Enstrophy dissipation in 2D incompressible fluids

Anna Mazzucato [1]   Milton Lopes Filho[2]   Helena Nussenzveig Lopes [2]

[1]Penn State University

[2]Universidade Federal do Rio de Janeiro

Workshop “Mathematics of Turbulence", IPAM, October 3, 2014

Work partially supported by the National Science Foundation.

Archive for Rational Mechanics and Analysis **179** (2006), n.3, 353-387

Motivation

- Reconcile Kraichnan - Batchelor theory of $2D$ *decaying* turbulence with properties of solutions to the $2D$ Euler equations.
- 2D turbulent flows characterized by forward cascade of enstrophy $\Omega := L^2$ norm squared of vorticity from large to small scale (and backward cascade of energy from small to large scales).
- Regular Euler flows conserve energy and enstrophy *exactly*.

  Energy dissipation rate $\sim 2\nu\Omega \to 0$ as $\nu \to 0$.

  Enstrophy dissipation rate should not vanish as $\nu \to 0$ to balance growth of vorticity gradients, i.e., palinstrophy (Batchelor 1969).
- Longt-time dynamics dominated by coherent vortices (McWilliams 1984).

Motivation

- Reconcile Kraichnan - Batchelor theory of $2D$ *decaying* turbulence with properties of solutions to the $2D$ Euler equations.
- 2D turbulent flows characterized by forward cascade of enstrophy $\Omega := L^2$ norm squared of vorticity from large to small scale (and backward cascade of energy from small to large scales).
- Regular Euler flows conserve energy and enstrophy *exactly*.

  Energy dissipation rate $\sim 2\nu\Omega \to 0$ as $\nu \to 0$.

  Enstrophy dissipation rate should not vanish as $\nu \to 0$ to balance growth of vorticity gradients, i.e., palinstrophy (Batchelor 1969).
- Longt-time dynamics dominated by coherent vortices (McWilliams 1984).

Motivation

- Reconcile Kraichnan - Batchelor theory of $2D$ *decaying* turbulence with properties of solutions to the $2D$ Euler equations.
- 2D turbulent flows characterized by forward cascade of enstrophy $\Omega := L^2$ norm squared of vorticity from large to small scale (and backward cascade of energy from small to large scales).
- Regular Euler flows conserve energy and enstrophy *exactly*.

  Energy dissipation rate $\sim 2\nu\Omega \to 0$ as $\nu \to 0$.

  Enstrophy dissipation rate should not vanish as $\nu \to 0$ to balance growth of vorticity gradients, i.e., palinstrophy (Batchelor 1969).
- Longt-time dynamics dominated by coherent vortices (McWilliams 1984).

Motivation cont.

Our aim:

- Study whether enstrophy dissipation sustained by *irregular* (weak) 2D Euler solutions as energy dissipation is for 3D Euler solutions (Onsager 1949, Majda 1993);
- Investigate uniqueness of weak solutions to 2D Euler (vorticity in L^p, $2 \leq p < \infty$): enstrophy dissipation as entropy condition.

• Prior results by Eyink (2001). Independent results from ours by Tran & Dritschel, with bound on dissipation rate $\nu\langle|\nabla\omega|^2\rangle \sim \ln(Re)^{-1/2}$.

• For *damped-driven* flows, enstrophy balance holds as $\nu \to 0$ in the long-time average (Constantin & Ramos 2007).

• Some results in the stochastic setting for related models (Bessaih & Ferrario 2012).

Motivation cont.

Our aim:

- Study whether enstrophy dissipation sustained by *irregular* (weak) 2D Euler solutions as energy dissipation is for 3D Euler solutions (Onsager 1949, Majda 1993);
- Investigate uniqueness of weak solutions to 2D Euler (vorticity in L^p, $2 \leq p < \infty$): enstrophy dissipation as entropy condition.

• Prior results by Eyink (2001). Independent results from ours by Tran & Dritschel, with bound on dissipation rate $\nu\langle|\nabla\omega|^2\rangle \sim \ln(Re)^{-1/2}$.

• For *damped-driven* flows, enstrophy balance holds as $\nu \to 0$ in the long-time average (Constantin & Ramos 2007).

• Some results in the stochastic setting for related models (Bessaih & Ferrario 2012).

Vorticity-velocity form of 2D Euler

Vorticity $\omega = \text{curl}\, u$-velocity u formulation to 2D Euler in $\mathbb{R}^2$:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1a}$$

$$u(x) = K * \omega(x) = \int_{\mathbb{R}^2} K(x-y)\,\omega(y)\,dy, \tag{1b}$$

where $K(x) \equiv \dfrac{x^\perp}{2\pi |x|^2}$ is the Biot-Savart kernel.

(1a) is a transport equation for $\omega \Rightarrow$ if u is regular,

$$\omega(\Phi(x,t),t) = \omega(x,0), \quad \frac{d\Phi}{dt}(x,t) = u(\Phi(x,t),t).$$

$\Rightarrow$ Regular flows conserve spatial integral of $f(\omega)$ ($f \in C^2$).

In particular conserve enstrophy $\Omega(t) = \frac{1}{2}\|\omega(t)\|_{L^2}^2$.

Vorticity-velocity form of 2D Euler

Vorticity $\omega = \text{curl}\, u$-velocity u formulation to 2D Euler in $\mathbb{R}^2$:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1a}$$

$$u(x) = K * \omega(x) = \int_{\mathbb{R}^2} K(x-y)\,\omega(y)\,dy, \tag{1b}$$

where $K(x) \equiv \dfrac{x^\perp}{2\pi |x|^2}$ is the Biot-Savart kernel.

(1a) is a transport equation for $\omega \Rightarrow$ if u is regular,

$$\omega(\Phi(x,t),t) = \omega(x,0), \qquad \frac{d\Phi}{dt}(x,t) = u(\Phi(x,t),t).$$

$\Rightarrow$ Regular flows conserve spatial integral of $f(\omega)$ ($f \in C^2$).

In particular conserve enstrophy $\Omega(t) = \dfrac{1}{2}\|\omega(t)\|^2_{L^2}$.

Function spaces

- $W^{s,p}(\mathbb{R}^2) =: W^{s,p}$, $s \in \mathbb{Z}_+$, $1 \leq p \leq \infty$, Sobolev space of functions with s derivatives in L^p. Set $W^{0,p} = L^p$, and $W^{s,2} =: H^s$ and $H^{-s} := (H^s)^*$.

- $B^s_{p,\infty}(\mathbb{R}^2) =: B^s_{p,\infty}$, $s \in \mathbb{R}$, Besov space of *distributions* with:

$$\|f\|_{B^s_{p,\infty}} := \|S_0 f\|_{L^p} + \sup_{j \in \mathbb{Z}_+} 2^{js} \|\Delta_j f\|_{L^p} < \infty,$$

where S_0 is a low-pass filter and Δ_j is a band filter at scale 2^j.

Δ_j is a sum of (almost orthogonal) wavelets at scale $\ell \sim 2^{-j}$.

In particular, $B^\alpha_{\infty,\infty} \equiv C^\alpha$, the space of α-Hölder continuous functions (if α not an integer).

Function spaces

- $W^{s,p}(\mathbb{R}^2) =: W^{s,p}$, $s \in \mathbb{Z}_+$, $1 \leq p \leq \infty$, Sobolev space of functions with s derivatives in L^p. Set $W^{0,p} = L^p$, and $W^{s,2} =: H^s$ and $H^{-s} := (H^s)^*$.

- $B^s_{p,\infty}(\mathbb{R}^2) =: B^s_{p,\infty}$, $s \in \mathbb{R}$, Besov space of *distributions* with:

$$\|f\|_{B^s_{p,\infty}} := \|S_0 f\|_{L^p} + \sup_{j \in \mathbb{Z}_+} 2^{js} \|\Delta_j f\|_{L^p} < \infty,$$

where S_0 is a low-pass filter and Δ_j is a band filter at scale 2^j.

Δ_j is a sum of (almost orthogonal) wavelets at scale $\ell \sim 2^{-j}$.

