesimal past.

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha^2 A^2 + \alpha^2(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A\nabla u), \tag{2}$$

$$A = A(u) = \nabla u + (\nabla u)^t.$$

ν is viscosity; α^2 is material parameter,

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect) *only* infinitesimal past.

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha^2 A^2 + \alpha^2(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A \nabla u), \tag{2}$$

$$A = A(u) = \nabla u + (\nabla u)^t.$$

ν is viscosity; α^2 is material parameter, encodes elastic response.

Second grade equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = \operatorname{div} \mathbb{S}, \\ \operatorname{div} u = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\mathbb{S}$ is the stress tensor.

$\mathbb{S} \to$ memory (elastic effect) *only* infinitesimal past.

$$\mathbb{S} = -p\mathbb{I} + \nu A - \alpha^2 A^2 + \alpha^2(\partial_t A + u \cdot \nabla A + (\nabla u)^t A + A \nabla u), \tag{2}$$

$$A = A(u) = \nabla u + (\nabla u)^t.$$

ν is viscosity; α^2 is material parameter, encodes elastic response.

OBS. The term $u \cdot \nabla A$ contains second-order derivative of u.

Question addressed here:

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$?

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$?

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α. Consider *same boundary condition* as for second grade –

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α. Consider *same boundary condition* as for second grade – mathematically interesting.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α. Consider *same boundary condition* as for second grade – mathematically interesting.

For Euler, *non-penetration*

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α. Consider *same boundary condition* as for second grade – mathematically interesting.

For Euler, *non-penetration* $u \cdot \hat{n} = 0$ on $\partial\Omega$.

Question addressed here: is solution to Euler approximated by Euler-α solutions, as $\alpha \to 0$? By second grade flows, as $\nu \to 0$? "OK" in full space/full plane.

Interested in *boundary effects*. Study second grade/Euler-α in a bounded domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

Study two different kinds of boundary condition for second grade:

(1) *Wall slip* i.e. Navier boundary condition This will be baseline result.

(2) *No slip* i.e. $u = 0$ on $\partial\Omega$.

No physically meaningful boundary condition for Euler-α. Consider *same boundary condition* as for second grade – mathematically interesting.

For Euler, *non-penetration* $u \cdot \hat{n} = 0$ on $\partial\Omega$.

Motivation: inviscid limit for Navier-Stokes, bounded domain.

Some background

Early results Well-posedness (local/large data, global/small data) for second grade:

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip.

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993,

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997,

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997, Galdi & Sequeira, 1994,

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997, Galdi & Sequeira, 1994, Busuioc, 2002, 2004.

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997, Galdi & Sequeira, 1994, Busuioc, 2002, 2004.

Busuioc, 2002 Vanishing viscosity, fixed α, second grade to Euler-α. Full space.

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997, Galdi & Sequeira, 1994, Busuioc, 2002, 2004.

Busuioc, 2002 Vanishing viscosity, fixed α, second grade to Euler-α. Full space.

Busuioc-Ratiu, 2003 Well-posedness for second grade and Euler-α, bounded domain, Navier slip conditions.

Some background

Early results Well-posedness (local/large data, global/small data) for second grade: Cioranescu & Ouazar, 1984, bounded domain, no slip. Also Galdi & Grobbelaar-Van Dalsen & Sauer, 1993, Cioranescu & Girault, 1997, Galdi & Sequeira, 1994, Busuioc, 2002, 2004.

Busuioc, 2002 Vanishing viscosity, fixed α, second grade to Euler-α. Full space.

Busuioc-Ratiu, 2003 Well-posedness for second grade and Euler-α, bounded domain, Navier slip conditions.

Iftimie, 2002 Convergence, fixed $\nu > 0$, $\alpha \to 0$, of second grade fluid to NS. H^1 data, convergence is weak in L^2. Full space.

Bardos-Linshiz-Titi, 2008 Global existence, 2D, Euler-α with vortex sheet initial data. Full plane. No Kelvin-Helmholtz.

Bardos-Linshiz-Titi, 2008 Global existence, 2D, Euler-α with vortex sheet initial data. Full plane. No Kelvin-Helmholtz.

Bardos-Linshiz-Titi, 2009 Convergence, $\alpha \to 0$, Euler-α with vortex sheet data (Birkhoff-Rott-α).

Bardos-Linshiz-Titi, 2008 Global existence, 2D, Euler-α with vortex sheet initial data. Full plane. No Kelvin-Helmholtz.

Bardos-Linshiz-Titi, 2009 Convergence, $\alpha \to 0$, Euler-α with vortex sheet data (Birkhoff-Rott-α).

Jiu-Niu-Titi-Xin, 2009 Global existence for Euler-α, axisymmetric no swirl, $\frac{1}{r}$ curl v_0 Radon measure. Uniqueness $\frac{1}{r}$ curl $v_0 \in L^p_c(\mathbb{R}^3)$ with $p > 3/2$. Full space.

Bardos-Linshiz-Titi, 2008 Global existence, 2D, Euler-α with vortex sheet initial data. Full plane. No Kelvin-Helmholtz.

Bardos-Linshiz-Titi, 2009 Convergence, $\alpha \to 0$, Euler-α with vortex sheet data (Birkhoff-Rott-α).

Jiu-Niu-Titi-Xin, 2009 Global existence for Euler-α, axisymmetric no swirl, $\frac{1}{r}$ curl v_0 Radon measure. Uniqueness $\frac{1}{r}$ curl $v_0 \in L^p_c(\mathbb{R}^3)$ with $p > 3/2$. Full space.

Linshiz-Titi, 2010 Convergence and rate-of-convergence, $\alpha \to 0$, full space. Data (v_0) in H^3.

Bardos-Linshiz-Titi, 2008 Global existence, 2D, Euler-α with vortex sheet initial data. Full plane. No Kelvin-Helmholtz.

Bardos-Linshiz-Titi, 2009 Convergence, $\alpha \to 0$, Euler-α with vortex sheet data (Birkhoff-Rott-α).

Jiu-Niu-Titi-Xin, 2009 Global existence for Euler-α, axisymmetric no swirl, $\frac{1}{r}$ curl v_0 Radon measure. Uniqueness $\frac{1}{r}$ curl $v_0 \in L^p_c(\mathbb{R}^3)$ with $p > 3/2$. Full space.

Linshiz-Titi, 2010 Convergence and rate-of-convergence, $\alpha \to 0$, full space. Data (v_0) in H^3.

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Our context: flow in bounded domain Ω, smooth boundary $\partial\Omega$.

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Our context: flow in bounded domain Ω, smooth boundary $\partial\Omega$.

Boundary condition: Navier (friction) boundary condition.

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Our context: flow in bounded domain Ω, smooth boundary $\partial\Omega$.

Boundary condition: Navier (friction) boundary condition.

Tangential velocity proportional to surface shear stress at boundary, i.e.,

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Our context: flow in bounded domain Ω, smooth boundary $\partial\Omega$.

Boundary condition: Navier (friction) boundary condition.

Tangential velocity proportional to surface shear stress at boundary, i.e.,

$$(\mathbb{S}\hat{n} + \gamma u)_{tan} = 0, \tag{3}$$

Bounded domain flow, Navier conditions

Joint work with A. V. Busuioc (Univ. Lyon), D. Iftimie (Univ. Lyon) and M. C. Lopes Filho (IM-UFRJ).

Our context: flow in bounded domain Ω, smooth boundary $\partial\Omega$.

Boundary condition: Navier (friction) boundary condition.

