A Windblown Sea

What is wave turbulence?

Wave Turbulence is the study of the long time statistical behaviour of solutions of nonlinear field equations, usually conservative and Hamiltonian, describing a sea of weakly nonlinear interacting dispersive waves. In most interesting contexts, the system is nonisolated having both sources and sinks of energy and other conserved densities.

Why is it believed to be a "solved" problem?

- A natural asymptotic closure
- A closed kinetic equation for the particle or wave action density n_k from which all other quantities of interest such as *nonlinear frequency modification, higher order cumulants, spatial structure functions* can be calculated.
- In addition to the usual thermodynamic stationary solutions, the kinetic equation has finite flux (Kolmogorov-Zakharov) solutions which capture the flow of conserved densities (energy, waveaction) from sources to sinks.

But, it is not a “solved” problem and the story is far from over

- The KZ solutions are almost never uniformly valid in *k* space as we see with
  - Whitecaps
  - Filamentation of optical waves
- In some cases, often one dimensional situations, the KZ solutions are not valid at any *k* and the WT closure fails.
  - FPU, self induced transparency, MMT

PREMISES

Premise 1 (P1): First, we assume the fields are spatially homogeneous and that ensemble averages of fields evaluated at the set of points $\mathrm{x}, \mathrm{x}+\mathrm{r}_1, \mathrm{x}+\mathrm{r}_2, \cdots$ depend only on the separations $\mathrm{r}_1, \mathrm{r}_2, \cdots$.

Premise 2 (P2): Second, we assume that at some initial point in time, the moment at which the external driving, e.g. the storm over the sea surface, is initiated, the fields at distant points are uncorrected. This means that the physical space cumulants have the property that, as the separations $|\mathbf{r}_j|$ become large, the cumulants decay sufficiently rapidly that their Fourier transforms are ordinary functions. This is a mild assumption but it is necessary because in the evaluation of the long time behavior of integrals such as $\int f(x)\frac{\sin xt}{x}dx$ need to know that $f(x)$ is sufficiently smooth in wavenumber x so that this integral behaves in long time as $\pi \mathrm{sgn}(t) f(0)$.

PREMISES

Premise 3 (P3): Third, we must ensure that various asymptotic expansions for the slow evolution of such two point functions as the waveaction density $n_{\mathbf{k}}$ remain uniformly valid in wavenumber. In its simplest form this means that the ratio of linear (t_L) to nonlinear (t_{NL}) time scales is small at all wavenumbers. It also means that all asymptotic expansions for the slow evolution of the waveaction density $dn_{\mathbf{k}}/dt$, the frequency renormalization which accounts for the slow time behavior of the leading order, higher order cumulants, and for the structure functions remain uniformly valid in wavenumber on almost all relevant solutions. The reason that we require $\frac{t_L}{t_{NL}}(k) \ll 1$ (with $k = |k|$) is that when we look for the long time behavior of integrals such as $\int f(x)\frac{\sin xt}{x}dx$ we want to know that the multiplying function $f(x)$ which will contain products of the waveaction densities $n_{\mathbf{k}}$ is not only smooth in x (wavenumber) but also that it varies slowly in time.

PREMISES

Premise 4 (P4): This premise says that one must test the deterministic theory first. If the field remains asymptotically linear (which might be tested by numerical simulations), we might surmise that this would rule out the appearance of coherent structures also dominating the long time behavior of the random system. The thinking here is that the deterministic problem would rule out resonances creating secular behavior (because the wavepackets are finite in length and so resonances do not produce long time cumulative effects) but not the appearance of coherent structures. If the latter do not appear in the deterministic system, the argument is that they will play no role in statistical ensembles either.

Premise 5 (P5): All KZ solutions are stable against perturbations which spontaneously break the spatial homogeneity symmetry.