In particular, $B^\alpha_{\infty,\infty} \equiv C^\alpha$, the space of α-Hölder continuous functions (if α not an integer).

Function spaces

- $W^{s,p}(\mathbb{R}^2) =: W^{s,p}$, $s \in \mathbb{Z}_+$, $1 \le p \le \infty$, Sobolev space of functions with s derivatives in L^p. Set $W^{0,p} = L^p$, and $W^{s,2} =: H^s$ and $H^{-s} := (H^s)^*$.

- $B^s_{p,\infty}(\mathbb{R}^2) =: B^s_{p,\infty}$, $s \in \mathbb{R}$, Besov space of *distributions* with:

$$\|f\|_{B^s_{p,\infty}} := \|S_0 f\|_{L^p} + \sup_{j \in \mathbb{Z}_+} 2^{js} \|\Delta_j f\|_{L^p} < \infty,$$

where S_0 is a low-pass filter and Δ_j is a band filter at scale 2^j.

Δ_j is a sum of (almost orthogonal) wavelets at scale $\ell \sim 2^{-j}$.

In particular, $B^\alpha_{\infty,\infty} \equiv C^\alpha$, the space of α-Hölder continuous functions (if α not an integer).

Energy Spectrum

Energy spectrum:

$$E(t,\kappa) := \frac{1}{2}\int_{|k|=\kappa} |\widehat{u}(t,k)|^2 \, d\sigma(k),$$

(as Radon-Nikodym derivative of abs. cont. measure).

u fluid velocity, $\widehat{u}$ the Fourier Transform of u.
k frequency, $\kappa \sim 1/\ell$ wavenumber, where ℓ length scale.

• Batchelor-Kraichnan spectrum: $E(\kappa) \sim \eta^{2/3}\,\kappa^{-3}$ at high κ, η mean rate of enstropy dissipation.

Incomplete agreement with numerics (Sulem et al. 1988, Benzi et al. 1989, Dmitruk-Montgomery 2005).

Energy Spectrum

Energy spectrum:

$$E(t,\kappa) := \frac{1}{2}\int_{|k|=\kappa} |\widehat{u}(t,k)|^2 \, d\sigma(k),$$

(as Radon-Nikodym derivative of abs. cont. measure).

u fluid velocity, $\widehat{u}$ the Fourier Transform of u.
k frequency, $\kappa \sim 1/\ell$ wavenumber, where ℓ length scale.

- Batchelor-Kraichnan spectrum: $E(\kappa) \sim \eta^{2/3}\,\kappa^{-3}$ at high κ, η mean rate of enstrophy dissipation.

Incomplete agreement with numerics (Sulem et al. 1988, Benzi et al. 1989, Dmitruk-Montgomery 2005).

Energy spectrum cont.

- $u \in H^1$ ($\Leftrightarrow \omega \in L^2$) gives $E(\kappa) = o(\kappa^{-3})$, upper bound is Kraichnan spectrum.

- To have dissipation of enstrophy must go below L^2.

Look for function spaces X for the velocity such that:

1. $H^1 \subset X$ (strict inclusion);
2. energy spectrum of $u \in X$ respects Kraichnan spectrum (up to log factors perhaps).

- $E(\kappa) \sim \kappa^{-1} \|\Delta_j u\|_{L^2}^2$ with $\kappa \sim 2^j$ (Littlewood-Paley spectrum, P. Costantin).

- Choose $u \in X = B^1_{2,\infty} \Leftrightarrow \omega \in B^0_{2,\infty}$ (Eyink).

Question of existence of Euler solutions in this class.

Energy spectrum cont.

• $u \in H^1$ ($\Leftrightarrow \omega \in L^2$) gives $E(\kappa) = o(\kappa^{-3})$, upper bound is Kraichnan spectrum.

• To have dissipation of enstrophy must go below L^2.

Look for function spaces X for the velocity such that:

1. $H^1 \subset X$ (strict inclusion);
2. energy spectrum of $u \in X$ respects Kraichnan spectrum (up to log factors perhaps).

• $E(\kappa) \sim \kappa^{-1} \|\Delta_j u\|_{L^2}^2$ with $\kappa \sim 2^j$ (Littlewood-Paley spectrum, P. Costantin).

• Choose $u \in X = B^1_{2,\infty} \Leftrightarrow \omega \in B^0_{2,\infty}$ (Eyink).

Question of existence of Euler solutions in this class.

Energy spectrum cont.

• $u \in H^1$ ($\Leftrightarrow \omega \in L^2$) gives $E(\kappa) = o(\kappa^{-3})$, upper bound is Kraichnan spectrum.

• To have dissipation of enstrophy must go below L^2.

Look for function spaces X for the velocity such that:

1. $H^1 \subset X$ (strict inclusion);
2. energy spectrum of $u \in X$ respects Kraichnan spectrum (up to log factors perhaps).

• $E(\kappa) \sim \kappa^{-1} \|\Delta_j u\|_{L^2}^2$ with $\kappa \sim 2^j$ (Littlewood-Paley spectrum, P. Costantin).

• Choose $u \in X = B^1_{2,\infty} \Leftrightarrow \omega \in B^0_{2,\infty}$ (Eyink).

Question of existence of Euler solutions in this class.

Weak solutions

Consider initial vorticity with *compact support* and $\omega_0 \in L^p_c$, $p \geq 4/3$.

$\omega \in L^\infty([0,T); L^p)$ is an Euler weak solution with data ω_0 if

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \omega + \nabla\varphi \cdot u\,\omega \; dxdt + \int_{\mathbb{R}^2} \varphi(x,0)\omega_0(x)\; dx = 0, \quad \forall \phi \in C^\infty_c.$$

Uniqueness only for nearly bounded vorticity (Yudovich, Vishik).

Existence for measures ($\omega_0 \in (\mathcal{B}M_{c,+} + L^1_c) \cap H^{-1}_{\text{loc}}$),
e.g. vortex sheets (Delort, Majda, Schochet, Vecchi-Wu), using a *different* weak formulation ($\omega \in L^p$, $p < 4/3 \Rightarrow u\,\omega \notin L^1_{\text{loc}}$).

Weak solutions

Consider initial vorticity with *compact support* and $\omega_0 \in L^p_c$, $p \geq 4/3$.

$\omega \in L^\infty([0,T); L^p)$ is an Euler weak solution with data ω_0 if

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \omega + \nabla\varphi \cdot u\,\omega\; dxdt + \int_{\mathbb{R}^2} \varphi(x,0)\omega_0(x)\; dx = 0, \quad \forall \phi \in C^\infty_c.$$

Uniqueness only for nearly bounded vorticity (Yudovich, Vishik).

Existence for measures ($\omega_0 \in (\mathcal{B}M_{c,+} + L^1_c) \cap H^{-1}_{\text{loc}}$),
e.g. vortex sheets (Delort, Majda, Schochet, Vecchi-Wu), using a *different* weak formulation ($\omega \in L^p$, $p < 4/3 \Rightarrow u\,\omega \notin L^1_{\text{loc}}$).

Enstrophy dissipation

If vorticity ω is bounded, then flow of u is well-defined (i.e., log-Lipschitz) $\Rightarrow$ no enstrophy dissipation is possible.

One approach to reconcile observed behavior of the enstrophy is to account for the dependence of the enstrophy on *Re* in the spectrum of NS solutions (Dritschel-Tran-Scott 2007).

Another approach is to consider unbounded vorticity, corresponding to *irregular* velocities, and study how these velocities transport enstrophy (Eyink).

If u is a vector field with only Sobolev regularity, then its flow is not defined everywhere, but it is defined a.e. $\Rightarrow$

renormalized solutions of transport equations (DiPerna- P. L. Lions).

Enstrophy dissipation

If vorticity ω is bounded, then flow of u is well-defined (i.e., log-Lipschitz) $\Rightarrow$ no enstrophy dissipation is possible.