Tangential velocity proportional to surface shear stress at boundary, i.e.,

$$(\mathbb{S}\hat{n} + \gamma u)_{tan} = 0, \tag{3}$$

Above, γ = friction coefficient and $\hat{n}$ is unit normal to boundary.

Special case: $\gamma = 0$ = perfect slip.

Special case: $\gamma = 0$ = perfect slip.

Too complicated to deal with this Navier condition.

Special case: $\gamma = 0$ = perfect slip.

Too complicated to deal with this Navier condition.

Better to work with *Newtonian* (linear) Navier condition:

Special case: $\gamma = 0$ = perfect slip.

Too complicated to deal with this Navier condition.

Better to work with *Newtonian* (linear) Navier condition:

$$(A\hat{n})_{tan} = 0. \tag{4}$$

Special case: $\gamma = 0$ = perfect slip.

Too complicated to deal with this Navier condition.

Better to work with *Newtonian* (linear) Navier condition:

$$(A\hat{n})_{tan} = 0. \tag{4}$$

Background: previous work used this simpler Navier condition.

Relation between physical and simpler Navier conditions needs to be established.

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u \in C^\infty([0,T];\overline{\Omega})$ div–free, sth $u \cdot \hat{n} = 0$ on $[0,T] \times \partial\Omega$.

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u \in C^\infty([0,T];\overline{\Omega})$ *div–free, sth* $u \cdot \hat{n} = 0$ *on* $[0,T] \times \partial\Omega$.

Then

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u \in C^\infty([0,T];\overline{\Omega})$ *div–free, sth* $u \cdot \hat{n} = 0$ *on* $[0,T] \times \partial\Omega$.

Then

$$(A\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega \Longrightarrow (\mathbb{S}\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega.$$

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u \in C^\infty([0,T];\overline{\Omega})$ *div–free, sth* $u \cdot \hat{n} = 0$ *on* $[0,T] \times \partial\Omega$.

Then

$$(A\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega \Longrightarrow (\mathbb{S}\hat{n})_{tan} = 0 \qquad \text{on } [0,T] \times \partial\Omega.$$

If $[A(u_0)\hat{n}]_{tan} = 0$ *on* $\partial\Omega$, *where* $u_0 = u(0,x)$, *then*

Relation between physical and simpler Navier conditions needs to be established.

Subtle issue, because physical one has memory.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u \in C^\infty([0,T];\overline{\Omega})$ *div–free, sth* $u \cdot \hat{n} = 0$ *on* $[0,T] \times \partial\Omega$.

Then

$$(A\hat{n})_{tan} = 0 \qquad on\ [0,T] \times \partial\Omega \Longrightarrow (\mathbb{S}\hat{n})_{tan} = 0 \qquad on\ [0,T] \times \partial\Omega.$$

If $[A(u_0)\hat{n}]_{tan} = 0$ *on* $\partial\Omega$, *where* $u_0 = u(0,x)$, *then*

$$(\mathbb{S}\hat{n})_{tan} = 0 \qquad on\ [0,T] \times \partial\Omega \Longrightarrow (A\hat{n})_{tan} = 0 \qquad on\ [0,T] \times \partial\Omega.$$

Idea of proof:

Idea of proof:

First statement.

Idea of proof:

First statement.

Decompose $A\hat{n} = \beta\hat{n} + w$, with w tangent to boundary.

Idea of proof:

First statement.

Decompose $A\hat{n} = \beta\hat{n} + w$, with w tangent to boundary.

If $(A\hat{n})_{tan} = 0$ on the boundary then can show

$$(\mathbb{S}\hat{n})_{tan} = \alpha^2(u \cdot \nabla A + (\nabla u)^t A + A\nabla u)\hat{n}).$$

Idea of proof:

First statement.

Decompose $A\hat{n} = \beta\hat{n} + w$, with w tangent to boundary.

If $(A\hat{n})_{tan} = 0$ on the boundary then can show

$$(\mathbb{S}\hat{n})_{tan} = \alpha^2(u \cdot \nabla A + (\nabla u)^t A + A\nabla u)\hat{n}).$$

... many calculations later ... $(\mathbb{S}\hat{n})_{tan} = 0$.

Converse.

Converse.

As before, $A\hat{n} = \beta\hat{n} + w$.

Converse.

As before, $A\hat{n} = \beta\hat{n} + w$.

Deduce equation for w:

$$\partial_t w + C(\nu, \alpha) w + [u \cdot \nabla w + (\nabla u)^t w]_{tan} = 0.$$

Converse.

As before, $A\hat{n} = \beta\hat{n} + w$.

Deduce equation for w:

$$\partial_t w + C(\nu, \alpha)w + [u \cdot \nabla w + (\nabla u)^t w]_{tan} = 0.$$

Energy estimates give: $w_0 = 0 \Longrightarrow w(t, \cdot) = 0$,
as desired. ■

Converse.

As before, $A\hat{n} = \beta\hat{n} + w$.

Deduce equation for w:

$$\partial_t w + C(\nu, \alpha) w + [u \cdot \nabla w + (\nabla u)^t w]_{tan} = 0.$$

Energy estimates give: $w_0 = 0 \Longrightarrow w(t, \cdot) = 0$,
as desired. ∎

Remark. Known existence results are for simpler Navier boundary condition. Theorem provides dictionary, but for *well-prepared data*.

IBVP

The initial-boundary-value-problem we solve is:

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \quad \text{in } (0,T) \times \Omega, \\ \\ \\ \\ \end{cases}$$

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } [0, T] \times \Omega \\ \\ \\ \\ \end{cases}$$

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } [0, T] \times \Omega \\ (\mathbb{S}\hat{n})_{tan} = 0, & \text{on } [0, T] \times \partial\Omega, \\ & \\ & \end{cases}$$

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } [0,T] \times \Omega \\ (\mathbb{S}\hat{n})_{tan} = 0, & \text{on } [0,T] \times \partial\Omega, \\ v = u - \alpha^2 \Delta u, & \text{in } [0,T] \times \Omega, \end{cases}$$

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u\cdot\nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu\Delta u, & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0, & \text{in } [0,T]\times\Omega \\ (\mathbb{S}\hat{n})_{tan} = 0, & \text{on } [0,T]\times\partial\Omega, \\ v = u - \alpha^2\Delta u, & \text{in } [0,T]\times\Omega, \\ u(0,\cdot) = u_0(\cdot), & \text{at } \{t=0\}. \end{cases} \tag{5}$$

IBVP

The initial-boundary-value-problem we solve is:

$$\begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } [0,T] \times \Omega \\ (\mathbb{S}\hat{n})_{tan} = 0, & \text{on } [0,T] \times \partial\Omega, \\ v = u - \alpha^2 \Delta u, & \text{in } [0,T] \times \Omega, \\ u(0,\cdot) = u_0(\cdot), & \text{at } \{t = 0\}. \end{cases} \tag{5}$$

Assume $(A(u_0)\hat{n})_{tan} = 0$. Also, notice there are three derivatives on u $\to \nabla\Delta u$.