The theory in outline

The variables

$$u^s(\mathbf{x},t)(e.g.\ \eta(\mathbf{x},t),\varphi(\mathbf{x},z=\eta(\mathbf{x},t),t)) \leftrightarrow A^s_{\mathbf{k}};\ \ \dim\mathbf{k}=d.$$

The governing equations

$$\frac{dA^s_{\mathbf{k}}}{dt} - is\omega_{\mathbf{k}}A^s_{\mathbf{k}} =$$

$$\sum_{r=2}^{\infty}\varepsilon^{r-1}\int L^{ss_1\ldots s_r}_{kk_1\ldots k_r}A^{s_1}_{k_1}\ldots A^{s_r}_{k_r}\delta(k_1+\cdots+k_r-k)dk_1\ldots dk_r.$$

The theory in outline

The statistics

$$< A^s_{\mathbf{k}} A^{-s}_{\mathbf{k}'} > = \delta(\mathbf{k} + \mathbf{k}') N^s_{\mathbf{k}}$$ 2 point correlation

$$\{ A^s_{\mathbf{k}} A^{s'}_{\mathbf{k}'} \ldots \}$$ higher order cumulants

The theory in outline 2

The strategy

1. Form BBGKY hierarchy.
2. Solve iteratively in power series in ε.
3. Choose "slow" variations of leading approximations to make 2. uniformly valid for "long" times.

The outcome ($N_k(t) = n_k(t) + \text{corrections} + \cdots$)

$$\frac{dn_k}{dt} = \varepsilon^2 T_2[n_k] + \varepsilon^4 T_4[n_k] + \varepsilon^6 T_6[n_k] + \cdots$$

$$s\omega_k \longrightarrow s\omega_k + \varepsilon^2 \Omega_2^s[n_k] + \cdots$$

PREMISES

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Ocean gravity waves

Hasselmann (1962)

$$T_4[n_k] = \int |L_{kk_1k_2k_3}|^2 n_k n_{k_1} n_{k_2} n_{k_3} (\frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}})$$

$$\delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) dk_1 dk_2 dk_3$$

Ocean gravity waves

E (average energy per unit area of surface)=

$$\int e_k dk = \int \omega_k n_k dk = \int E_k dk = \int E_\omega d\omega$$

- $c g^{1/2} P^{1/3} k^{-7/2}$
- $g^{1/2} k^{1/2}$
- $c P^{1/3} k^{-4}$
- $2\pi c g^{1/2} P^{1/3} k^{-5/2}$
- $4\pi c g^{2} P^{1/3} \omega^{-4}$

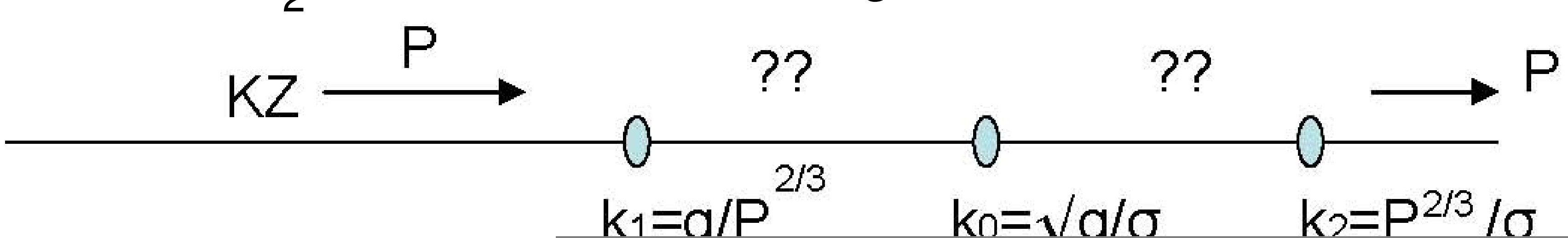
KZ sol

$T_4 [n_k]=0$

Breakdown!