One approach to reconcile observed behavior of the enstrophy is to account for the dependence of the enstrophy on *Re* in the spectrum of NS solutions (Dritschel-Tran-Scott 2007).

Another approach is to consider unbounded vorticity, corresponding to *irregular* velocities, and study how these velocities transport enstrophy (Eyink).

If u is a vector field with only Sobolev regularity, then its flow is not defined everywhere, but it is defined a.e. $\Rightarrow$

renormalized solutions of transport equations (DiPerna- P. L. Lions).

Enstrophy dissipation

If vorticity ω is bounded, then flow of *u* is well-defined (i.e., log-Lipschitz) $\Rightarrow$ no enstrophy dissipation is possible.

One approach to reconcile observed behavior of the enstrophy is to account for the dependence of the enstrophy on *Re* in the spectrum of NS solutions (Dritschel-Tran-Scott 2007).

Another approach is to consider unbounded vorticity, corresponding to *irregular* velocities, and study how these velocities transport enstrophy (Eyink).

If *u* is a vector field with only Sobolev regularity, then its flow is not defined everywhere, but it is defined a.e. $\Rightarrow$

renormalized solutions of transport equations (DiPerna- P. L. Lions).

Renormalized solutions

Definition

$u \in L^1([0,T], W^{1,1}_{loc})$, div $u = 0$, $\omega \in L^\infty([0,T], L^0)$ (essentially measurable).

ω is a renormalized solution to $\partial_t \omega + u \cdot \nabla \omega = 0$ if

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0, \qquad a.e.,$$

for all β in admissible class $\mathcal{A} = \{\beta \in C^1 \cap L^\infty, \beta \equiv 0 \text{ near } 0\}$.

- Renormalized solutions are unique given u.
- Distribution function is exactly preserved $\Rightarrow$ enstrophy $\Omega(t) = \Omega(0)$, if $\Omega(0)$ finite (initial vorticity $\omega_0 \in L^p$, $p \geq 2$) , since then weak solution of 2D Euler is renormalized solution of transport equation.
- If $p > 2$, can enlarge $\mathcal{A}$ to include $\beta(s) = s^2 \Rightarrow$ enstrophy density $\vartheta(x,t) = |\omega(x,t)|^2/2$ is exactly transported by u (Eyink).

Renormalized solutions

Definition

$u \in L^1([0,T], W^{1,1}_{\mathrm{loc}})$, $\operatorname{div} u = 0$, $\omega \in L^\infty([0,T], L^0)$ (essentially measurable).

ω is a renormalized solution to $\partial_t \omega + u \cdot \nabla \omega = 0$ if

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0, \qquad a.e.,$$

for all β in admissible class $\mathcal{A} = \{\beta \in C^1 \cap L^\infty, \beta \equiv 0 \text{ near } 0\}$.

- Renormalized solutions are unique given u.

- Distribution function is exactly preserved $\Rightarrow$ enstrophy $\Omega(t) = \Omega(0)$, if $\Omega(0)$ finite (initial vorticity $\omega_0 \in L^p$, $p \geq 2$) , since then weak solution of 2D Euler is renormalized solution of transport equation.

- If $p > 2$, can enlarge $\mathcal{A}$ to include $\beta(s) = s^2 \Rightarrow$ enstrophy density $\vartheta(x,t) = |\omega(x,t)|^2/2$ is exactly transported by u (Eyink).

Renormalized solutions

Definition

$u \in L^1([0,T], W^{1,1}_{\text{loc}})$, div $u = 0$, $\omega \in L^\infty([0,T], L^0)$ (essentially measurable).
ω is a renormalized solution to $\partial_t \omega + u \cdot \nabla \omega = 0$ if

$$\partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0, \qquad a.e.,$$

for all β in admissible class $\mathcal{A} = \{\beta \in C^1 \cap L^\infty, \beta \equiv 0 \text{ near } 0\}$.

- Renormalized solutions are unique given u.

- Distribution function is exactly preserved $\Rightarrow$ enstrophy $\Omega(t) = \Omega(0)$, if $\Omega(0)$ finite (initial vorticity $\omega_0 \in L^p$, $p \geq 2$) , since then weak solution of 2D Euler is renormalized solution of transport equation.

- If $p > 2$, can enlarge $\mathcal{A}$ to include $\beta(s) = s^2 \Rightarrow$ enstrophy density $\vartheta(x,t) = |\omega(x,t)|^2/2$ is exactly transported by u (Eyink).

Enstrophy dissipation cont.

Enstrophy dissipation only possible for vorticities in L^2 or below.

- Finite-enstrophy case, initial vorticity $\omega_0 \in L^2_c$: dissipation may arise from irregular transport of enstrophy density ϑ.

  Need to define non-linear term $u \cdot \nabla\vartheta = \text{div}\, u\, \theta$ in transport equation (u unbounded, $\theta \in L^1$). No cancellation between $\vartheta = \omega^2$ and u (when $\omega \in L^1$, cancellation between u and ω by antisymmetry of Biot- Savart Op.).

- Infinite-enstrophy case, initial vorticity $\omega_0 \in B^0_{2,\infty}$: define non-trivial enstrophy defects as limit of flux terms in local balance equation after regularization (Eyink).

  Regularization by vanishing viscosity and mollification. Give different enstrophy dissipation in the limit.

Enstrophy dissipation cont.

Enstrophy dissipation only possible for vorticities in L^2 or below.

- Finite-enstrophy case, initial vorticity $\omega_0 \in L^2_c$: dissipation may arise from irregular transport of enstrophy density ϑ.

  Need to define non-linear term $u \cdot \nabla\vartheta = \operatorname{div} u\,\theta$ in transport equation (u unbounded, $\theta \in L^1$). No cancellation between $\vartheta = \omega^2$ and u (when $\omega \in L^1$, cancellation between u and ω by antisymmetry of Biot- Savart Op.).

- Infinite-enstrophy case, initial vorticity $\omega_0 \in B^0_{2,\infty}$: define non-trivial enstrophy defects as limit of flux terms in local balance equation after regularization (Eyink).

  Regularization by vanishing viscosity and mollification. Give different enstrophy dissipation in the limit.

Enstrophy dissipation cont.

Enstrophy dissipation only possible for vorticities in L^2 or below.

- Finite-enstrophy case, initial vorticity $\omega_0 \in L^2_c$: dissipation may arise from irregular transport of enstrophy density ϑ.

  Need to define non-linear term $u \cdot \nabla \vartheta = \text{div}\, u\, \theta$ in transport equation (u unbounded, $\theta \in L^1$). No cancellation between $\vartheta = \omega^2$ and u (when $\omega \in L^1$, cancellation between u and ω by antisymmetry of Biot- Savart Op.).

- Infinite-enstrophy case, initial vorticity $\omega_0 \in B^0_{2,\infty}$: define non-trivial enstrophy defects as limit of flux terms in local balance equation after regularization (Eyink).

  Regularization by vanishing viscosity and mollification. Give different enstrophy dissipation in the limit.

Viscosity solutions: $\omega = \lim_{\nu \to 0} \omega_\nu$

ω_ν solution of 2D Navier-Stokes equations with $\omega_\nu(0) = \omega_0$:

$$\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0,$$
$$u_\nu = K * \omega_\nu,$$

Mollification: $\omega = \lim_{\epsilon \to 0} \omega_\epsilon$

where $\begin{cases} \omega_\epsilon = j_\epsilon * \omega, \\ u_\epsilon = j_\epsilon * u. \end{cases}$

$j_\epsilon(x) = \epsilon^{-n} j(\epsilon |x|)$ standard, radially-symmetric mollifiers.

Regularized enstrophy density $\theta_\nu = \dfrac{1}{2}\omega_\nu^2$ and $\theta_\epsilon = \frac{1}{2}\omega_\epsilon^2$ satisfies balance equations with non-trivial flux terms.