Let φ be a smooth (in x and t), div-free vector field, tangent to $\partial\Omega$. Multiply (5) by φ, integrate by parts, use the boundary condition $(A\hat{n})_{tan} = 0$ and the symmetry of $\mathbb{S}$, to get:

Let φ be a smooth (in x and t), div-free vector field, tangent to $\partial\Omega$. Multiply (5) by φ, integrate by parts, use the boundary condition $(A\hat{n})_{tan} = 0$ and the symmetry of $\mathbb{S}$, to get:

$$
\begin{gathered}
-\int_0^t \int_\Omega \left[u \cdot \partial_t \phi + \frac{\alpha^2}{2} A : \partial_t A(\phi) \right] + \int_\Omega \left[u(t) \cdot \phi(t) + \frac{\alpha^2}{2} A(t) \cdot A(\phi(t)) \right] \\
-\int_\Omega \left[u_0 \cdot \phi_0 + \frac{\alpha^2}{2} A(u_0) \cdot A(\phi_0) \right] + \int_0^t \int_\Omega u \cdot \nabla u \cdot \phi + \frac{\nu}{2} \int_0^t \int_\Omega A : A(\phi) \\
-\frac{\alpha^2}{2} \int_0^t \int_\Omega A^2 : A(\phi) - \frac{\alpha^2}{2} \int_0^t \int_\Omega u \cdot \nabla A(\phi) : A \\
+\frac{\alpha^2}{2} \int_0^t \int_\Omega \left[(\nabla u)^t A + A \nabla u \right] : A(\phi) = 0, \qquad (*)
\end{gathered}
$$

for all times t.

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0,T]; H^1(\Omega))$;

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0, T]; H^1(\Omega))$;
2. u is divergence free and tangent to the boundary;

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0, T]; H^1(\Omega))$;
2. u is divergence free and tangent to the boundary;
3. relation $(*)$ holds true for all vector fields $\phi \in C^0([0, T]; H^2(\Omega)) \cap C^1([0, T]; H^1(\Omega))$ which are divergence free and tangent to the boundary;

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0,T]; H^1(\Omega))$;
2. u is divergence free and tangent to the boundary;
3. relation $(*)$ holds true for all vector fields $\phi \in C^0([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega))$ which are divergence free and tangent to the boundary;
4. the following energy inequality holds true

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0,T]; H^1(\Omega))$;
2. u is divergence free and tangent to the boundary;
3. relation $(*)$ holds true for all vector fields $\phi \in C^0([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega))$ which are divergence free and tangent to the boundary;
4. the following energy inequality holds true

$$\int_\Omega \left(|u|^2 + \frac{\alpha^2}{2}|A|^2 \right)(t) + \nu \int_0^t \int_\Omega |A|^2 \leq \int_\Omega |u_0|^2 + \frac{\alpha^2}{2}|A(u_0)|^2, \quad (6)$$

Weak H^1 formulation

Definition

We say that u is a weak H^1 solution of the second grade fluid equations with perfect slip Navier boundary conditions on the time interval $[0, T]$ if and only if

1. $u \in C^0_w([0,T]; H^1(\Omega))$;
2. u is divergence free and tangent to the boundary;
3. relation $(*)$ holds true for all vector fields $\phi \in C^0([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega))$ which are divergence free and tangent to the boundary;
4. the following energy inequality holds true

$$\int_\Omega \left(|u|^2 + \frac{\alpha^2}{2}|A|^2 \right)(t) + \nu \int_0^t \int_\Omega |A|^2 \leq \int_\Omega |u_0|^2 + \frac{\alpha^2}{2}|A(u_0)|^2, \quad (6)$$

$\forall t \in [0, T]$.

Our convergence result is stated below.

Our convergence result is stated below.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u_0 \in H^3(\Omega)$ be div-free, tangent to $\partial\Omega$ and assume $(A(u_0)\hat{n})_{tan} = 0$. Let $T > 0$ be sth $u^E \in C^0([0,T];H^3) \cap C^1((0,T);H^2)$ is Euler solution, initial data u_0.
Suppose, additionally, there exists an H^1 weak solution $u^{\nu,\alpha}$ of (5), up to time T. Then

Our convergence result is stated below.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u_0 \in H^3(\Omega)$ be div-free, tangent to $\partial\Omega$ and assume $(A(u_0)\hat{n})_{tan} = 0$. Let $T > 0$ be sth $u^E \in C^0([0, T]; H^3) \cap C^1((0, T); H^2)$ is Euler solution, initial data u_0.

Suppose, additionally, there exists an H^1 weak solution $u^{\nu,\alpha}$ of (5), up to time T. Then

$$\lim_{\nu,\alpha\to 0} \|u^{\nu,\alpha} - u^E\|_{L^\infty(0,T;L^2(\Omega))} = 0.$$

Our convergence result is stated below.

Theorem (Busuioc, Iftimie, Lopes Filho, NL, 2010)

Let $u_0 \in H^3(\Omega)$ be div-free, tangent to $\partial\Omega$ and assume $(A(u_0)\hat{n})_{tan} = 0$. Let $T > 0$ be sth $u^E \in C^0([0,T]; H^3) \cap C^1((0,T); H^2)$ is Euler solution, initial data u_0.
Suppose, additionally, there exists an H^1 weak solution $u^{\nu,\alpha}$ of (5), up to time T. Then

$$\lim_{\nu,\alpha\to 0} \|u^{\nu,\alpha} - u^E\|_{L^\infty(0,T;L^2(\Omega))} = 0.$$

OBS. Note that limits in $\nu \to 0$ and $\alpha \to 0$ taken independently.

Idea of proof:

Idea of proof:

Write $w = u^{\nu,\alpha} - u^E$. Formal PDE for w is:

Idea of proof:

Write $w = u^{\nu,\alpha} - u^E$. Formal PDE for w is:

$$\partial_t w + (w \cdot \nabla) w + (w \cdot \nabla) u^E + (u^E \cdot \nabla) w = \mathrm{div}(\mathbb{S}) + \nabla \overline{p}.$$

Idea of proof:

Write $w = u^{\nu,\alpha} - u^E$. Formal PDE for w is:

$$\partial_t w + (w \cdot \nabla) w + (w \cdot \nabla) u^E + (u^E \cdot \nabla) w = \mathrm{div}(\mathbb{S}) + \nabla \overline{p}.$$

Cannot multiply by w and integrate – w only H^1, not enough regularity (problem with term $w\mathbb{S}$).

Idea of proof:

Write $w = u^{\nu,\alpha} - u^E$. Formal PDE for w is:

$$\partial_t w + (w \cdot \nabla) w + (w \cdot \nabla) u^E + (u^E \cdot \nabla) w = \mathrm{div}(\mathbb{S}) + \nabla \overline{p}.$$

Cannot multiply by w and integrate – w only H^1, not enough regularity (problem with term $w\mathbb{S}$).

Instead: we start from the *energy inequality* in the Definition of weak solution, together with energy estimates for the limit equation.

Observe: PDE(w) = PDE($u^{\nu,\alpha}$) - PDE(u^E).

Observe: PDE(w) = PDE($u^{\nu,\alpha}$) - PDE(u^E).

Hence

$$w \cdot \mathrm{PDE}(w) = (u^{\nu,\alpha} - u^E) \cdot (\mathrm{PDE}(u^{\nu,\alpha}) - \mathrm{PDE}(u^E))$$

Observe: PDE(w) = PDE($u^{\nu,\alpha}$) - PDE(u^E).

Hence

$$w \cdot \mathrm{PDE}(w) = (u^{\nu,\alpha} - u^E) \cdot (\mathrm{PDE}(u^{\nu,\alpha}) - \mathrm{PDE}(u^E))$$

$$= u^{\nu,\alpha} \cdot \mathrm{PDE}(u^{\nu,\alpha}) - u^{\nu,\alpha} \cdot \mathrm{PDE}(u^E) - u^E \cdot \mathrm{PDE}(u^{\nu,\alpha}) + u^E \cdot \mathrm{PDE}(u^E).$$

Observe: PDE(w) = PDE($u^{\nu,\alpha}$) - PDE(u^E).