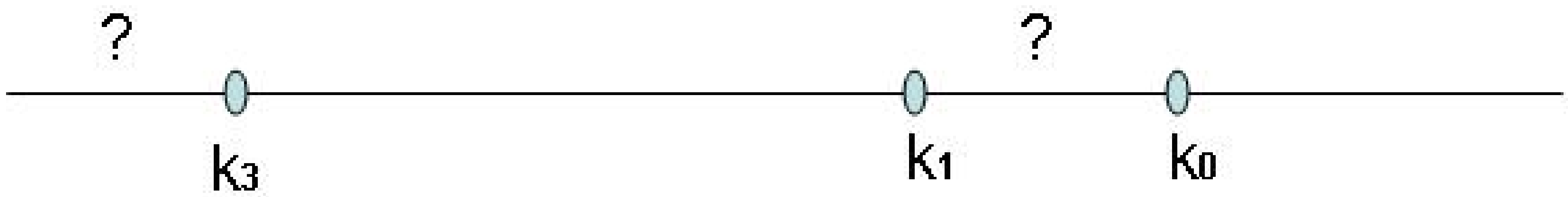
The markers of a successful closure are

- $\frac{t_L}{t_{NL}} = \frac{1}{\omega_k}\frac{1}{n_k}\frac{dn_k}{dt} \simeq c^2 P^{2/3}\frac{k}{g} \ll 1$
- $\frac{T_6[n_k]}{T_4[n_k]}, \frac{\Omega_k[n_k]}{w_k}, \ldots, \simeq cP^{1/3}\frac{k^{1/2}}{g^{1/2}} \ll 1$
- $\frac{S_4 - 3S_2^2}{S_2^2}, \ldots, \simeq cP^{1/3}\frac{1}{g^{1/2}r^{1/2}} \ll 1$



Breakdown!

The finite flux solutions of wave turbulence are almost never uniformly valid for all wavenumbers. They almost always fail at very high or very low wavenumbers.



Optical waves ← of diffraction and formation of condensates/collapses

Ocean waves and whitecapping

What new solutions obtain in breakdown regions?

The generalized Phillips' spectrum (GPS)

We (ACN VEZ, Phys. Lett. A 372, 4230-4233 (2008)) argue that the GPS can play a central role. It has four important properties.

- It is the only spectrum for which symmetries of the original governing equations are inherited by the asymptotic statistics. This in fact will serve as the definition of the GPS.
- It is the unique spectrum on which wave turbulence is uniformly valid at all wavenumbers.

The generalized Phillips' spectrum (GPS)

- It is a solution of $T_4[n_k] - \gamma_k[n_k] = 0$, i.e. a balance of nonlinear transfer and dissipation and independent of energy flux.(i.e. can absorb any excess flux)
- It is connected with whitecapping and Phillips' picture.

Inheriting symmetries

$$\frac{dA_k^s}{dt} - is\omega_k A_k^s = \sum_{r=2} \int L^{s\ldots s_r}_{k\ldots k_r} A^{s_1}_{k_1}\ldots A^{s_r}_{k_r}\,\delta(k_1+\ldots+k_r-k)\,dk_1\ldots dk_r$$

α (degree of ω_k), γ_r (degree of L) — **degrees of homogeneity**

$$k \to K = \lambda k, \qquad t \to T = \lambda^{-\alpha} t, \qquad A^s_k \to \lambda^b B^s_K$$

$$\lambda^{b-\alpha} \qquad \lambda^{-\gamma_r} \quad \lambda^{rb} \qquad \lambda^{-(r-1)d}$$

$$\Longrightarrow b = d + \frac{\gamma_r - \alpha}{r-1}, \text{ all } r$$

$$< A^s_k\ A^{-s}_{k'} > = \delta(\mathbf{k}+\mathbf{k}')\, k^{-\alpha x}$$ ← n_k

$$\Longrightarrow \lambda^{2b} < B^s_K\ B^{-s}_{K'} > = \lambda^d \delta(\mathbf{K}+\mathbf{K}')K^{-\alpha x}\lambda^{-\alpha x}$$

$$\Longrightarrow \alpha x = 2b-d = d + 2\frac{\gamma_r - \alpha}{r-1} \qquad \text{GPS}$$

Inheriting symmetries

Gravity waves:
$\alpha = \frac{1}{2}, \gamma_2 = \frac{7}{4}, \gamma_3 = 3, d = 2 \Rightarrow n_k \sim k^{-9/2}, < \eta_k^2 > \sim k^{-4}.$

Capillary waves:
$\alpha = \frac{3}{2}, \gamma_2 = \frac{9}{4}, \gamma_3 = 3, d = 2 \Rightarrow n_k \sim k^{-7/2}, < \eta_k^2 > \sim k^{-4}.$

WT U valid all k: $n_k \sim k^{-\alpha x}$, $\alpha x = ?$

$$\frac{dn_k}{dt} = T_2[n_k] + T_4[n_k] + T_6[n_k] + \cdots$$

$$s\omega_k \longrightarrow s\omega_k + \Omega_k^s[n_k] + \cdots \qquad U \text{ valid in } k?$$

$$\frac{t_L}{t_{NL}} = \frac{1}{\omega_k}\frac{1}{n_k}\frac{dn_k}{dt} = \frac{s_k^i}{\omega_k} \ll 1 \text{ all } k \ ?$$