Two regularization

Viscosity solutions: $\omega = \lim_{\nu \to 0} \omega_\nu$

ω_ν solution of 2D Navier-Stokes equations with $\omega_\nu(0) = \omega_0$:

$$\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0,$$
$$u_\nu = K * \omega_\nu,$$

Mollification: $\omega = \lim_{\epsilon \to 0} \omega_\epsilon$

where $$\begin{cases} \omega_\epsilon = j_\epsilon * \omega, \\ u_\epsilon = j_\epsilon * u. \end{cases}$$

$j_\epsilon(x) = \epsilon^{-n} j(\epsilon |x|)$ standard, radially-symmetric mollifiers.

Regularized enstrophy density $\theta_\nu = \dfrac{1}{2}\omega_\nu^2$ and $\theta_\epsilon = \frac{1}{2}\omega_\epsilon^2$ satisfies balance equations with non-trivial flux terms.

Enstrophy defects

Transport enstrophy defect: let ω any weak solution to Euler
$\Rightarrow \vartheta_\epsilon(x,t)$ satisfies:

$$\partial_t \vartheta_\epsilon + \text{div}\left[u_\epsilon \vartheta_\epsilon + \omega_\epsilon\left(j_\epsilon * (u\omega) - u_\epsilon \omega_\epsilon\right)\right] = -Z_\epsilon(\omega), \tag{2}$$

Measure dissipation due to irregular transport :

$$Z^T(\omega) := \lim_{\epsilon \to 0} Z_\epsilon(\omega) = \lim_{\epsilon \to 0} [-\nabla \omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)]$$

Viscous enstrophy defect: let ω viscosity solution
$\Rightarrow \vartheta_\nu(x,t)$ satisfies:

$$\partial_t \vartheta_\nu + u_\nu \cdot \nabla \vartheta_\nu - \nu \Delta \vartheta_\nu = -Z^\nu(\omega_\nu), \tag{3}$$

Measure dissipation due to viscosity:

$$Z^V(\omega) \equiv \lim_{\nu \to 0} Z^\nu(\omega) = \lim_{\nu \to 0} \nu |\nabla \omega_\nu|^2 \geq 0,$$

Enstrophy defects

Transport enstrophy defect: let ω any weak solution to Euler
$\Rightarrow \vartheta_\epsilon(x,t)$ satisfies:

$$\partial_t \vartheta_\epsilon + \mathrm{div}\left[u_\epsilon \vartheta_\epsilon + \omega_\epsilon \left(j_\epsilon * (u\omega) - u_\epsilon \omega_\epsilon\right)\right] = -Z_\epsilon(\omega), \tag{2}$$

Measure dissipation due to irregular transport :

$$Z^T(\omega) := \lim_{\epsilon\to 0} Z_\epsilon(\omega) = \lim_{\epsilon\to 0}\left[-\nabla\omega_\epsilon \cdot ((u\omega)_\epsilon - u_\epsilon \omega_\epsilon)\right]$$

Viscous enstrophy defect: let ω viscosity solution
$\Rightarrow \vartheta_\nu(x,t)$ satisfies:

$$\partial_t \vartheta_\nu + u_\nu \cdot \nabla \vartheta_\nu - \nu \Delta \vartheta_\nu = -Z^\nu(\omega_\nu), \tag{3}$$

Measure dissipation due to viscosity:

$$Z^V(\omega) \equiv \lim_{\nu\to 0} Z^\nu(\omega) = \lim_{\nu\to 0} \nu |\nabla\omega_\nu|^2 \geq 0,$$

Energy dissipation in 3D flows

• Onsager's conjecture: no energy dissipation for Euler solutions in $C([0,T), C^\alpha)$, $\alpha > 1/3$, $\exists$ dissipative solutions for $\alpha \leq 1/3$.

No energy dissipation proved in C^α (Eyink 1994);
for $u \in C([0,T), B^\alpha_{3,\infty})$, $\alpha > 1/3$ (Constantin-E-Titi 1994);
refinement of this result (Cheskidov-Constantin-Friedlander-Shvydkoy 2008).

Energy dissipation for: $u \in L^3([0,T), L^3)$ (Duchon-Robert 2000);
$u \in L^2([0,T), L^2)$, (Scheffer 1993, Shnirelman 2000);
$u \in L^1([0,T); C^{1/3-\epsilon})$), $\epsilon > 0$ (De Lellis-Szekelyhidi 2007-2012, Isett 2012, Buckmaster-De Lellis-Szekelyhidi 2013-14), compact support (finite energy).

• Rigorous estimates on energy spectrum for NS solutions (Doering-Titi 1995, Doering-Foias 2002, Doering-Gibbon 2005).

Energy dissipation in 3D flows

• Onsager's conjecture: no energy dissipation for Euler solutions in $C([0,T),C^\alpha)$, $\alpha > 1/3$, $\exists$ dissipative solutions for $\alpha \leq 1/3$.

No energy dissipation proved in C^α (Eyink 1994);

for $u \in C([0,T),B^\alpha_{3,\infty})$, $\alpha > 1/3$ (Constantin-E-Titi 1994);

refinement of this result (Cheskidov-Constantin-Friedlander-Shvydkoy 2008).

Energy dissipation for: $u \in L^3([0,T),L^3)$ (Duchon-Robert 2000);

$u \in L^2([0,T),L^2)$, (Scheffer 1993, Shnirelman 2000);

$u \in L^1([0,T);C^{1/3-\epsilon}))$, $\epsilon > 0$ (De Lellis-Szekelyhidi 2007-2012, Isett 2012, Buckmaster-De Lellis-Szekelyhidi 2013-14), compact support (finite energy).

• Rigorous estimates on energy spectrum for NS solutions (Doering-Titi 1995, Doering-Foias 2002, Doering-Gibbon 2005).

Energy dissipation in 3D flows

• Onsager's conjecture: no energy dissipation for Euler solutions in $C([0,T), C^\alpha)$, $\alpha > 1/3$, $\exists$ dissipative solutions for $\alpha \leq 1/3$.

No energy dissipation proved in C^α (Eyink 1994);

for $u \in C([0,T), B^\alpha_{3,\infty})$, $\alpha > 1/3$ (Constantin-E-Titi 1994);

refinement of this result (Cheskidov-Constantin-Friedlander-Shvydkoy 2008).

Energy dissipation for: $u \in L^3([0,T), L^3)$ (Duchon-Robert 2000);

$u \in L^2([0,T), L^2)$, (Scheffer 1993, Shnirelman 2000);

$u \in L^1([0,T); C^{1/3-\epsilon}))$, $\epsilon > 0$ (De Lellis-Szekelyhidi 2007-2012, Isett 2012, Buckmaster-De Lellis-Szekelyhidi 2013-14), compact support (finite energy).

• Rigorous estimates on energy spectrum for NS solutions (Doering-Titi 1995, Doering-Foias 2002, Doering-Gibbon 2005).

Finite enstrophy data

If $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p \geq 2$, total enstrophy is conserved (flow is measure-preserving). If $p > 2$, enstrophy density $\vartheta = \frac{1}{2}|\omega|^2$ transported by u (DiPerna-Lions).

If $\omega_0 \in L^2_c$, ϑ may not satisfy the advection equation:

$$\vartheta_t + \text{div}(u\,\vartheta) = 0.$$

Non-zero enstrophy defects may exist solely from irregular transport.

Simple energy estimates on $2D$ Navier-Stokes imply

$$Z^V(\omega) = \lim_{\nu \to 0_+} \nu \|\omega\|^2_{L^2} \equiv 0.$$

To control transport of θ need more regularity on ω.

Finite enstrophy data

If $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p \geq 2$, total enstrophy is conserved (flow is measure-preserving). If $p > 2$, enstrophy density $\vartheta = \frac{1}{2}|\omega|^2$ transported by u (DiPerna-Lions).

If $\omega_0 \in L^2_c$, ϑ may not satisfy the advection equation:

$$\vartheta_t + \text{div}(u\,\vartheta) = 0.$$

Non-zero enstrophy defects may exist solely from irregular transport.

Simple energy estimates on $2D$ Navier-Stokes imply

$$Z^V(\omega) = \lim_{\nu \to 0_+} \nu \|\omega\|^2_{L^2} \equiv 0.$$

To control transport of θ need more regularity on ω.