Hence

$$w \cdot \mathrm{PDE}(w) = (u^{\nu,\alpha} - u^E) \cdot (\mathrm{PDE}(u^{\nu,\alpha}) - \mathrm{PDE}(u^E))$$

$$= u^{\nu,\alpha} \cdot \mathrm{PDE}(u^{\nu,\alpha}) - u^{\nu,\alpha} \cdot \mathrm{PDE}(u^E) - u^E \cdot \mathrm{PDE}(u^{\nu,\alpha}) + u^E \cdot \mathrm{PDE}(u^E).$$

We use this idea, substituting $u^{\nu,\alpha} \cdot \mathrm{PDE}(u^{\nu,\alpha})$ by energy inequality. Let H^1_α be H^1 with the norm

Observe: PDE(w) = PDE($u^{\nu,\alpha}$) - PDE(u^E).

Hence

$$w \cdot \text{PDE}(w) = (u^{\nu,\alpha} - u^E) \cdot (\text{PDE}(u^{\nu,\alpha}) - \text{PDE}(u^E))$$

$$= u^{\nu,\alpha} \cdot \text{PDE}(u^{\nu,\alpha}) - u^{\nu,\alpha} \cdot \text{PDE}(u^E) - u^E \cdot \text{PDE}(u^{\nu,\alpha}) + u^E \cdot \text{PDE}(u^E).$$

We use this idea, substituting $u^{\nu,\alpha} \cdot \text{PDE}(u^{\nu,\alpha})$ by energy inequality. Let H^1_α be H^1 with the norm

$$\|w\|^2_{H^1_\alpha(\Omega)} = \int_\Omega \left(|w|^2 + \frac{\alpha^2}{2} |A(w)|^2 \right).$$

The procedure above yields the estimate:

The procedure above yields the estimate:

$$\begin{aligned}\|w(t)\|^2_{H^1_\alpha(\Omega)} \leq & C\alpha^2\left(\|u^E\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|^2_{H^1(\Omega)}\right) \\ & + C\alpha T\|u_0\|_{H^1(\Omega)}\|u^E\|_{C^1([0,T];H^1(\Omega))} \\ & + C\nu T\|u^E\|^2_{L^\infty(0,T;H^1(\Omega))} \\ & + C\alpha^2 T\|u^E\|^3_{L^\infty(0,T;H^3(\Omega))} \\ & + C\|u^E\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t \|w\|^2_{H^1_\alpha(\Omega)}\end{aligned}$$

The procedure above yields the estimate:

$$
\begin{aligned}
\|w(t)\|^2_{H^1_\alpha(\Omega)} \leq & C\alpha^2\left(\|u^E\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u_0\|^2_{H^1(\Omega)}\right) \\
& + C\alpha T\|u_0\|_{H^1(\Omega)}\|u^E\|_{C^1([0,T];H^1(\Omega))} \\
& + C\nu T\|u^E\|^2_{L^\infty(0,T;H^1(\Omega))} \\
& + C\alpha^2 T\|u^E\|^3_{L^\infty(0,T;H^3(\Omega))} \\
& + C\|u^E\|_{L^\infty(0,T;H^3(\Omega))}\int_0^t \|w\|^2_{H^1_\alpha(\Omega)} \\
\equiv & \; C_1(T,u^E,u_0)(\alpha^2+\alpha+\nu) + C_2(T,u^E)\int_0^t \|w\|^2_{H^1_\alpha(\Omega)}.
\end{aligned}
$$

By Gronwall's Lemma we find

By Gronwall's Lemma we find

$$\|w(t)\|^2_{H^1_\alpha} \leq (\alpha^2 + \alpha + \nu) C(T, u^E, u_0),$$

By Gronwall's Lemma we find

$$\|w(t)\|^2_{H^1_\alpha} \leq (\alpha^2 + \alpha + \nu) C(T, u^E, u_0),$$

which implies the desired conclusion. ■

By Gronwall's Lemma we find

$$\|w(t)\|^2_{H^1_\alpha} \leq (\alpha^2 + \alpha + \nu) C(T, u^E, u_0),$$

which implies the desired conclusion. ∎

OBS. No need for boundary corrector.

In 2D flow a uniform existence time has been established by Busuioc-Ratiu (2003), independent of ν and α.

In 2D flow a uniform existence time has been established by Busuioc-Ratiu (2003), independent of ν and α.

In 3D flow, we have global existence under special symmetry. We show:

In 2D flow a uniform existence time has been established by Busuioc-Ratiu (2003), independent of ν and α.

In 3D flow, we have global existence under special symmetry. We show:

Proposition

Suppose u_0 is axisymmetric, no swirl, belongs to H^3, is div-free and tangent to the boundary, and satisfies $(A(u_0)\hat{n})_{tan} = 0$. Suppose also $\frac{1}{r}\,\mathrm{curl}(u_0 - \alpha^2 \Delta u_0) \in L^2$. Then there exists a global H^3 no swirl axisymmetric solution of (5).

In 2D flow a uniform existence time has been established by Busuioc-Ratiu (2003), independent of ν and α.

In 3D flow, we have global existence under special symmetry. We show:

Proposition

Suppose u_0 is axisymmetric, no swirl, belongs to H^3, is div-free and tangent to the boundary, and satisfies $(A(u_0)\hat{n})_{tan} = 0$. Suppose also $\frac{1}{r}\,\mathrm{curl}(u_0 - \alpha^2 \Delta u_0) \in L^2$. Then there exists a global H^3 no swirl axisymmetric solution of (5).

Work in progress: show that solutions of Euler-α, bounded domain, Navier condition, with same initial data, exist on a time interval independent of α.

In 2D flow a uniform existence time has been established by Busuioc-Ratiu (2003), independent of ν and α.

In 3D flow, we have global existence under special symmetry. We show:

Proposition

Suppose u_0 is axisymmetric, no swirl, belongs to H^3, is div-free and tangent to the boundary, and satisfies $(A(u_0)\hat{n})_{tan} = 0$. Suppose also $\frac{1}{r}\,\mathrm{curl}(u_0 - \alpha^2 \Delta u_0) \in L^2$. Then there exists a global H^3 no swirl axisymmetric solution of (5).

Work in progress: show that solutions of Euler-α, bounded domain, Navier condition, with same initial data, exist on a time interval independent of α. Uses co-normal spaces.

Bounded domain, no slip, Euler-α 2D

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\left\{ \right.$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \quad \text{in } \Omega \times (0, \infty), \end{cases}$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, & \text{in } \Omega \times (0, \infty), \\ \\ \operatorname{div} u = 0, & \text{in } \Omega \times [0, \infty), \\ \\ \\ \\ \\ \end{cases}$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, & \text{in } \Omega \times (0, \infty), \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \\ & \\ & \end{cases}$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \\ u = 0, & \text{on } [0, T] \times \partial\Omega, \end{cases}$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \\ u = 0, & \text{on } [0, T] \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), & \text{at } \{t = 0\}. \end{cases}$$

Bounded domain, no slip, Euler-α 2D

Consider first Euler-α, bounded *planar* domain, with *Dirichlet condition*:

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \sum_j v_j \nabla u_j = -\nabla p, & \text{in } \Omega\times(0,\infty), \\ \text{div } u = 0, & \text{in } \Omega\times[0,\infty), \\ v = u - \alpha^2\Delta u, & \text{in } \Omega\times[0,\infty), \\ u = 0, & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot) = u_0(\cdot), & \text{at } \{t=0\}. \end{cases}$$

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999,

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

A priori estimates – 2D

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

A priori estimates – 2D

Solution u^α satisfies

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

A priori estimates – 2D

Solution u^α satisfies

$$\|u^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 = \|u_0^\alpha\|_{L^2}^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2$$

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

A priori estimates – 2D

Solution u^α satisfies

$$\|u^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 = \|u_0^\alpha\|_{L^2}^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2$$

and

If $u_0^\alpha \in H^3 \cap H_0^1$, div-free, $\exists\,!$ solution of Euler-α, global in time, see Cioranescu and Girault, 1994, Busuioc, 1999, Marsden, Ratiu and Shkoller, 2000.