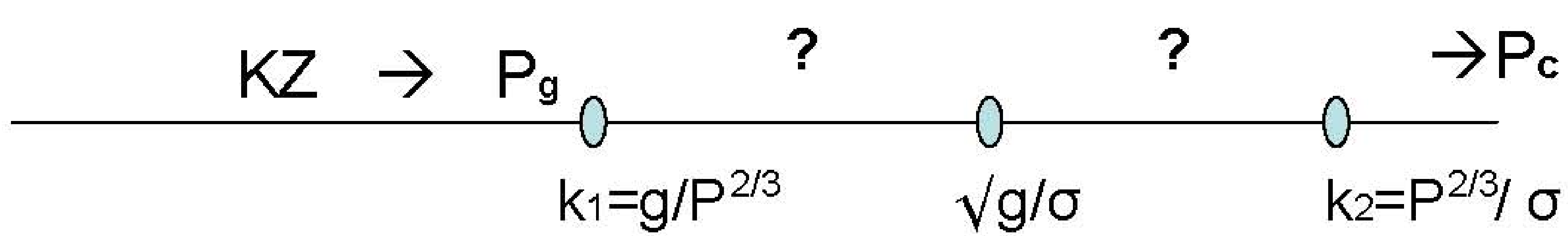
$$\frac{S_4 - 3S_2^2}{S_2^2} \ll 1 \text{ all } r?$$

$$\frac{t_L}{t_{NL}} \quad n_k = ck^{-\alpha x} \quad \frac{T_4}{\omega_k n_k} \sim \frac{k^{2\gamma r} c^3 k^{-3\alpha x} k^{-\alpha} k^{2d}}{k^\alpha ck^{-\alpha x}} \sim c^2 k^{2(\gamma_3 + d - \alpha - \alpha x)}$$

WT U valid all k: $n_k \sim k^{-\alpha x}$, $\alpha x = ?$

On GPS, $\alpha x = \gamma_3 + d - \alpha$, $\frac{t_L}{t_{NL}}$ k independent

On KZ, $\alpha x = \frac{2\gamma_3}{3} + d$, $\frac{t_L}{t_{NL}} \sim cP^{2/3}k^{2(\frac{\gamma_3}{3}-\alpha)} \sim P^{2/3}\frac{k}{g}$



GPS solves $T[n_k] - \gamma[n_k] = 0$

From $\frac{dn_k}{dt} = T[n_k] - \gamma[n_k]$, angle average to obtain

$$\frac{dE_k}{dt} = -\frac{\partial p(k)}{\partial k} - \gamma_k$$

where

$$\int \omega_k n_k dk = \int E_k dk, \quad -\frac{\partial p(k)}{\partial k} =< \omega_k T[n_k] >$$

Let RHS=0 in integral sense

$$\int_{k_1}^{\infty} -\frac{\partial p(k)}{\partial k} dk = p(k_1) = \int_{k_1}^{\infty} \gamma_k dk = P$$

breakdown wavenumber

all dissipation between k_1 and ∞ absorbs flux P

GPS solves $T[n_k] - \gamma[n_k] = 0$

But on $n_k = Ck^{-\alpha x}, p(k_1) = C^3\mathbf{I}(x)k_1^{3\alpha(x_{KZ}-x)}$

$$C^3\mathbf{I}(x)k_1^{3\alpha(x_{KZ}-x)} = P = k_1^{-\gamma_3+3\alpha}$$

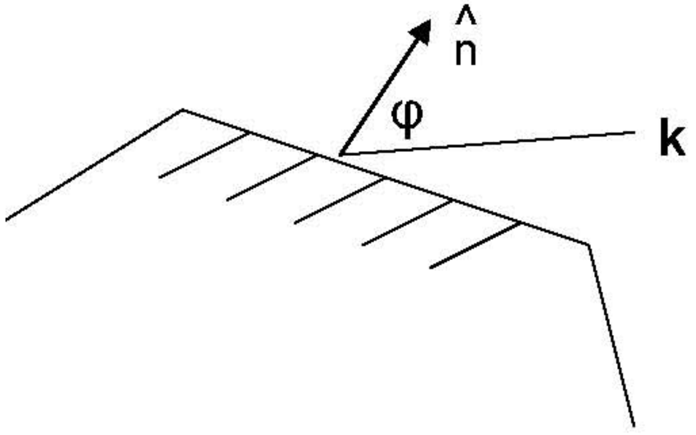
$$\Rightarrow \alpha x = \gamma_3 + d - \alpha, \; GPS.$$