Finite enstrophy data

If $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p \geq 2$, total enstrophy is conserved (flow is measure-preserving). If $p > 2$, enstrophy density $\vartheta = \frac{1}{2}|\omega|^2$ transported by u (DiPerna-Lions).

If $\omega_0 \in L^2_c$, ϑ may not satisfy the advection equation:

$$\vartheta_t + \text{div}(u\,\vartheta) = 0.$$

Non-zero enstrophy defects may exist solely from irregular transport.

Simple energy estimates on $2D$ Navier-Stokes imply

$$Z^V(\omega) = \lim_{\nu \to 0_+} \nu \|\omega\|^2_{L^2} \equiv 0.$$

To control transport of θ need more regularity on ω.

Pick ω_0 in $L^2_c \log L^{1/4}$, logarithmic refinement of $L^2 \Rightarrow$ norm invariant under measure-preserving flows, and $u\,\theta \in L^1_{\text{loc}}$.

Theorem 1

If $\omega \in L^\infty([0,T); L^2(\log L)^{1/4})$ is a viscosity solution of 2D Euler in L^2, then

$$\partial_t(\theta) + \operatorname{div}(u\,\theta) = 0,$$

where $u = K * \omega$,

Finite enstrophy data - Results

Pick ω_0 in $L^2_c \log L^{1/4}$, logarithmic refinement of $L^2 \Rightarrow$ norm invariant under measure-preserving flows, and $u\,\theta \in L^1_{\text{loc}}$.

Theorem 1

If $\omega \in L^\infty([0,T); L^2(\log L)^{1/4})$ is a viscosity solution of 2D Euler in L^2, then

$$\partial_t(\theta) + \operatorname{div}(u\,\theta) = 0,$$

where $u = K * \omega$,

Finite enstrophy results cont.

Theorem 2

If $\omega \in L^\infty([0,T); L^2(\log L)^{1/4}) \cap L^1$ is a weak solution of 2D Euler, then $Z^T(\omega)$ exists (as a distribution).

If ω is, in addition, a viscosity solution, then $Z^T(\omega) \equiv 0$.

Euler solutions ω in L^2 such that $Z^T(\omega) > 0$ would give agreement with Kraichnan-Batchelor and indicate that non-linear interactions are responsible for enstrophy dissipation at very low viscosity, if enstrophy is finite.

Non uniqueness of solutions to 2D Euler with L^2-regularity follows if there exists a solution $\omega \in L^2 \log L^{1/4}$ such that $Z^T(\omega) \neq 0$.

Finite enstrophy results cont.

Theorem 2

If $\omega \in L^{\infty}([0,T); L^2(\log L)^{1/4}) \cap L^1$ is a weak solution of 2D Euler, then $Z^T(\omega)$ exists (as a distribution).

If ω is, in addition, a viscosity solution, then $Z^T(\omega) \equiv 0$.

Euler solutions ω in L^2 such that $Z^T(\omega) > 0$ would give agreement with Kraichnan-Batchelor and indicate that non-linear interactions are responsible for enstrophy dissipation at very low viscosity, if enstrophy is finite.

Non uniqueness of solutions to 2D Euler with L^2-regularity follows if there exists a solution $\omega \in L^2 \log L^{1/4}$ such that $Z^T(\omega) \neq 0$.

Proof of Theorem 1:

- ω viscosity solution. Show that θ transported by u.
  Pass to the limit $\nu \to 0$ in

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta_\nu \, dxdt + \int_0^T \int_{\mathbb{R}^2} \nabla\varphi \cdot u_\nu \vartheta_\nu \, dxdt = \int_0^T \int_{\mathbb{R}^2} \nu \Delta\varphi \, \vartheta_\nu \, dxdt - \int_0^T \int_{\mathbb{R}^2} \varphi Z^\nu(\omega_\nu) \, dxdt.$$

- $\omega_\nu(t) \to \omega(t)$ strongly in $L^2(\mathbb{R}^2) \Rightarrow$
  $\vartheta_\nu \to \vartheta$ and $Z^\nu(\omega_\nu) \to 0$ in $L^1([0,T) \times \mathbb{R}^2)$

- In the limit, θ satisfies:

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta \, dxdt + \lim_{\nu\to 0} \int_0^T \int_{\mathbb{R}^2} \nabla\phi \cdot u \, \theta \, dxdt = 0.$$

Proof of Theorem 1:

- ω viscosity solution. Show that θ transported by u.
  Pass to the limit $\nu \to 0$ in

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta_\nu \, dxdt + \int_0^T \int_{\mathbb{R}^2} \nabla\varphi \cdot u_\nu \vartheta_\nu \, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^2} \nu \Delta\varphi \, \vartheta_\nu \, dxdt - \int_0^T \int_{\mathbb{R}^2} \varphi Z^\nu(\omega_\nu) \, dxdt.$$

- $\omega_\nu(t) \to \omega(t)$ strongly in $L^2(\mathbb{R}^2) \Rightarrow$
  $\vartheta_\nu \to \vartheta$ and $Z^\nu(\omega_\nu) \to 0$ in $L^1([0,T) \times \mathbb{R}^2)$

- In the limit, θ satisfies:

$$\int_0^T \int_{\mathbb{R}^2} \varphi_t \vartheta \, dxdt + \lim_{\nu\to 0} \int_0^T \int_{\mathbb{R}^2} \nabla\phi \cdot u \, \theta \, dxdt = 0.$$

Proof cont.

- Analyze behavior of non-linear term as $\nu \to 0$:

$$-\int_0^T \int_{\mathbb{R}^2} \omega_\nu(y,t) \int_{\mathbb{R}^2} K(y-x) \cdot \nabla\varphi(x,t)\vartheta_\nu(x,t)\, dx\, dydt.$$

- $\vartheta_\nu \to \vartheta$ strongly in $L^1([0,T) \times \mathbb{R}^2) \Rightarrow K * (\nabla\phi\vartheta_\nu) \to K * (\nabla\phi\vartheta)$ weakly in $\mathcal{D}'$.

- Uniform bound for $\omega_\nu(t)$ in $L^2 \log L^{1/4}$ from divergence-free condition on u and convexity of $[s \log^{1/4}(2+s)]^2$:

$$\|\omega_\nu(t)\|_{L^2 \log L^{1/4}} \leq \|\omega_0\|_{L^2 (\log L^{1/4}}.$$

- Product estimate:

$$\|f\, g\|_{L \log L^{1/2}} \leq 4(\max\{\|f\|_{L^2 \log L^{1/4}}, \|g\|_{L^2 \log L^{1/4}}\})^2.$$

Product estimate + uniform bound $\Rightarrow \{\nabla\phi\theta_\nu\}$ bounded in $L^\infty((0,T), L \log L^{1/2})$.

Proof cont.

- Analyze behavior of non-linear term as $\nu \to 0$:

$$-\int_0^T \int_{\mathbb{R}^2} \omega_\nu(y,t) \int_{\mathbb{R}^2} K(y-x) \cdot \nabla\varphi(x,t)\vartheta_\nu(x,t)\, dx\, dydt.$$

- $\vartheta_\nu \to \vartheta$ strongly in $L^1([0,T) \times \mathbb{R}^2) \Rightarrow K * (\nabla\phi\vartheta_\nu) \to K * (\nabla\phi\vartheta)$ weakly in $\mathcal{D}'$.

- Uniform bound for $\omega_\nu(t)$ in $L^2 \log L^{1/4}$ from divergence-free condition on u and convexity of $[s \log^{1/4}(2+s)]^2$:

$$\|\omega_\nu(t)\|_{L^2 \log L^{1/4}} \leq \|\omega_0\|_{L^2(\log L^{1/4}}.$$

- Product estimate:

$$\|f\,g\|_{L \log L^{1/2}} \leq 4(\max\{\|f\|_{L^2 \log L^{1/4}}, \|g\|_{L^2 \log L^{1/4}}\})^2.$$

Product estimate + uniform bound $\Rightarrow \{\nabla\phi\theta_\nu\}$ bounded in $L^\infty((0,T), L \log L^{1/2})$.