A priori estimates – 2D

Solution u^α satisfies

$$\|u^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 = \|u_0^\alpha\|_{L^2}^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2$$

and

$$\|\text{curl } u^\alpha(t) - \alpha^2 \text{ curl } \Delta u^\alpha(t)\|_{L^2}^2 = \|\text{curl } u_0^\alpha - \alpha^2 \text{ curl } \Delta u_0^\alpha\|_{L^2}^2.$$

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$.

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0$, as $\alpha \to 0$, in $L^2(\Omega)$,

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0$, as $\alpha \to 0$, in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1})$, as $\alpha \to 0$, and

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0,$ as $\alpha \to 0,$ in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1}),$ as $\alpha \to 0,$ and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3}),$ as $\alpha \to 0.$

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0,$ as $\alpha \to 0,$ in $L^2(\Omega),$
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1}),$ as $\alpha \to 0,$ and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3}),$ as $\alpha \to 0.$

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0$, as $\alpha \to 0$, in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1})$, as $\alpha \to 0$, and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3})$, as $\alpha \to 0$.

Note. For any $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0$, as $\alpha \to 0$, in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1})$, as $\alpha \to 0$, and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3})$, as $\alpha \to 0$.

Note. For any $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$ can construct family of suitable approximations.

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0$, as $\alpha \to 0$, in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1})$, as $\alpha \to 0$, and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3})$, as $\alpha \to 0$.

Note. For any $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$ can construct family of suitable approximations.

For suitable approximations get:

Let $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Introduce *suitable* approximation:

1. u_0^α vanishes on $\partial\Omega$,
2. $u_0^\alpha \to u_0,$ as $\alpha \to 0,$ in $L^2(\Omega)$,
3. $\|\nabla u_0^\alpha\|_{L^2} = o(\alpha^{-1}),$ as $\alpha \to 0,$ and
4. $\|u_0^\alpha\|_{H^3} = \mathcal{O}(\alpha^{-3}),$ as $\alpha \to 0$.

Note. For any $u_0 \in H^3(\Omega)$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$ can construct family of suitable approximations.

For suitable approximations get:

$$\|u^\alpha(t)\|_{H^3} \leq \frac{C}{\alpha^3} \text{ and } \|\nabla u^\alpha(t)\|_{L^2} \leq \frac{C}{\alpha}.$$

With this setup, have the following result, vanishing α limit.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be suitable approximations.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be suitable approximations. Set $u^\alpha \in C([0,T]; H^3)$ solution to Euler-α with initial data u_0^α.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be suitable approximations. Set $u^\alpha \in C([0,T]; H^3)$ solution to Euler-α with initial data u_0^α. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0.

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be suitable approximations. Set $u^\alpha \in C([0,T]; H^3)$ solution to Euler-α with initial data u_0^α. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

With this setup, have the following result, vanishing α limit.

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be suitable approximations. Set $u^\alpha \in C([0,T]; H^3)$ solution to Euler-α with initial data u_0^α. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

$$\sup_{t\in[0,T]} \left(\|u^\alpha(t) - u^E(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha\|_{L^2}^2\right) \xrightarrow[\alpha\to 0]{} 0. \tag{7}$$

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$.

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W,

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W,

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω,

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω, then on $(0, T)$.

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω, then on $(0, T)$.
Obtain:

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω, then on $(0, T)$.
Obtain:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s,$$

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω, then on $(0, T)$.
Obtain:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ \int_0^t \int_\Omega \mathrm{div}\, \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s,$$

where

Proof in the spirit of Kato criterion for vanishing viscosity limit of Navier-Stokes.

Introduce $W^\alpha = u^\alpha - u^E$. Write equation for W, multiply by W, integrate in Ω, then on $(0, T)$.
Obtain:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ \int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s,$$

where

$$\operatorname{div} \sigma^\alpha = \alpha^2 \partial_t \Delta u^\alpha + \alpha^2 (u^\alpha \cdot \nabla)\Delta u^\alpha + \alpha^2 \sum_{j=1}^{2} (\Delta u_j^\alpha) \nabla u_j^\alpha.$$

Using the fact that

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \,\mathrm{d}x\,\mathrm{d}s = 0,$$

we find

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \, \mathrm{d}x \, \mathrm{d}s = 0,$$

we find

$$\int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s =$$

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \, \mathrm{d}x \, \mathrm{d}s = 0,$$

we find

$$\int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s = \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \,\mathrm{d}x\,\mathrm{d}s = 0,$$

we find

$$\begin{aligned} \int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^E \,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \, \mathrm{d}x \, \mathrm{d}s = 0,$$

we find

$$\begin{aligned} \int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s = \alpha^2 & \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\ - \alpha^2 & \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^E \, \mathrm{d}x \, \mathrm{d}s \\ - \alpha^2 & \int_0^t \int_\Omega \sum_{j=1}^{2} (\Delta u_j^\alpha) \nabla u_j^\alpha \cdot u^E \, \mathrm{d}x \, \mathrm{d}s \end{aligned}$$

Using the fact that

$$\alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s + \alpha^2 \int_0^t \int_\Omega \Delta u^\alpha \cdot [(u^\alpha \cdot \nabla) u^\alpha] \, \mathrm{d}x \, \mathrm{d}s = 0,$$

we find

$$\begin{aligned}
\int_0^t \int_\Omega \operatorname{div} \sigma^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \alpha^2 \int_0^t \int_\Omega [(u^\alpha \cdot \nabla)\Delta u^\alpha] \cdot u^E \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \alpha^2 \int_0^t \int_\Omega \sum_{j=1}^2 (\Delta u_j^\alpha) \nabla u_j^\alpha \cdot u^E \, \mathrm{d}x \, \mathrm{d}s \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

Summarizing, we have:

Summarizing, we have:

$$\frac{1}{2}\|W^{\alpha}(t)\|^2 =$$

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla) u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ I_1(t) + I_2(t) + I_3(t).$$

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ I_1(t) + I_2(t) + I_3(t).$$

Easy estimates:

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s$$
$$+I_1(t) + I_2(t) + I_3(t).$$

Easy estimates:

$$\boxed{-\int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \le K \int_0^t \|W^\alpha\|^2 \,\mathrm{d}s.}$$

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ I_1(t) + I_2(t) + I_3(t).$$

Easy estimates:

$$\boxed{- \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \leq K \int_0^t \|W^\alpha\|^2 \, \mathrm{d}s.}$$

Integration by parts + Ladyzhenskaya inequality yield

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s$$
$$+ I_1(t) + I_2(t) + I_3(t).$$

Easy estimates:

$$\boxed{-\int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \leq K \int_0^t \|W^\alpha\|^2 \, \mathrm{d}s.}$$

Integration by parts + Ladyzhenskaya inequality yield

$$\boxed{I_2(t) + I_3(t) \leq K\alpha^2 \left(\int_0^t \|\nabla u^\alpha\|^2 \, \mathrm{d}s + T \right).}$$

Summarizing, we have:

$$\frac{1}{2}\|W^\alpha(t)\|^2 = \frac{1}{2}\|W^\alpha(0)\|^2 - \int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s$$
$$+ I_1(t) + I_2(t) + I_3(t).$$

Easy estimates:

$$\boxed{-\int_0^t \int_\Omega [(W^\alpha \cdot \nabla)u^E] \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \le K \int_0^t \|W^\alpha\|^2 \,\mathrm{d}s.}$$

Integration by parts + Ladyzhenskaya inequality yield

$$\boxed{I_2(t) + I_3(t) \le K\alpha^2 \left(\int_0^t \|\nabla u^\alpha\|^2 \,\mathrm{d}s + T \right).}$$

Left with $I_1(t)$.