Connection with whitecaps

Consider sea surface dominated by a series of wedge shapes of finite length.

$$\eta(x, y) \sim \frac{1}{2s} e^{-s|x|} e^{-\frac{y^2}{L^2}}$$

$$< \eta_k^2 > \propto \frac{1}{(s^2 + k^2 \cos^2 \varphi)^2} e^{-\frac{k^2 L^2 \sin^2 \varphi}{2}}$$



Connection with whitecaps

For s fixed, average over $\varphi \Rightarrow k^{-5}$, not Phillips'!!!
But averaging over s, $s > k_1, \ldots,$

$$\frac{L}{k^3}\int_0^{\frac{\pi}{2}} e^{-\frac{k^2L^2\sin^2\varphi}{2}} \frac{1}{\cos^3\varphi}[\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\frac{k_1}{k\cos\varphi}-\frac{1}{2}\frac{kk_1\cos\varphi}{k_1^2+k^2\cos^2\varphi}]d\varphi$$

for $kL \gg 1, < \eta_k^2 > \sim k^{-4}$, Phillips'!!!

MMT equation

$$i\frac{\partial u}{\partial t} = Lu + \lambda u^2 u^* \ , \ Le^{ikx} = \omega_k e^{ikx}, \omega_k = \sqrt{g|k|}.$$

$$\begin{aligned} T_4[n_{\mathbf{k}}] = 4\pi\lambda^2 \ \int \ & n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \\ \times \ & \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}} \right) \\ \times \ & \delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ \times \ & \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123} \end{aligned}$$

$$\tilde{\omega}_k = \omega_k + 2\lambda \int n_{\mathbf{k}} d\mathbf{k}$$

PREMISES

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Premise 5 (P5): All KZ solutions are stable against perturbations which spontaneously break the spatial homogeneity symmetry.

$$n_{\mathbf{k}} = n_0(k) = DP^{1/3}k^{-d}$$

$$\frac{\partial n_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}}\tilde{\omega}_k \cdot \nabla_{\mathrm{x}} n_{\mathbf{k}} - \nabla_{\mathrm{x}}\tilde{\omega}_k \cdot \nabla_{\mathbf{k}} n_{\mathbf{k}} = \sum_{r=1} T_{2r}[n_{\mathbf{k}}]$$

Inserting $n_0(k) + \Delta(\mathbf{k})\exp(i\mathbf{K}\cdot\mathbf{x} - i\Omega t)$,

$$i\Delta(\mathbf{k})(\omega_k'\hat{\mathbf{k}}\cdot\hat{\mathbf{K}} - c)K - 2i\hat{\mathbf{k}}\cdot\hat{\mathbf{K}}Kn_0'(k)\lambda\int\Delta(\mathbf{k})d\mathbf{k} = \delta T_4$$

where $K = |\mathbf{K}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $\hat{\mathbf{K}} = \mathbf{K}/K$, $n_0'(k) = dn_0/dk$. $c = \Omega/K$ is the phase velocity of the modulation and $\omega_k' = d\omega_k/dk$ is the group velocity of linear waves. Complex values of c and Ω will signal instability. δT_4 is a linear functional of $\Delta(\mathbf{k})$. To begin we ignore δT_4, and we will discuss its small influence later. Integrating over $\mathbf{k}$ we obtain

$$1 = 2\lambda\int\frac{n_0'(k)\hat{\mathbf{k}}\cdot\hat{\mathbf{K}}}{\omega_k'\hat{\mathbf{k}}\cdot\hat{\mathbf{K}} - c}d\mathbf{k}$$

$$1 = 2\lambda \int_{-\infty}^{\infty} \frac{n_0(k)\omega_k''}{(\omega_k' - c)^2} dk.$$

$$1 + \alpha \int_1^{\infty} \frac{z^2}{\sigma^2} \left(\frac{1}{(z-\sigma)^2} + \frac{1}{(z+\sigma)^2} \right) d\sigma = 0.$$

$$z = \frac{g}{2c\omega_0} = \omega_0'/c\,,\ \alpha = 4\lambda D\, P^{1/3}/\omega_0$$

$$1 + \alpha \left(2 + \frac{2}{z} \ln \frac{1-z}{1+z} + \frac{1}{1-z} + \frac{1}{1+z} \right) = 0.$$