Proof cont.

- Analyze behavior of non-linear term as $\nu \to 0$:

$$-\int_0^T \int_{\mathbb{R}^2} \omega_\nu(y,t) \int_{\mathbb{R}^2} K(y-x) \cdot \nabla\varphi(x,t)\vartheta_\nu(x,t)\,dx\,dydt.$$

- $\vartheta_\nu \to \vartheta$ strongly in $L^1([0,T) \times \mathbb{R}^2) \Rightarrow K * (\nabla\phi\vartheta_\nu) \to K * (\nabla\phi\vartheta)$ weakly in $\mathcal{D}'$.

- Uniform bound for $\omega_\nu(t)$ in $L^2 \log L^{1/4}$ from divergence-free condition on u and convexity of $[s \log^{1/4}(2+s)]^2$:

$$\|\omega_\nu(t)\|_{L^2 \log L^{1/4}} \leq \|\omega_0\|_{L^2(\log L^{1/4}}.$$

- Product estimate:

$$\|f\,g\|_{L \log L^{1/2}} \leq 4(\max\{\|f\|_{L^2 \log L^{1/4}}, \|g\|_{L^2 \log L^{1/4}}\})^2.$$

Product estimate + uniform bound $\Rightarrow \{\nabla\phi\theta_\nu\}$ bounded in $L^\infty((0,T), L \log L^{1/2})$.

Proof cont.

- Biot-Savart operator K smoothing of order 1 and $L^\infty((0,T), L(\log L)^{1/2}) \hookrightarrow L^\infty((0,T), H^{-1}_{\text{loc}}) \Rightarrow$

$$\{K * \nabla\phi\vartheta_\nu\} \text{ is bounded in } L^\infty((0,T); L^2_{\text{loc}}).$$

- Fix $R > 0$ large, do "near" and "far field" expansion of non-linear term. Have "weak-strong" pairs in each:

$$\int_0^T \int_{B_{2R}} \omega_\nu K * (\nabla\phi\vartheta_\nu)\, dx\, dt + \int_0^T \int_{B^c_{2R}} \omega_\nu K * (\nabla\phi\vartheta_\nu)\, dx\, dt$$

$$\to \langle \omega, K * (\nabla\phi\,\vartheta)\rangle = \langle u\,\omega, \nabla\phi\rangle,$$

- If Supp $\phi \subset B_R$, show $|K * (\nabla\phi\vartheta_\nu)(y,t)| \lesssim \Omega/|y|$, $|y| > 2R \Rightarrow$ $K * \nabla\phi\vartheta_\nu \to K * (\nabla\phi\vartheta)$ weakly in $L^\infty((0,T) \times B(0,2R)^c)$.

- $\omega_\nu \to \omega$ strongly in $L^1([0,T] \times \mathbb{R}^2)$ from uniform bound on L^1 norm (maximum principle) and strong convergence in L^2.

Proof cont.

- Biot-Savart operator K smoothing of order 1 and $L^\infty((0,T), L(\log L)^{1/2}) \hookrightarrow L^\infty((0,T), H^{-1}_{\text{loc}}) \Rightarrow$

$$\{K * \nabla\phi\vartheta_\nu\} \text{ is bounded in } L^\infty((0,T); L^2_{\text{loc}}).$$

- Fix $R > 0$ large, do "near" and "far field" expansion of non-linear term. Have "weak-strong" pairs in each:

$$\int_0^T \int_{B_{2R}} \omega_\nu\, K * (\nabla\phi\vartheta_\nu)\, dx\, dt + \int_0^T \int_{B^c_{2R}} \omega_\nu\, K * (\nabla\phi\vartheta_\nu)\, dx\, dt$$

$$\to \langle \omega, K * (\nabla\phi\, \vartheta)\rangle = \langle u\,\omega, \nabla\phi\rangle,$$

- If $\operatorname{Supp}\phi \subset B_R$, show $|K * (\nabla\phi\vartheta_\nu)(y,t)| \lesssim \Omega/|y|$, $|y| > 2R \Rightarrow$ $K * \nabla\phi\vartheta_\nu \to K * (\nabla\phi\vartheta)$ weakly in $L^\infty((0,T) \times B(0,2R)^c)$.

- $\omega_\nu \to \omega$ strongly in $L^1([0,T] \times \mathbb{R}^2)$ from uniform bound on L^1 norm (maximum principle) and strong convergence in L^2.

Proof cont.

- Biot-Savart operator K smoothing of order 1 and $L^\infty((0,T), L(\log L)^{1/2}) \hookrightarrow L^\infty((0,T), H^{-1}_{\text{loc}}) \Rightarrow$

$$\{K * \nabla\phi\vartheta_\nu\} \text{ is bounded in } L^\infty((0,T); L^2_{\text{loc}}).$$

- Fix $R > 0$ large, do "near" and "far field" expansion of non-linear term. Have "weak-strong" pairs in each:

$$\int_0^T \int_{B_{2R}} \omega_\nu \, K * (\nabla\phi\vartheta_\nu)\, dx\, dt + \int_0^T \int_{B^c_{2R}} \omega_\nu \, K * (\nabla\phi\vartheta_\nu)\, dx\, dt$$

$$\to \langle \omega, K * (\nabla\phi\,\vartheta)\rangle = \langle u\,\omega, \nabla\phi\rangle,$$

- If $\operatorname{Supp}\phi \subset B_R$, show $|K * (\nabla\phi\vartheta_\nu)(y,t)| \lesssim \Omega/|y|$, $|y| > 2R \Rightarrow$ $K * \nabla\phi\vartheta_\nu \to K * (\nabla\phi\vartheta)$ weakly in $L^\infty((0,T) \times B(0,2R)^c)$.

- $\omega_\nu \to \omega$ strongly in $L^1([0,T] \times \mathbb{R}^2)$ from uniform bound on L^1 norm (maximum principle) and strong convergence in L^2.

Near optimality of main results

Regularity $\omega \in L^2_c \log L^{1/4}$ is close to optimal $\Rightarrow$ there exist $\omega \in L^2 \log L^{1/6}$ such that $(K * \omega)\,\omega^2 = u\,\vartheta \notin \mathcal{D}'$.

But ω not the solution of 2D Euler.

Define $\omega^{\pm}(x) = \dfrac{1}{|x||\log|x||^{\alpha}}\,\chi_{D^{\pm}(0;1/3)}(x)$, $u^{\pm} = K * \omega^{\pm}$, where

$$1/2 < \alpha < 1, \qquad \begin{cases} D^{+}(0;1/3) = D(0,1/3) \cap \{x_2 > 0\}, \\ D^{-}(0;1/3) = D(0,1/3) \cap \{x_2 < 0\}. \end{cases}$$

Define $\tilde{\omega}(x) = \dfrac{1}{|x||\log|x||^{\alpha}}\,\chi_{D(0;1/3)}(x)$, $\tilde{u} = K * \tilde{\omega}$.

Near optimality of main results

Regularity $\omega \in L^2_c \log L^{1/4}$ is close to optimal $\Rightarrow$ there exist $\omega \in L^2 \log L^{1/6}$ such that $(K * \omega)\,\omega^2 = u\,\vartheta \notin \mathcal{D}'$.

But ω not the solution of 2D Euler.

Define $\omega^{\pm}(x) = \dfrac{1}{|x||\log|x||^{\alpha}}\,\chi_{D^{\pm}(0;1/3)}(x)$, $u^{\pm} = K * \omega^{\pm}$, where

$$1/2 < \alpha < 1, \qquad \begin{cases} D^{+}(0;1/3) = D(0,1/3) \cap \{x_2 > 0\}, \\ D^{-}(0;1/3) = D(0,1/3) \cap \{x_2 < 0\}. \end{cases}$$

Define $\tilde{\omega}(x) = \dfrac{1}{|x||\log|x||^{\alpha}}\,\chi_{D(0;1/3)}(x)$, $\tilde{u} = K * \tilde{\omega}$.