Must use boundary corrector v^δ –

Must use boundary corrector v^δ – Kato's boundary corrector:

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂,

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, φ

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂.

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$.

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E,

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$,

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector $v^\delta = \nabla^\perp(\varphi^\delta \psi)$.

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector $v^\delta = \nabla^\perp(\varphi^\delta \psi)$.

Estimates for v^δ:

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector $v^\delta = \nabla^\perp(\varphi^\delta \psi)$.

Estimates for v^δ:

$$\|v^\delta\|_{L^\infty(L^\infty)} = \mathcal{O}(1),$$

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp\psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector $v^\delta = \nabla^\perp(\varphi^\delta\psi)$.

Estimates for v^δ:

$$\|v^\delta\|_{L^\infty(L^\infty)} = \mathcal{O}(1),$$

$$\|v^\delta\|_{L^\infty(L^2)} = \mathcal{O}(\delta^{1/2}),$$

Must use boundary corrector v^δ – Kato's boundary corrector:

Choose smooth cutoff near ∂, $\varphi = 1$ in neighborhood of ∂. Set $\varphi^\delta(\cdot) = \varphi(\cdot/\delta)$. Let ψ be stream function for u^E, $u^E = \nabla^\perp \psi$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$.

Boundary corrector $v^\delta = \nabla^\perp(\varphi^\delta \psi)$.

Estimates for v^δ:

$$\|v^\delta\|_{L^\infty(L^\infty)} = \mathcal{O}(1),$$

$$\|v^\delta\|_{L^\infty(L^2)} = \mathcal{O}(\delta^{1/2}),$$

$$\|\nabla v^\delta\|_{L^\infty(L^2) = \mathcal{O}(\delta^{-1/2})}.$$

With this we can now perform the **key estimate**:

With this we can now perform the **key estimate**:

$I_1(t) =$

With this we can now perform the **key estimate**:

$$I_1(t) = \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s$$

$$=$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s
\end{aligned}
$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta) \,\mathrm{d}x\,\mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta) \, \mathrm{d}x \, \mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \, \mathrm{d}x \, \mathrm{d}s \\
&\leq -\frac{\alpha^2}{4} \|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2} \|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2 \, \mathrm{d}s \\
&- \alpha^2 \int_\Omega \nabla u_0^\alpha \cdot \nabla u_0 \, \mathrm{d}x +
\end{aligned}
$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta) \, \mathrm{d}x \, \mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \, \mathrm{d}x \, \mathrm{d}s \\
&\leq -\frac{\alpha^2}{4} \|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2} \|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2 \, \mathrm{d}s \\
&- \alpha^2 \int_\Omega \nabla u_0^\alpha \cdot \nabla u_0 \, \mathrm{d}x + K\alpha\delta^{-1/2} + K\alpha^2 + K\alpha^2\delta^{-1} + K\delta^{1/2}.
\end{aligned}
$$

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \, \mathrm{d}x \, \mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta) \, \mathrm{d}x \, \mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \, \mathrm{d}x \, \mathrm{d}s \\
&\leq -\frac{\alpha^2}{4} \|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2} \|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2 \, \mathrm{d}s \\
&- \alpha^2 \int_\Omega \nabla u_0^\alpha \cdot \nabla u_0 \, \mathrm{d}x + K\alpha\delta^{-1/2} + K\alpha^2 + K\alpha^2\delta^{-1} + K\delta^{1/2}.
\end{aligned}
$$

Choose $\delta = \mathcal{O}(\alpha)$.

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta) \,\mathrm{d}x\,\mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \,\mathrm{d}x\,\mathrm{d}s \\
&\leq -\frac{\alpha^2}{4} \|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2} \|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2 \,\mathrm{d}s \\
&- \alpha^2 \int_\Omega \nabla u_0^\alpha \cdot \nabla u_0 \,\mathrm{d}x + K\alpha\delta^{-1/2} + K\alpha^2 + K\alpha^2\delta^{-1} + K\delta^{1/2}.
\end{aligned}
$$

Choose $\delta = \mathcal{O}(\alpha)$. Then find

With this we can now perform the **key estimate**:

$$
\begin{aligned}
I_1(t) &= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot W^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&= \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot u^\alpha \,\mathrm{d}x\,\mathrm{d}s - \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot (u^E - v^\delta)\,\mathrm{d}x\,\mathrm{d}s \\
&- \alpha^2 \int_0^t \int_\Omega \partial_s \Delta u^\alpha \cdot v^\delta \,\mathrm{d}x\,\mathrm{d}s \\
&\leq -\frac{\alpha^2}{4}\|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2}\|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2\,\mathrm{d}s \\
&- \alpha^2 \int_\Omega \nabla u_0^\alpha \cdot \nabla u_0 \,\mathrm{d}x + K\alpha\delta^{-1/2} + K\alpha^2 + K\alpha^2\delta^{-1} + K\delta^{1/2}.
\end{aligned}
$$

Choose $\delta = \mathcal{O}(\alpha)$. Then find

$$
\boxed{I_1(t) \leq -\frac{\alpha^2}{4}\|\nabla u^\alpha(t)\|^2 + \frac{\alpha^2}{2}\|\nabla u_0^\alpha\|^2 + K\alpha^2 \int_0^t \|\nabla u^\alpha\|^2\,\mathrm{d}s + o(1).}
$$

End up with

End up with

$$\|W(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 \leq K_1(\|W_0\|_{L^2}^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2)$$

End up with

$$\|W(t)\|_{L^2}^2 + \alpha^2\|\nabla u^\alpha(t)\|_{L^2}^2 \leq K_1(\|W_0\|_{L^2}^2 + \alpha^2\|\nabla u_0^\alpha\|_{L^2}^2)$$

$$+K_2\int_0^t(\|W(s)\|_{L^2}^2 + \alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2)\,ds + o(1).$$

End up with

$$\|W(t)\|_{L^2}^2 + \alpha^2\|\nabla u^\alpha(t)\|_{L^2}^2 \leq K_1(\|W_0\|_{L^2}^2 + \alpha^2\|\nabla u_0^\alpha\|_{L^2}^2)$$

$$+K_2\int_0^t(\|W(s)\|_{L^2}^2 + \alpha^2\|\nabla u^\alpha(s)\|_{L^2}^2)\,ds + o(1).$$

Use Gronwall to conclude the proof.

End up with

$$\|W(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(t)\|_{L^2}^2 \leq K_1(\|W_0\|_{L^2}^2 + \alpha^2 \|\nabla u_0^\alpha\|_{L^2}^2)$$

$$+K_2 \int_0^t (\|W(s)\|_{L^2}^2 + \alpha^2 \|\nabla u^\alpha(s)\|_{L^2}^2)\, ds + o(1).$$

Use Gronwall to conclude the proof.

It works!

We call this boundary layer indifference.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L-y), 0)$.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L-y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L-y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

For Euler-α have: stationary parallel Euler-α flow in channel $(\varphi(y), 0)$,

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L - y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

For Euler-α have: stationary parallel Euler-α flow in channel $(\varphi(y), 0)$, any φ such that $\varphi(0) = \varphi(L) = 0$.