In 2D

$$\frac{1}{8\pi\alpha} = \frac{1}{z} \sin^{-1} z - \frac{1}{4} \frac{1}{\sqrt{1-z^2}} - \frac{3}{4}$$

for $z = \omega_0'/c$, or with $z = \sin(\zeta)$,

$$\frac{1}{8\pi\alpha} = \zeta \operatorname{cosec}(\zeta) - \frac{1}{4} \sec(\zeta) - \frac{3}{4}.$$

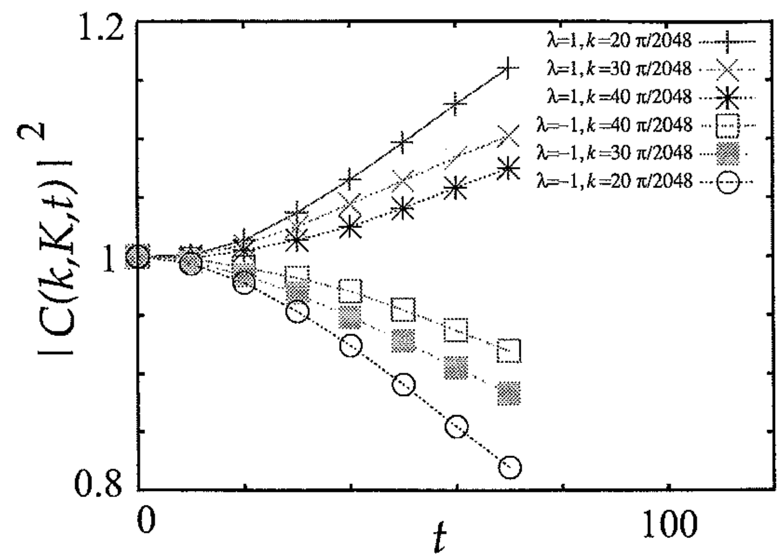

FIG. 2: Time evolution of the correlation $|C(k,K,t)|^2 = |\langle A_k(t)A^*_{k+K}(t)\rangle|^2/|\langle A_k(0)A^*_{k+K}(0)\rangle|^2$ with $K = 6\pi/2048$ for an ensemble of 400,000 trajectories for the equation (2) with $\lambda = 1$ and with $\lambda = -1$. The initial conditions are Kolmogorov-Zakharov distributed $n_k \sim k^{-1}$ with a Gaussian amplitude distribution and random phases for $|k| \geq 20\pi/2048$ and $n_k = 0$ for $|k| < 20\pi/2048$ (the wavenumber space is $-\pi < k \leq \pi$). A small spatial modulation is superimposed on these random initial conditions, and this modulation is the same for each member of the ensemble. The system is not externally damped or driven. This correlation grows for $\lambda = 1$, reflecting an instability of wave turbulence against spatially inhomogeneous perturbations. There is no such instability for $\lambda = -1$, and the correlation decays.

Open Challenges

Acoustic Turbulence, Isotropy or Shocks?
The resonant manifolds for the dispersion relation $\omega = c|k|$ are rays in wavevector space. The first closure transfers spectral energy along but not between the rays. Given an initial anisotropic energy distribution, do the nonlinear interactions of the next closure lead to an isotropic distribution or to condensation along particular rays which would likely produce fully nonlinear shocks (L'vov et al. (1997))? Or, to use a more colorful vernacular: Were the dinosaurs frozen or fried?

Energy Exchange Times.
For a discrete set of interacting triads, the nonlinear energy exchange time is ϵ^{-1}. For a continuum set of such triads, "cancellations" cause this time to be extended to ϵ^{-2}. Why?

Condensate Formation,
modeled by the defocussing ($\lambda = -1$) NLS equation, is an open and hot topic.

Wave. Turbulence in Astrophysics.
Magnetized plasmas, found in the solar corona, solar wind and earth's magneto-sphere support waves and, like ocean waves, have a continuum of scales (up to 18 decades!) and are a natural playground for wave turbulence.

Continuum Limit of Finite Dimensional Wave Turbulence.

Open Challenges

A Priori Conditions for Wave Turbulence.
Can one find mathematically rigorous a priori conditions on the governing equation (1.1) or its statistical hierarchy which guarantees that wave turbulence theory will obtain?

Homogeneity.
Is broken spatial homogeneity (PI) a potential problem for all turbulence theories?

Anomalous Exponents.
Are all finite capacity Kolmogorov solutions reached with anomalous exponents? Do they have anything to do with positive entropy production?