Optimality cont.

Prove $u^+ (\omega^+)^2 \notin L^1_{\text{loc}}(\mathbb{R}^2)$, and, if ω_n^+ truncation of ω, then $u_n^+ (\omega_n^+)^2$, $n \to \infty$, does not converge in $\mathcal{D}'$.

To do so, show $|u^+(x)| \geq C|\log|x||^{1-\alpha}$ near origin.

Proof:

Note $u^+ = \tilde{u} - u^-$, where $\tilde{u}$ radial and u^- harmonic in $\mathbb{H}^+$.

$\tilde{u} = K * \tilde{\omega}$ bounded, because $\tilde{\omega} \in L^2$ radial.

Obtain growth of u^- by evaluating $K * \omega^-$ directly on real axis and using potential estimates on its harmonic extension to $\mathbb{H}^+$.

Set $\omega = \omega^+$. $\square$

Optimality cont.

Prove $u^+ (\omega^+)^2 \notin L^1_{\text{loc}}(\mathbb{R}^2)$, and, if ω_n^+ truncation of ω, then $u_n^+ (\omega_n^+)^2$, $n \to \infty$, does not converge in $\mathcal{D}'$.

To do so, show $|u^+(x)| \geq C|\log|x||^{1-\alpha}$ near origin.

Proof:

Note $u^+ = \tilde{u} - u^-$, where $\tilde{u}$ radial and u^- harmonic in $\mathbb{H}^+$.

$\tilde{u} = K * \tilde{\omega}$ bounded, because $\tilde{\omega} \in L^2$ radial.

Obtain growth of u^- by evaluating $K * \omega^-$ directly on real axis and using potential estimates on its harmonic extension to $\mathbb{H}^+$.

Set $\omega = \omega^+$. □

Optimality cont.

Prove $u^+(\omega^+)^2 \notin L^1_{\text{loc}}(\mathbb{R}^2)$, and, if ω_n^+ truncation of ω, then $u_n^+(\omega_n^+)^2$, $n \to \infty$, does not converge in $\mathcal{D}'$.

To do so, show $|u^+(x)| \geq C|\log|x||^{1-\alpha}$ near origin.

<u>Proof:</u>

Note $u^+ = \tilde{u} - u^-$, where $\tilde{u}$ radial and u^- harmonic in $\mathbb{H}^+$.

$\tilde{u} = K * \tilde{\omega}$ bounded, because $\tilde{\omega} \in L^2$ radial.

Obtain growth of u^- by evaluating $K * \omega^-$ directly on real axis and using potential estimates on its harmonic extension to $\mathbb{H}^+$.

Set $\omega = \omega^+$. □

Optimality cont.

Prove $u^+ (\omega^+)^2 \notin L^1_{\text{loc}}(\mathbb{R}^2)$, and, if ω_n^+ truncation of ω, then $u_n^+ (\omega_n^+)^2$, $n \to \infty$, does not converge in $\mathcal{D}'$.

To do so, show $|u^+(x)| \geq C|\log|x||^{1-\alpha}$ near origin.

Proof:

Note $u^+ = \tilde{u} - u^-$, where $\tilde{u}$ radial and u^- harmonic in $\mathbb{H}^+$.

$\tilde{u} = K * \tilde{\omega}$ bounded, because $\tilde{\omega} \in L^2$ radial.

Obtain growth of u^- by evaluating $K * \omega^-$ directly on real axis and using potential estimates on its harmonic extension to $\mathbb{H}^+$.

Set $\omega = \omega^+$. □

Infinite-enstrophy data

Consider initial vorticity ω_0 in the Besov space $B^0_{2,\infty} \supset L^p_c$, $p \geq 2$

$\Rightarrow$ velocity $u_0 = K * \omega_0$ can have energy spectrum $E(\kappa) \sim \kappa^{-3}$.

If upper bound on spectrum as $\nu \to 0$ given by Batchelor-Kraichnan spectrum:

$$\sup_{\nu > 0} \|\omega^\nu\|_{L^2([0,T], B^0_{2,\infty})} < C,$$

then viscosity solutions $\omega = \lim_{\nu \to 0} \omega^\nu$ exist by compactness (Eyink).

Conjecture (Eyink)

Let ω be a viscosity solution with data $\omega_0 \in B^0_{2,\infty}(\mathbb{R}^2)$. Then:

$$Z^T(\omega), Z^V(\omega) \geq 0, \qquad \text{in } \mathcal{D}'.$$

Moreover there exist initial data for which defect is strictly positive.

Infinite-enstrophy data

Consider initial vorticity ω_0 in the Besov space $B^0_{2,\infty} \supset L^p_c$, $p \geq 2$

$\Rightarrow$ velocity $u_0 = K * \omega_0$ can have energy spectrum $E(\kappa) \sim \kappa^{-3}$.

If upper bound on spectrum as $\nu \to 0$ given by Batchelor-Kraichnan spectrum:

$$\sup_{\nu>0} \|\omega^\nu\|_{L^2([0,T],B^0_{2,\infty})} < C,$$

then viscosity solutions $\omega = \lim_{\nu\to 0} \omega^\nu$ exist by compactness (Eyink).

Conjecture (Eyink)

Let ω be a viscosity solution with data $\omega_0 \in B^0_{2,\infty}(\mathbb{R}^2)$. Then:

$$Z^T(\omega), Z^V(\omega) \geq 0, \qquad \text{in } \mathcal{D}'.$$

Moreover there exist initial data for which defect is strictly positive.

Dissipative solutions

Construct exact steady viscosity solution ω to 2D Euler such that

$$Z^T(\omega) = 0, \qquad Z^V(\omega) = \frac{4\pi^3}{t}\delta_o, \qquad t > 0.$$

data in $L^{2,\infty} \cap B^0_{2,\infty}$, rearrangement-invariant space of functions with KB energy spectrum $\Rightarrow$

- Strictly dissipative solutions exist.
- Notion of enstrophy defect **depends** on the approximating sequence.

Dissipative solutions

Construct exact steady viscosity solution ω to 2D Euler such that

$$Z^T(\omega) = 0, \qquad Z^V(\omega) = \frac{4\pi^3}{t}\delta_o, \qquad t > 0.$$

data in $L^{2,\infty} \cap B^0_{2,\infty}$, rearrangement-invariant space of functions with KB energy spectrum $\Rightarrow$

- Strictly dissipative solutions exist.
- Notion of enstrophy defect **depends** on the approximating sequence.

Rankine-like Vortices

Let $\omega(x) = \tilde{\omega}(r)$, $r = |x|$. Set

$$u(x) = r^{-2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \int_0^r s\tilde{\omega}(s)\, ds.$$

$u(x)$ is an exact steady solution to 2D Euler $\Rightarrow u(x) \perp \nabla\omega(x)$.
Example of coherent vortex (DiPerna-Majda).

If ω_ν is solution to 2D heat equation with initial data ω, same formula give rise to exact solution to 2D Navier-Stokes.

Take $\omega = \phi(x)\frac{1}{|x|}$, ϕ positive bump function on $B(0,1)$.

Rankine-like Vortices

Let $\omega(x) = \tilde{\omega}(r)$, $r = |x|$. Set

$$u(x) = r^{-2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \int_0^r s\tilde{\omega}(s)\, ds.$$

$u(x)$ is an exact steady solution to 2D Euler $\Rightarrow u(x) \perp \nabla\omega(x)$.
Example of coherent vortex (DiPerna-Majda).

If ω_ν is solution to 2D heat equation with initial data ω, same formula give rise to exact solution to 2D Navier-Stokes.

Take $\omega = \phi(x)\dfrac{1}{|x|}$, ϕ positive bump function on $B(0,1)$.

Rankine-like Vortices

Let $\omega(x) = \tilde{\omega}(r)$, $r = |x|$. Set

$$u(x) = r^{-2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \int_0^r s\tilde{\omega}(s)\, ds.$$

$u(x)$ is an exact steady solution to 2D Euler $\Rightarrow u(x) \perp \nabla\omega(x)$. Example of coherent vortex (DiPerna-Majda).