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L - y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

For Euler-α have: stationary parallel Euler-α flow in channel $(\varphi(y), 0)$, any φ such that $\varphi(0) = \varphi(L) = 0$. However, pressure is

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L-y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

For Euler-α have: stationary parallel Euler-α flow in channel $(\varphi(y), 0)$, any φ such that $\varphi(0) = \varphi(L) = 0$. However, pressure is

$$p = -\frac{\varphi^2 - \alpha^2(\varphi')^2}{2}.$$

We call this boundary layer indifference.

Notice: mismatched boundary conditions leads to boundary layer. Can still pass to limit. Why? Illustrative example.

Stationary, parallel Navier-Stokes flow in channel $\mathbb{R} \times (0, L)$: must be of the form $u^\nu = (Cy(L - y), 0)$. Stationary, parallel Euler flow in channel: $u^E = (\varphi(y), 0)$, any φ.

For Euler-α have: stationary parallel Euler-α flow in channel $(\varphi(y), 0)$, any φ such that $\varphi(0) = \varphi(L) = 0$. However, pressure is

$$p = -\frac{\varphi^2 - \alpha^2(\varphi')^2}{2}.$$

Hence Euler solution approximates Euler-α solution outside of boundary layer of arbitrary width.

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$\left\{ \right.$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, \quad \text{in } \Omega \times (0, \infty), \\ \\ \\ \\ \end{cases}$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$
\begin{cases}
\partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } \Omega \times (0, \infty), \\
\text{div } u = 0, & \text{in } \Omega \times [0, \infty),
\end{cases}
$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } \Omega \times (0, \infty), \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \end{cases}$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$
\begin{cases}
\partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } \Omega \times (0,\infty), \\
\text{div } u = 0, & \text{in } \Omega \times [0,\infty), \\
v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0,\infty), \\
u = 0, & \text{on } [0,T] \times \partial\Omega,
\end{cases}
$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } \Omega \times (0, \infty), \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \\ u = 0, & \text{on } [0, T] \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), & \text{at } \{t = 0\}. \end{cases}$$

Bounded domain, second grade, 2D, no slip

Consider the second grade equations, bounded planar domain, Dirichlet on the boundary:

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p + \nu \Delta u, & \text{in } \Omega \times (0, \infty), \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty), \\ v = u - \alpha^2 \Delta u, & \text{in } \Omega \times [0, \infty), \\ u = 0, & \text{on } [0, T] \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot), & \text{at } \{t = 0\}. \end{cases}$$

First result:

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

$$\sup_{t\in[0,T]} (\|u^{\nu,\alpha}(t) - u^E(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^{\nu,\alpha}\|_{L^2}^2) \xrightarrow[\alpha\to 0]{} 0. \tag{8}$$

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

$$\sup_{t\in[0,T]} \left(\|u^{\nu,\alpha}(t) - u^E(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^{\nu,\alpha}\|_{L^2}^2\right) \xrightarrow[\alpha\to 0]{} 0. \tag{8}$$

OBS. Have $\nu \to 0$, $\alpha \to 0$, but $\nu = \mathcal{O}(\alpha^2)$.

First result:

Theorem (L^2, Titi and Zang, 2014)

Fix $T > 0$. Let $u_0 \in H^3$, div-free, $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let $\{u_0^\alpha\}_\alpha \subset H^3$ be a suitable approximation. Set $u^{\nu,\alpha} \in C([0,T]; H^3)$ solution to second grade equations with initial data u_0^α and assume $\nu = \mathcal{O}(\alpha^2)$. Set $u^E \in C([0,T]; H^3) \cap C^1([0,T]; H^2)$ the unique smooth solution to Euler, initial data u_0. Then:

$$\sup_{t\in[0,T]} \left(\|u^{\nu,\alpha}(t) - u^E(t)\|_{L^2}^2 + \alpha^2 \|\nabla u^{\nu,\alpha}\|_{L^2}^2\right) \xrightarrow[\alpha\to 0]{} 0. \tag{8}$$

OBS. Have $\nu \to 0$, $\alpha \to 0$, but $\nu = \mathcal{O}(\alpha^2)$. Not independent limits.

Proof is a variant of Euler-α case.

Proof is a variant of Euler-α case. Need only estimate dissipative term, Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation;

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy.

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall:

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall: T. Kato, 1984 proved

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall: T. Kato, 1984 proved

$$\boxed{u^\nu \xrightarrow[\nu\to 0^+]{} u^E \text{ in } L^\infty(L^2) \Longleftrightarrow \nu \int_0^T \int_{\Gamma_{C\nu}} |\nabla u^\nu|^2 \, dxds \xrightarrow[\nu\to 0^+]{} 0.}$$

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall: T. Kato, 1984 proved

$$\boxed{u^\nu \xrightarrow[\nu\to 0^+]{} u^E \text{ in } L^\infty(L^2) \Longleftrightarrow \nu \int_0^T \int_{\Gamma_{C\nu}} |\nabla u^\nu|^2 \, dxds \xrightarrow[\nu\to 0^+]{} 0.}$$

For second grade, have two results,

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall: T. Kato, 1984 proved

$$\boxed{u^\nu \xrightarrow[\nu\to 0^+]{} u^E \text{ in } L^\infty(L^2) \Longleftrightarrow \nu \int_0^T \int_{\Gamma_{C\nu}} |\nabla u^\nu|^2 \, dxds \xrightarrow[\nu\to 0^+]{} 0.}$$

For second grade, have two results, one *near* **Euler-α**,

Proof is a variant of Euler-α case. Need only estimate dissipative term, use boundary corrector + interpolation; easy. Condition $\nu = \mathcal{O}(\alpha^2)$ natural.

Next results are *criteria* for convergence, like the Kato criterion.

Recall: T. Kato, 1984 proved

$$\boxed{u^\nu \xrightarrow[\nu\to 0^+]{} u^E \text{ in } L^\infty(L^2) \Longleftrightarrow \nu \int_0^T \int_{\Gamma_{c\nu}} |\nabla u^\nu|^2 \, dxds \xrightarrow[\nu\to 0^+]{} 0.}$$

For second grade, have two results, one *near* **Euler-α**, the other *near* **Navier-Stokes**.

For second grade, *close to* Euler-α:

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$.

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0.

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade, initial data* u_0^α.

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. Let $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, suitable family of approximations for u_0. Let $u^{\alpha,\nu} \in C([0,T];H^3)$ solution of second grade, initial data u_0^α. Let $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ solution of Euler with initial data u_0.

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T]; H^3)$ *solution of second grade, initial data* u_0^α. *Let* $u^E \in C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega))$ *solution of Euler with initial data* u_0. *Assume:*

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T]; H^3)$ *solution of second grade, initial data* u_0^α. *Let* $u^E \in C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega))$ *solution of Euler with initial data* u_0. *Assume:*

$$\lim_{\alpha \to 0} \frac{\nu}{\alpha^2} = \infty,$$

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T]; H^3)$ *solution of second grade, initial data* u_0^α. *Let* $u^E \in C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega))$ *solution of Euler with initial data* u_0. *Assume:*

$$\lim_{\alpha\to 0} \frac{\nu}{\alpha^2} = \infty, \quad \textit{and} \quad \nu = \mathcal{O}(\alpha^{6/5}).$$

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$*,* $u_0 \in H^3(\Omega) \cap H$*. Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$*, suitable family of approximations for* u_0*. Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade, initial data* u_0^α*. Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *solution of Euler with initial data* u_0*. Assume:*

$$\lim_{\alpha\to 0}\frac{\nu}{\alpha^2} = \infty, \quad \textit{and} \quad \nu = \mathcal{O}(\alpha^{6/5}).$$