If ω_ν is solution to 2D heat equation with initial data ω, same formula give rise to exact solution to 2D Navier-Stokes.

Take $\omega = \phi(x)\dfrac{1}{|x|}$, ϕ positive bump function on $B(0, 1)$.

Positive enstrophy defect

Mollifier j_ϵ is radial $\Rightarrow$ $\nabla\omega_\epsilon \perp u_\epsilon$ so that $Z_\epsilon \equiv 0 \Rightarrow Z^T \equiv 0$.

Recall $\omega_\nu = e^{\nu t \Delta}\,\omega$.

Homogeneity of ω give

$$\|Z^\nu(\omega)\|_{L^1([0,t]\times\mathbb{R}^2)} = \frac{4\pi^3}{t} + l.o.t., \qquad \nu \to 0_+,\ t > 0. \qquad \text{(A)}$$

In fact, if $\omega_\nu = e^{\nu t\Delta}\omega$:

$$\|Z^\nu(\omega)(t)\|_{L^1} = \nu\|\nabla e^{\nu t\,\Delta}\omega\|^2_{L^2} \le C/t\,\|\omega\|^2_{B^0_{2,\infty}}, \qquad 0 < \nu t \le 1,$$

Uniform bound in $L^1(\mathbb{R}^2)$ for each $t > 0$ $\Rightarrow$ $\exists\ \nu_k$ and a Radon measure $\mu(t)$ such that $Z^{\nu_k}(\omega)(t) \rightharpoonup \mu(t)$.

Positive enstrophy defect

Mollifier j_ϵ is radial $\Rightarrow \nabla\omega_\epsilon \perp u_\epsilon$ so that $Z_\epsilon \equiv 0 \Rightarrow Z^T \equiv 0$.

Recall $\omega_\nu = e^{\nu t\,\Delta}\,\omega$.

Homogeneity of ω give

$$\|Z^\nu(\omega)\|_{L^1([0,t]\times\mathbb{R}^2)} = \frac{4\pi^3}{t} + l.o.t., \qquad \nu \to 0_+,\ t > 0. \qquad \text{(A)}$$

In fact, if $\omega_\nu = e^{\nu t\Delta}\omega$:

$$\|Z^\nu(\omega)(t)\|_{L^1} = \nu\|\nabla e^{\nu t\,\Delta}\omega\|_{L^2}^2 \leq C/t\,\|\omega\|_{B^0_{2,\infty}}^2, \qquad 0 < \nu t \leq 1,$$

Uniform bound in $L^1(\mathbb{R}^2)$ for each $t > 0 \;\Rightarrow\; \exists\,\nu_k$ and a Radon measure $\mu(t)$ such that $Z^{\nu_k}(\omega)(t) \rightharpoonup \mu(t)$.

Positive enstrophy defect

Mollifier j_ϵ is radial $\Rightarrow$ $\nabla\omega_\epsilon \perp u_\epsilon$ so that $Z_\epsilon \equiv 0 \Rightarrow Z^T \equiv 0$.

Recall $\omega_\nu = e^{\nu t \Delta}\omega$.

Homogeneity of ω give

$$\|Z^\nu(\omega)\|_{L^1([0,t]\times\mathbb{R}^2)} = \frac{4\pi^3}{t} + l.o.t., \qquad \nu \to 0_+,\ t > 0. \qquad \text{(A)}$$

In fact, if $\omega_\nu = e^{\nu t\Delta}\omega$:

$$\|Z^\nu(\omega)(t)\|_{L^1} = \nu\|\nabla e^{\nu t \Delta}\omega\|_{L^2}^2 \leq C/t\,\|\omega\|_{B^0_{2,\infty}}^2, \qquad 0 < \nu t \leq 1,$$

Uniform bound in $L^1(\mathbb{R}^2)$ for each $t > 0$ $\Rightarrow$ $\exists\, \nu_k$ and a Radon measure $\mu(t)$ such that $Z^{\nu_k}(\omega)(t) \rightharpoonup \mu(t)$.

Enstrophy defect cont.

Near and far-field estimate on the heat kernel $\Rightarrow$ Every convergent subsequence $Z^{\nu_k}(t) := Z^{\nu_k}(\omega)(t)$ converges to a distribution supported at the origin ($Z^{\nu} \geq 0$, hence limit is a measure):

$$Z^{\nu_k}(t) \rightharpoonup C_{\{\nu_k\}}(t)\,\delta.$$

Show $C_{\{\nu_k\}}(t) = \frac{4\pi^3}{t}$.

$Z^{\nu_k}(t)$ is a *tight* family of functions in $L^1(\mathbb{R}^2) \Rightarrow$

$$\int Z^{\nu_k}(t,x)\,dx \approx < Z^{\nu_k}(t), \phi >,$$

for ϕ a bump function on large ball.

Then, pass to limit $\nu \to 0$, use weak convergence of defects and explicit uniform L^1 bound.

Enstrophy defect cont.

Near and far-field estimate on the heat kernel $\Rightarrow$ Every convergent subsequence $Z^{\nu_k}(t) := Z^{\nu_k}(\omega)(t)$ converges to a distribution supported at the origin ($Z^\nu \geq 0$, hence limit is a measure):

$$Z^{\nu_k}(t) \rightharpoonup C_{\{\nu_k\}}(t)\,\delta.$$

Show $C_{\{\nu_k\}}(t) = \frac{4\pi^3}{t}$.

$Z^{\nu_k}(t)$ is a *tight* family of functions in $L^1(\mathbb{R}^2) \Rightarrow$

$$\int Z^{\nu_k}(t,x)\,dx \approx < Z^{\nu_k}(t), \phi >,$$

for ϕ a bump function on large ball.

Then, pass to limit $\nu \to 0$, use weak convergence of defects and explicit uniform L^1 bound.

Enstrophy defect cont.

Near and far-field estimate on the heat kernel $\Rightarrow$ Every convergent subsequence $Z^{\nu_k}(t) := Z^{\nu_k}(\omega)(t)$ converges to a distribution supported at the origin ($Z^\nu \geq 0$, hence limit is a measure):

$$Z^{\nu_k}(t) \rightharpoonup C_{\{\nu_k\}}(t)\,\delta.$$

Show $C_{\{\nu_k\}}(t) = \frac{4\pi^3}{t}$.

$Z^{\nu_k}(t)$ is a *tight* family of functions in $L^1(\mathbb{R}^2) \Rightarrow$

$$\int Z^{\nu_k}(t,x)\,dx \approx < Z^{\nu_k}(t), \phi >,$$

for ϕ a bump function on large ball.

Then, pass to limit $\nu \to 0$, use weak convergence of defects and explicit uniform L^1 bound.

This should be typical behavior in $B^0_{2,\infty}$.

Expect estimate to follow from using a Littlewood-Paley decomposition and splitting high and low frequencies at $k \approx \log \nu t$.

Main difficulty is that the Besov norm is not rearrangement invariant, hence not preserved by the flow.

$Z^\nu(\omega)$ convergence weakly-$*$ to a (non-negative) measure by a diagonal argument.

This should be typical behavior in $B^0_{2,\infty}$.

Expect estimate to follow from using a Littlewood-Paley decomposition and splitting high and low frequencies at $k \approx \log \nu t$.

Main difficulty is that the Besov norm is not rearrangement invariant, hence not preserved by the flow.

$Z^\nu(\omega)$ convergence weakly-$*$ to a (non-negative) measure by a diagonal argument.

This should be typical behavior in $B^0_{2,\infty}$.

Expect estimate to follow from using a Littlewood-Paley decomposition and splitting high and low frequencies at $k \approx \log \nu t$.

Main difficulty is that the Besov norm is not rearrangement invariant, hence not preserved by the flow.

$Z^\nu(\omega)$ convergence weakly-$*$ to a (non-negative) measure by a diagonal argument.

THANK YOU!