Then

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade, initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *solution of Euler with initial data* u_0. *Assume:*

$$\lim_{\alpha\to 0} \frac{\nu}{\alpha^2} = \infty, \quad \textit{and} \quad \nu = \mathcal{O}(\alpha^{6/5}).$$

Then $u^{\alpha,\nu}$ *converges strongly to* u^E *in* $C([0,T];(L^2(\Omega))^2)$ *as* $\alpha \to 0$ *if and only if*

$$\lim_{\alpha\to 0} \nu \int_0^T \int_{\Omega_\delta} |\nabla u^{\nu,\alpha}|^2 \, dx \, dt = 0, \tag{9}$$

For second grade, *close to* Euler-α:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega) \cap H$. Let $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, suitable family of approximations for u_0. Let $u^{\alpha,\nu} \in C([0,T];H^3)$ solution of second grade, initial data u_0^α. Let $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ solution of Euler with initial data u_0. Assume:

$$\lim_{\alpha\to 0} \frac{\nu}{\alpha^2} = \infty, \quad \textit{and} \quad \nu = \mathcal{O}(\alpha^{6/5}).$$

Then $u^{\alpha,\nu}$ converges strongly to u^E in $C([0,T];(L^2(\Omega))^2)$ as $\alpha \to 0$ if and only if

$$\lim_{\alpha\to 0} \nu \int_0^T \int_{\Omega_\delta} |\nabla u^{\nu,\alpha}|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{9}$$

where Ω_δ is a $\delta-$neighborhood of $\partial\Omega$ with $\delta = C\dfrac{\alpha^3}{\nu^{\frac{3}{2}}}$.

For second grade, *close to* Navier-Stokes:

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$.

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha > 0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0.

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T]; H^3)$ *solution of second grade with initial data* u_0^α.

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0.

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0. *Assume that*

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0. *Assume that*

$$\alpha = \mathcal{O}(\nu^{3/2}) \quad \textit{as} \quad \nu \to 0. \tag{10}$$

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0. *Assume that*

$$\alpha = \mathcal{O}(\nu^{3/2}) \quad \textit{as} \quad \nu \to 0. \tag{10}$$

Then

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0. *Assume that*

$$\alpha = \mathcal{O}(\nu^{3/2}) \quad \text{as} \quad \nu \to 0. \tag{10}$$

Then $u^{\alpha,\nu}$ *converges strongly to* u^E *in* $C([0,T];(L^2(\Omega))^2)$ *as* $\nu \to 0$ *if and only if*

$$\lim_{\nu \to 0} \nu \int_0^T \int_{\Omega_{c\nu}} |\nabla u^{\nu,\alpha}|^2 \, dx \, dt = 0, \tag{11}$$

For second grade, *close to* Navier-Stokes:

Theorem

Fix $T > 0$, $u_0 \in H^3(\Omega)$. *Let* $\{u_0^\alpha\}_{\alpha>0} \subset H^3(\Omega)$, *suitable family of approximations for* u_0. *Let* $u^{\alpha,\nu} \in C([0,T];H^3)$ *solution of second grade with initial data* u_0^α. *Let* $u^E \in C([0,T];H^3(\Omega)) \cap C^1([0,T];H^2(\Omega))$ *Euler solution with initial data* u_0. *Assume that*

$$\alpha = \mathcal{O}(\nu^{3/2}) \quad \text{as} \quad \nu \to 0. \tag{10}$$

Then $u^{\alpha,\nu}$ *converges strongly to* u^E *in* $C([0,T];(L^2(\Omega))^2)$ *as* $\nu \to 0$ *if and only if*

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega_{C\nu}} |\nabla u^{\nu,\alpha}|^2 \, dx \, dt = 0, \tag{11}$$

where $\Omega_{C\nu}$ *is a* $C\nu-$*neighborhood of* $\partial\Omega$.

Schematically, we obtained:

Schematically, we obtained:

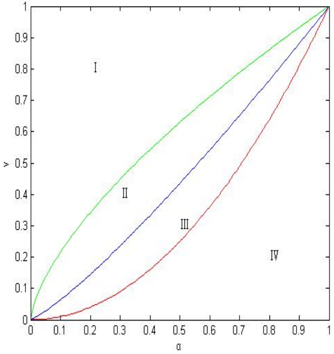

Figure: Curve between region I and region II: $\nu = \alpha^{2/3}$; between II and III: $\nu = \alpha^{6/5}$; between III and IV: $\nu = \alpha^2$.

Summary observations

Initial motivation:

Summary observations

Initial motivation: probe relation between

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found:

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α.

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Euler-α, no-slip $\xrightarrow[\alpha \to 0]{}$ Euler– nonlinear

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Euler-α, no-slip $\xrightarrow[\alpha \to 0]{}$ Euler– nonlinear and

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Euler-α, no-slip $\xrightarrow[\alpha \to 0]{}$ Euler– nonlinear and non-compatible ∂ conditions .

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Euler-α, no-slip $\xrightarrow[\alpha \to 0]{}$ Euler– nonlinear and non-compatible ∂ conditions . Similarly, in some cases, for second grade $\xrightarrow[\nu, \alpha \to 0]{}$ Euler.

Summary observations

Initial motivation: probe relation between

$$\boxed{\text{inviscid limit for Navier-Stokes}} \times \boxed{\alpha \to 0 \text{ limit for Euler-}\alpha}.$$

Explore diverse nature of boundary layer and use second grade as interpolant.

We found: change in behavior depending on relation ν and α. Also found region where no equivalence criteria obtained.

Wrt inviscid limit for Navier-Stokes:

Euler-α, no-slip $\xrightarrow[\alpha\to 0]{}$ Euler– nonlinear and non-compatible ∂ conditions . Similarly, in some cases, for second grade $\xrightarrow[\nu,\alpha\to 0]{}$ Euler.

Why did it work???

Concluding remarks

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time).

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade!

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE,

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability. Vortex sheet problem (Birkhoff-Rott-α) well-posed for Euler-α, see Bardos, Linshiz and Titi, 2008 and 2009.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability. Vortex sheet problem (Birkhoff-Rott-α) well-posed for Euler-α, see Bardos, Linshiz and Titi, 2008 and 2009.

Mismatched ∂ conditions lead to vortex sheet on boundary;

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability. Vortex sheet problem (Birkhoff-Rott-α) well-posed for Euler-α, see Bardos, Linshiz and Titi, 2008 and 2009.

Mismatched ∂ conditions lead to vortex sheet on boundary; this is why absence of Kelvin-Helmholtz instability helps.

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability. Vortex sheet problem (Birkhoff-Rott-α) well-posed for Euler-α, see Bardos, Linshiz and Titi, 2008 and 2009.

Mismatched ∂ conditions lead to vortex sheet on boundary; this is why absence of Kelvin-Helmholtz instability helps. **Kato criterion** designed to rule out Kelvin-Helmholtz near boundary /

Concluding remarks

Remark 1. Euler-α is not *regularizing* (in time). Similarly for second grade! Hence, more like ODE, different from Navier-Stokes.

Remark 2. Euler and NS have special kind of nonlinearity.

Shear flow / vortex sheet + Euler nonlinearity $\Longrightarrow$ Kelvin-Helmholtz instability.

Euler-α **not subject** to Kelvin-Helmholtz instability. Vortex sheet problem (Birkhoff-Rott-α) well-posed for Euler-α, see Bardos, Linshiz and Titi, 2008 and 2009.

Mismatched ∂ conditions lead to vortex sheet on boundary; this is why absence of Kelvin-Helmholtz instability helps. **Kato criterion** designed to rule out Kelvin-Helmholtz near boundary / creation (and amplification) of small scales at boundary.

Thank you!